PROPERTIES OF THE JULIA SET OF A RATIONAL MAP

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Introduction

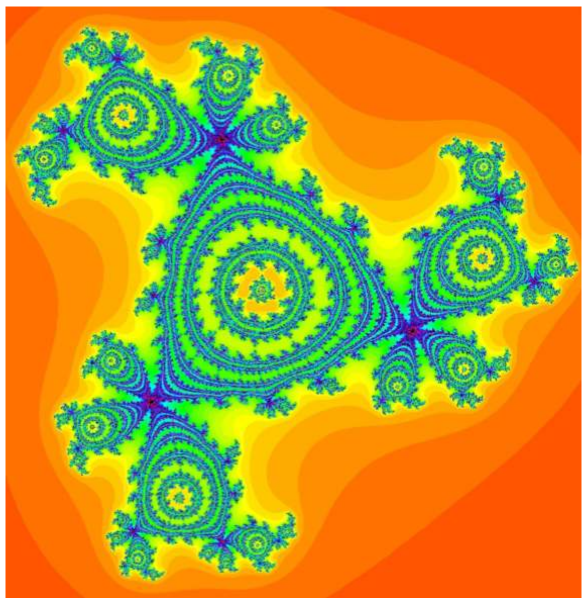

FIGURE 0.1. The Julia set for $z^3 + \lambda/z^3 + c$ when $c = 0.583840.27022i$ and $\lambda = 0.0000001$ [4].

The theory of iteration of rational maps is a relatively new area in mathematics, which has enjoyed a bit of a renaissance the last three decades thanks to computer images that reveal the beauty of the Mandelbrot set and various Julia sets. I first stumbled upon images of some Julia sets and the Mandelbrot set while searching for images of the Cantor set and the Sierpinski triangle. Certainly, like many, I was first attracted to this area of mathematics because of the obvious complexity of these sets that these images revealed. Though what ultimately hooked me in was the complex analysis involved. I had never heard of pointwise convergence or considered the notion of a sequence of functions before my study of iterating rational functions began. I would have had too much fun in analysis with so many different types of epsilon arguments to choose from. Indeed I am quite pleased with the mathematics behind the theory of iteration of rational maps.

To this end, we seek out some basic results, where for the most part the theme involves either a family of maps or the image of a set under a rational map.

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1. Preliminaries

In our work it is important to be able to handle expressions which we could not make sense of if we restricted ourselves to just the complex plane with the standard Euclidean metric. To this end, we introduce an abstract point known as the *point at infinity*, which we denote by ∞, and adjoin it to the complex plane. The *extended complex plane* is then the union

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

In order to have a geometric model for $\widehat{\mathbb{C}}$, we note that if we remove any single point from the unit sphere in the Euclidean space $\mathbb{R}^3$, then the resulting space is homeomorphic with the complex plane $\mathbb{C}$. Let S^2 be the unit sphere in $\mathbb{R}^3$ centered around the origin,

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\},$$

and let $N = (0, 0, 1)$ denote the north pole of S^2. We use *stereographic projection* to homeomorphically map $\mathbb{C}$ onto $S^2 \setminus N$. We then declare the point at infinity to be the north pole, and obtain the desired model known as the *Riemann sphere*. The metric we obtain from this process is known as the *chordal metric*, which we denote by σ. The explicit formula for σ is

$$\sigma(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}} \quad \text{when } z, w \in \mathbb{C};$$

in the case w is ∞ we take the limit, $\lim_{w\to\infty} \sigma(z, w)$, and obtain

$$\sigma(z, \infty) = \lim_{w\to\infty} \sigma(z, w) = \frac{2}{\sqrt{1 + |z|^2}}.$$

We note that $\widehat{\mathbb{C}}$ is a compact space under the topology induced by the metric σ, this is a key fact which we will often exploit.

Before we may give our definition of the Fatou and Julia sets of a rational map, we introduce the notion of an equicontinuous family of functions.

Definition 1. *Let $\mathcal{F} = \{f_\alpha\}$ be a family of maps from a metric space (X, d) to a metric space (X', d'). The family $\mathcal{F}$ of functions is* ***equicontinuous*** *on a set $S \subseteq X$ if and only if, given $x_0 \in S$, for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in S$ and $f \in \mathcal{F}$, if $d(x_o, x) < \delta$, then $d'(f(x_o), f(x)) < \varepsilon$.*

Notable in this definition is that δ depends, in general, on both ε and $x_0 \in S$, but is chosen independent of the function $f \in \mathcal{F}$. If δ may be chosen independent of the point $x_0 \in S$, then we say that the family $\mathcal{F}$ is uniformly equicontinuous on S. In the case a family $\mathcal{F}$ of functions is equicontinuous on a compact set K, we may say that $\mathcal{F}$ is uniformly equicontinuous on K. Obviously any finite family of continuous functions is equicontinuous, but if the family is infinite then things become less apparent.

In our context, we are of course considering the family of iterates $\{\, R^n \mid n \in \mathbb{N} \,\}$ of a rational map R from $(\widehat{\mathbb{C}}, \sigma)$ into itself. We may now give a definition of the Fatou and Julia sets of a rational map R in terms of equicontinuity.

Definition 2. *Let R be a non-constant rational function. The* ***Fatou set*** *of R is the maximal open subset of $\widehat{\mathbb{C}}$ on which the family of iterates $\{ R^n \,|\, n \in \mathbb{N}\}$ is equicontinuous, and the* ***Julia set*** *of R is the complement of the Fatou set in $\widehat{\mathbb{C}}$.*

We denote the Fatou and Julia sets of a rational map R by $F(R)$ and $J(R)$, respectively.

Notice that the Julia set is compact, as it is defined to be the complement of an open set in the compact space $(\widehat{\mathbb{C}}, \sigma)$. This is another key fact which at times we will exploit.

Proposition 1.1. *The Julia set of the map $P(z) = z^2$ is the unit cirlce S^1.*

Proof. Suppose $z_0 \notin S^1$. We will show that the family of iterates $\mathcal{F} = \{P^n \,|\, n \in \mathbb{N}\}$ is equicontinuous at z_0. Let $\varepsilon > 0$ be given. We consider the two possible cases.

Case 1. $|z_0| < 1$, so z_0 is in the unit disk. Let $\bar{D}$ denote the closed disk with center at z_0 and radius $\frac{1-|z_0|}{2}$, $\bar{D} = \bar{D}(z_0, \frac{1-|z_0|}{2})$. Given $z \in \bar{D}$ we have that $|z| \leq \frac{1+|z_0|}{2} < 1$, and so $|z| = \frac{1}{1+r}$ for some number $r > 0$. It clearly follows from Bernoulli's Inequality that $\frac{1}{(1+r)^{2n}} < \frac{1}{1+2nr}$ for all $n \in \mathbb{N}$. By the Archimedean property, we may find $N \in \mathbb{N}$ such that $\varepsilon N > 1/r$. Hence we have

$$|P^n(z)| = \frac{1}{(1+r)^{2n}} < \frac{1}{1+2nr} < \frac{\varepsilon}{2} \text{ for all } n \geq N.$$

So the sequence $\{P^n\}$ of iterates converges pointwise on $\bar{D}$.

Now consider the set $U_n = \{z \in \mathbb{C} : |P^n(z)| < \frac{\varepsilon}{2}\}$. Because each P^n is continuous, we have that U_n is open for all $n \in \mathbb{N}$. Clearly if $z \in U_N$, then $z \in U_n$ for all $n \geq N$, and so the sets U_n are nested,

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq U_{n+1} \subseteq \cdots .$$

From the pointwise convergence of the sequence $\{P^n\}$, we know that for each $z \in \bar{D}$ there is a positive integer $N(z) \in \mathbb{N}$ so that $z \in U_{N(z)}$. Hence the family $\{U_n\}_{n=1}^{\infty}$ covers $\bar{D}$. By compactness, there is a positive integer $N \in \mathbb{N}$ so that the finite subcollection $\{U_n\}_{n=1}^{N}$ covers $\bar{D}$, but since the sets are nested we have that $U_N \supset \bar{D}$. Thus for all but a finite number of the iterates $P^n(z)$, we have that

$$|P^n(z) - P^n(z_0)| \leq |P^n(z)| + |P^n(z_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for all } z \in \bar{D}.$$

Because each function in the set $\{P^n \,|\, n = 1, 2, \ldots, N-1\}$ is continuous on $\mathbb{C}$, we know that for each of these P^n there is a number $\delta_n > 0$ so that $|P^n(z) - P^n(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta_n$. Setting $\delta = min\{\frac{1-|z_0|}{2}, \delta_1, \delta_2, \ldots, \delta_{N-1}\}$ we obtain

$$|P^n(z) - P^n(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta \text{ for all } n \in \mathbb{N}.$$

Thus we have shown that $\mathcal{F}$ is equicontinuous on the unit disk.

Case 2. $|z_0| > 1$. Note that the image of the set $A = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$ under the Möbius map $M(z) = \frac{1}{z}$ is the punctured unit disk $D \setminus \{0\}$. By the continuity of $M(z)$ on A, if $K \subset A$ is compact then $M(K) \subset D \setminus \{0\}$ is compact. From our work in

Case 1, it now follows that the sequence $\{P^n \circ M\}$ of functions converges uniformly to the constant function $f(z) = 0$ on every compact subset $K \subset A$. Notice

$$\frac{1}{2}\sigma(P^n M(z), 0) = \frac{1}{\sqrt{|z|^{4n} + 1}} \leq \frac{1}{|z^{2n}|} = |P^n M(z)| \text{ for all } z \in A \text{ and } n \in \mathbb{N}.$$

So the sequence $\{P^n \circ M\}$ converges uniformly to the constant function $f(z) = 0$ on compact subsets of A with respect to the chordal metric as well.

Because A is open, we may find $r > 0$ such that A contains the closed disk of center z_0 and radius r, we denote this disk by $\bar{D}(z_0, r)$. Since M is an isometry of the complex sphere, we have that $\sigma(M(z), M(0)) = \sigma(z, 0)$. Since $\bar{D}(z_0, r) \subset A$ is compact, we see that for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that

$$\sigma(P^n(z), \infty) = \sigma(MP^n M(z), M(0)) < \varepsilon \quad \text{whenever } n \geq N \text{ for all } z \in \bar{D}(z_0, r).$$

Thus the sequence $\{P^n\}$ converges uniformly to the constant function $g(z) = \infty$ on every compact set $K \subset A$. We will soon see that such convergence implies that $\mathcal{F}$ is equicontinuous on every compact set $K \subset A$. Therefore, in a less elegant manner than that of Case 1, we conclude that $\mathcal{F}$ is equicontinuous on A.

These cases show that $z_0 \in F(P)$ (Definition 2). Therefore $z_0 \notin J(P)$, and so $J(P) \subseteq S^1$.

Suppose $z_0 \in S^1$, so $z_0 = e^{i\theta_0}$ for some number $\theta_0 \in (0, 2\pi]$. We will show that $\mathcal{F}$ is not equicontinuous at z_0. Set $\hat{\varepsilon} = 1/2$ and let $\delta > 0$ be given. Define $\hat{r} = max\{1 - \delta/2, 1/2\}$ and let $\hat{z} = \hat{r}e^{i\theta_0}$. So $|\hat{z} - z_0| < \delta$ and $|\hat{z}| < 1$. From our work in case 1, we know that for this number $\hat{\varepsilon}$ there exists a positive integer $M \in \mathbb{N}$ so that $|P^n(\hat{z})| < \hat{\varepsilon}$ for all $n \geq M$. Hence we have

$$|P^n(\hat{z}) - P^n(z_0)| \geq ||P^n(\hat{z})| - |P^n(z_0)|| > 1/2 = \hat{\varepsilon} \text{ for all } n \geq M\,.$$

Therefore $\mathcal{F}$ is not equicontinuous at z_0, and so $J(P) = S^1$. □

Theorem 1.2. *For any non-constant rational map R, and any positive integer m, $F(R^m) = F(R)$ and $J(R^m) = J(R)$.*

Proof. Let $S = R^m$ and let $\mathcal{F}_0 = \{\, S^n \,|\, n \geq 0\,\}$. Since $\mathcal{F}_0$ is a subfamily of the family of iterates $\{R^n\}$, $\mathcal{F}_0$ is equicontinuous wherever $\{R^n\}$ is equicontinuous, and so $F(R) \subseteq F(S)$.

To show $F(S) \subseteq F(R)$, suppose that $z_0 \in F(S)$. Let k be any positive integer and let $\varepsilon > 0$ be given. Since R is continuous on $\widehat{\mathbb{C}}$, the composite R^k is also continuous on $\widehat{\mathbb{C}}$. Next for each $z \in \widehat{\mathbb{C}}$, let $U_z = \{w \in \widehat{\mathbb{C}} : \sigma(R^k(z), R^k(w)) < \varepsilon/2\}$. By the continuity of R^k, we know that each set U_z is open. So the family $\{U_z\}_{z\in\widehat{\mathbb{C}}}$ is an open covering of $\widehat{\mathbb{C}}$. Since $\widehat{\mathbb{C}}$ is compact, there is a number $\lambda > 0$ so that for every $w \in \widehat{\mathbb{C}}$ the disk $D(w, \lambda)$ is contained in one of the open sets of the family $\{U_z\}_{z\in\widehat{\mathbb{C}}}$ (Lebesgue's Number Lemma). Thus for any $v, w \in \widehat{\mathbb{C}}$ satisfying $\sigma(v, w) < \lambda$, there is a $z \in \widehat{\mathbb{C}}$ so that $v, w \in U_z$. We combine to see that for this given positive number ε there is a number $\lambda > 0$ so that, for any $v, w \in \widehat{\mathbb{C}}$,

$$\sigma(R^k(v), R^k(w)) \leq \sigma(R^k(v), R^k(z)) + \sigma(R^k(z), R^k(w)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $\sigma(v, w) < \lambda$.

Since $\mathcal{F}_0$ is equicontinuous at z_0, we know that for this positive number λ, there is a number $\delta > 0$ such that, for all $n \in \mathbb{N}$, $\sigma(S^n(z_0), S^n(z)) < \lambda$ whenever $\sigma(z_0, z) < \delta$. Hence for all $n \in \mathbb{N}$ we have that $\sigma(R^k S^n(z_0), R^k S^n(z)) < \varepsilon$ whenever $\sigma(z_0, z) < \delta$. This gives us that each family $\mathcal{F}_k = \{ R^k \circ S^n \mid n \geq 0 \}$ is equicontinuous at z_0, and so too is the finite union

$$\bigcup_{n=0}^{m-1} \mathcal{F}_n \, .$$

Since this union is the family $\{ R^n \mid n \in \mathbb{N} \}$, we have that $z_0 \in F(R)$ and so $F(S) = F(R)$. Notice that since the Julia set is defined as the complement of the Fatou set in $\widehat{\mathbb{C}}$, it follows that $J(S) = J(R)$.

□

Before we develop our knowledge of the Julia and Fatou sets further, it is important that we introduce the notion of invariance.

Definition 3. *Let $f : X \to X$ be a self-map and let $A \subseteq X$. We say that A under f is:*

(i) ***forward invariant*** *if $f(A) = A$;*
(ii) ***backward invariant*** *if $f^{-1}(A) = A$;*
(iii) ***completely invariant*** *if A is both forward and backward invariant.*

Notice that for a non-constant rational map R, the Fundamental Theorem of Algebra gives us that $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is surjective. Now suppose A is backward invariant under a non-constant rational map R. Then it follows from the definition of backward invariant that $R(R^{-1}(A)) = R(A)$. By surjectivity, $R(R^{-1}(A)) = A$, and so we combine to conclude that $A = R(A)$. Hence A is also forward invariant under R. This shows that a set $A \subseteq \widehat{\mathbb{C}}$ is completely invariant under a rational map R if and only if it is backward invariant.

We give the following theorem to formalize an important notion regarding the division of $\widehat{\mathbb{C}}$ determined by a rational map R into the sets $F(R)$ and $J(R)$.

Theorem 1.3. *For any rational map R the Fatou set F and the Julia set J are completely invariant under R.*

Proof. By surjectivity, it suffices to show that $R^{-1}(J) = J$ and $R^{-1}(F) = F$. Since $F = \widehat{\mathbb{C}} \setminus J$, it follows that $R^{-1}(F) = R^{-1}(\widehat{\mathbb{C}} \setminus J) = \widehat{\mathbb{C}} \setminus R^{-1}(J)$, and vice versa. Since the case where R is constant is trivial, we only consider the case where R is non-constant. Hence by the open mapping, R is an open map.

Suppose $z_0 \in R^{-1}(F)$, or equivalently $z_0 \notin R^{-1}(J)$. Let $w_0 = R(z_0)$, so $w_0 \in F$. Let $\varepsilon > 0$ be given. By definition of F, for this number ε there is a number $\delta > 0$ such that $\sigma(R^n(w), R^n(w_0)) < \varepsilon$ whenever $\sigma(w, w_0) < \delta$ for all $n \in \mathbb{N}$. Because R is continuous at z_0, for this positive number δ there is a number $\rho > 0$ so that $\sigma(R(z), w_0) < \delta$ whenever $\sigma(z, z_0) < \rho$. We combine to see that there is a number $\rho > 0$ such that

$$\sigma(R^{n+1}(z), R^{n+1}(z_0)) = \sigma(R^n(R(z)), R^n(w_0)) < \varepsilon \text{ whenever } \sigma(z, z_0) < \rho \text{ for all } n \in \mathbb{N}.$$

So the family $\{R^{n+1} : n \in N\}$ is equicontinuous at z_0. Clearly we may add the single continuous function $R(z)$ to this family and the resulting family will still be equicontinuous at z_0. Thus $z_0 \in F$, and hence we have that $R^{-1}(F) \subseteq F$ and that $J \subseteq R^{-1}(J)$.

Suppose $z_0 \in F$, or equivalently $z_0 \notin J$. Let $w_0 = R(z_0)$, and let $\varepsilon > 0$ be given. By definition of F, for this number ε there exists $\delta > 0$ such that $\sigma(R^{n+1}(z), R^{n+1}(z_0)) < \varepsilon$ whenever $\sigma(z, z_0) < \delta$ for all $n \in \mathbb{N}$. Let D denote the open disk $D = \{z \in \widehat{\mathbb{C}} : \sigma(z, z_0) < \delta\}$. Since $R(D)$ is open and $w_0 \in R(D)$, we may find $\rho > 0$ so that the open disk $D(w_0, \rho) = \{w \in \widehat{\mathbb{C}} : \sigma(w, w_0) < \rho\}$ is completely contained in $R(D)$. Notice that if $w \in D(w_0, \rho)$, then $w = R(z)$ for some $z \in D$. Thus we have that

$$\sigma(R^n(w), R^n(w_0)) = \sigma(R^{n+1}(z), R^{n+1}(z_0)) < \varepsilon \text{ whenever } \sigma(w, w_0) < \rho \text{ for all } n \in \mathbb{N}.$$

Hence $w_0 \in F$, and so $z_0 \in R^{-1}(F)$. We conlude that $R^{-1}(F) = F$ and $R^{-1}(J) = J$, as required. □

Before we give our next result we must introduce the notion of valency. We define the valency $v_R(z_0)$ of a rational map R at z_0 to be the number of solutions to the equation $R(z) = R(z_0)$ at z_0, counting multiplicity. To illustrate we consider the simplest rational function of degree d, namely $R(z) = z^d$. Since all d zeros of $R(z)$ are at 0, we have that $v_R(0) = d$. Notice that for any non-constant rational map R we have $v_R(z) \geq 1$ for all $z \in \widehat{\mathbb{C}}$. What is less clear is the fact that there are only finitely many $z \in \widehat{\mathbb{C}}$ for which $v_R(z) > 1$, this is fundamental. The relationship between the degree of a non-constant rational map R and the number of $z \in \widehat{\mathbb{C}}$ for which $v_R(z) > 1$ is expressed by the *Riemann-Hurwitz relation*:

$$\sum [v_R(z) - 1] = 2deg(R) - 2.$$

We may now obtain a result which will be of great avail to us later.

Theorem 1.4. *Let R be a rational map of degree at least two. If a finite nonempty set E is completely invariant under R, then E has at most two elements.*

Proof. Suppose E has k elements for some positive integer k. From the definition of completely invariant, it follows that R restricted to E is a bijection. Since R must act as a permutation on the k elements of E, there exists some positive integer j so that $R^j(e) = e$ for all $e \in E$. We claim that given $e_0 \in E$, all solutions to the equation $R^j(z) = R^j(e_0)$ are at e_0. Since E is completely invariant under R and since R^j restricted to E is the identity mapping, the claim follows. Now let d denote the degree of R^j. Since all d solutions to the equation $R^j(z) = R^j(e_0)$ are at e_0, we have that $v_R(e_0) = d$. Applying the Riemann-Hurwitz relation to R^j, we obtain

$$\sum_{e \in E} [v_{R^j}(e) - 1] = k(d-1) \leq 2(d-1).$$

Since $d \geq 2$, we conclude that $k \leq 2$. □

2. Normal Families

In this section we shall introduce the notion of a normal family of functions. It will become quite clear that an equicontinuous family is closely related to a normal family.

Definition 4. *Let $\mathcal{F} = \{f_\alpha\}$ be a family of maps from a metric space (X, d) to a metric space (X', d'). The family $\mathcal{F}$ of functions is* ***normal*** *on a set $S \subseteq X$ if and only if every sequence $\{f_n\}$ of functions in $\mathcal{F}$ contains a subsequence $\{f_{n_k}\}$ which converges uniformly on every compact subset of S.*

We may now give the theorem that will establish the desired connection between normality and equicontinuity. In the statement of the theorem we say that the functions f in the family $\mathcal{F}$ are functions from a subdomain $D \subset \mathbb{C}$ into $\widehat{\mathbb{C}}$, that is $f : D \to \widehat{\mathbb{C}}$. However, out of preference, for the first half of the proof we actually assume that the functions f in $\mathcal{F}$ are maps into $\mathbb{C}$. Hence we use only the standard Euclidean metric on $\mathbb{C}$ to prove that normality implies equicontinuity.

Theorem 2.1 (Arzela-Ascoli Theorem)**.** *Let D be a subdomain of $\mathbb{C}$. A family $\mathcal{F}$ of continuous functions from D into $(\widehat{\mathbb{C}}, \sigma)$ is normal in D if and only if $\mathcal{F}$ is equicontionuous on every compact set $E \subset D$.*

Proof. Assume $\mathcal{F}$ is normal in D and suppose there is a compact set $E \subset D$ on which $\mathcal{F}$ is not equicontinuous. Since $\mathcal{F}$ fails to be equicontinuous on E there exists a number $\varepsilon_0 > 0$, two sequences of points $\{z_n\}$ and $\{w_n\}$ in E, and a sequence of functions $\{f_n\}$ in $\mathcal{F}$ such that $|z_n - w_n| < 1/n$ but $|f_n(z_n) - f_n(w_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Because $\mathcal{F}$ is normal in D there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges uniformly on E. Since E is compact, by the Bolzano-Weierstrass Theorem there is a subsequence $\{z_p\}$ of $\{z_{n_k}\}$ that converges to an element $z_0 \in E$. Notice that given $\varepsilon > 0$ we may find $N_0 \in \mathbb{N}$ such that $|z_p - w_p| < \varepsilon/2$ and $|z_p - z_0| < \varepsilon/2$ for all $p \geq N_0$. Since

$$|w_p - z_0| \leq |w_p - z_p| + |z_p - z_0|,$$

it follows that the corresponding subsequence $\{w_p\}$ of $\{w_{n_k}\}$ also converges to z_0. Since $\{f_p\}$ is a subsequence of $\{f_{n_k}\}$ it has the same limit f. By the uniform limit theorem, the limit function f of $\{f_p\}$ is continuous on E, and hence it is uniformly continuous on E. Set $\varepsilon' = \varepsilon_0/4$, then there is a number $\delta > 0$ so that $|f(z) - f(w)| < \varepsilon'$ whenever $|z - w| < \delta$ for all $z, w \in E$. For this number δ there is $N_1 \in \mathbb{N}$ such that $|z_p - z_0| < \delta$ and $|w_p - z_0| < \delta$ for all $p \geq N_1$. Also, we may find $N_2 \in \mathbb{N}$ such that $|f_p(z) - f(z)| < \varepsilon'$ for all $p \geq N_2$ and $z \in E$. Setting $N = max\{N_1, N_2\}$ we obtain

$$|f_p(z_p) - f_p(w_p)| \leq |f_p(z_p) - f(z_p)| + |f(z_p) - f(z_0)| + |f(z_0) - f(z_p)| + |f(w_p) - f_p(w_p)| < \varepsilon_0$$

for all $p \geq N$. This is a contradiction, and thus $\mathcal{F}$ is equicontinuous on E.

Now assume $\mathcal{F}$ is equicontinuous on every compact set $E \subset D$. Let $E \subset D$ be a compact subset of D and let $\{f_n\}$ be a sequence in $\mathcal{F}$. Let $E_{\mathbb{Q}} = \{x + iy \in E \,|\, x, y \in \mathbb{Q}\}$ denote the subset of points in E with both real and imaginary parts rational. Since $E_{\mathbb{Q}}$ is countable, we may let $\{\zeta_1, \zeta_2, \ldots, \zeta_k, \zeta_{k+1}, \ldots\}$ be an enumeration of $E_{\mathbb{Q}}$. From the given sequence $\{f_n\}$ we are going to extract a subsequence which converges at all points $\zeta_k \in E_{\mathbb{Q}}$. We may find a subsequence that converges at a given point ζ_k since the values of the sequence $\{f_n(\zeta_k)\}$ lie in $\widehat{\mathbb{C}}$ which is compact. For the point ζ_1, we label the subsequence as indicated and denote it by

$$\{f_{n_{1j}}(\zeta_1)\} = \{f_{n_{11}}(\zeta_1), f_{n_{12}}(\zeta_1), \dots, f_{n_{1j}}(\zeta_1), \dots\}.$$

Next consider $\{f_{n_{1j}}(\zeta_2)\}$, this sequence of points in $\widehat{\mathbb{C}}$ has a convergent subsequence which we denote by $\{f_{n_{2j}}(\zeta_2)\}$. Continuing this process of constructing successive subsequences gives rise to an array of functions,

$$\begin{array}{ccccc} f_{n_{11}} & f_{n_{12}} & \cdots & f_{n_{1j}} & \cdots \\ f_{n_{21}} & f_{n_{22}} & \cdots & f_{n_{2j}} & \cdots \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\vdots\vdots \\ f_{n_{k1}} & f_{n_{k2}} & \cdots & f_{n_{kj}} & \cdots \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\vdots\vdots \end{array}$$

Since each row of functions in this array is containted in the preceding one, it follows that the diagonal sequence $\{f_{n_{jj}}\} = \{f_{n_{11}}, f_{n_{22}}, \dots, f_{n_{jj}}, \dots\}$ of functions is eventually a subsequence of each row. Combining this observation with the fact that $\lim_{j\to\infty} f_{n_{kj}}(\zeta_k)$ exists for all $\zeta_k \in E_{\mathbb{Q}}$, gives us that $\{f_{n_{jj}}\}$ is a subsequence of $\{f_n\}$ which converges at every point ζ_k. We simplify notation by setting $f_{n_{jj}} = g_j$.

Let $\varepsilon > 0$ be given. Since $\mathcal{F}$ is equicontinuous on the compact set E, we know that there is a number $\delta > 0$ such that $\sigma(f(z), f(w)) < \varepsilon/3$ whenever $|z - w| < \delta$, for all $f \in \mathcal{F}$ and $z, w \in E$. Since the family of open discs $\{D(z, \frac{\delta}{2})\}_{z\in E}$ is a covering of E, it admits a finite subcovering $\{D(z_n, \frac{\delta}{2})\}_{n=1}^{N}$. Since $E_{\mathbb{Q}}$ is everywhere dense in E, for each open disk $D(z_n, \frac{\delta}{2})$ of this finite cover there exists a point $\zeta_{k(n)} \in E_{\mathbb{Q}}$ such that $\zeta_{k(n)} \in D(z_n, \frac{\delta}{2})$. Since $\{g_j(\zeta_{k(n)})\}$ is a convergent sequence, it is a Cauchy sequence. So given one of these points $\zeta_{k(n)}$, we know that there exists a positive integer $J_{k(n)} \in \mathbb{N}$ such that $\sigma(g_i(\zeta_{k(n)}), g_j(\zeta_{k(n)})) < \varepsilon/3$ whenever $i, j \geq J_{k(n)}$. We set $J = max\{J_{k(n)}\}$ to obtain

$$\begin{aligned} \sigma(g_i(z), g_j(z)) &\leq \sigma(g_i(z), g_i(\zeta_{k(n)})) + \sigma(g_i(\zeta_{k(n)}), g_j(\zeta_{k(n)})) + \sigma(g_j(\zeta_{k(n)}), g_j(z)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{whenever } i, j \geq J \text{ for all } z \in E. \end{aligned}$$

Therefore $\{g_j\}$ is uniformly convergent on E, as it is uniformly Cauchy in the compact and hence complete metric space $\widehat{\mathbb{C}}$. □

We give two theorems on normality that will make our interest in the Arzela-Ascoli Theorem apparent. These results are fundamental in the study of iteration of rational maps. A full proof of the following theorem is contained in [1], p.60. The proof is so substantial that it requires its own appendix, and so we omit the proof.

Theorem 2.2. *Let $\mathcal{F}$ be a family of maps, each analytic on a common domain $D \subset \widehat{\mathbb{C}}$. If each function f in $\mathcal{F}$ omits the three values 0, 1, and ∞ in D, then the family $\mathcal{F}$ is normal in D.*

The proof of our next result dealing with normality shows how Theorem 2.2 produces a stronger variation of similar theme. However, before we may give the next result on normality, we will need the following lemma.

Lemma 2.3. *Let m be a given positive number. Then the family $\mathcal{G}$ of Möbius transformations g which satisfy*

$$\sigma(g(0), g(1)) \geq m, \quad \sigma(g(1), g(\infty)) \geq m, \quad \sigma(g(\infty), g(0)) \geq m,$$

also satisfies the uniform Lipschitz condition

$$\sigma(g(z), g(w)) \leq (\pi/m^3)\sigma(z, w) \text{ for all } g \in \mathcal{G} \text{ and } z, w \in \widehat{\mathbb{C}}.$$

Again we unfortunately do not provide proof of the above result, but note that a proof is contained in [1], p.34. Now that we have liberally taken the previous two results, we may state and give proof of the following theorem.

Theorem 2.4 (Montel's Theorem)**.** *Let $\mathcal{F}$ be a family of maps, each analytic on a common domain $D \subset \widehat{\mathbb{C}}$. If for each $f \in \mathcal{F}$ there exist three values $a_f, b_f, c_f \in \mathbb{C}_\infty$ such that:*

(i) there exist $m > 0$ so that $min\{\sigma(a_f, b_f), \sigma(c_f, b_f), \sigma(c_f, a_f)\} \geq m$; and
(ii) f omits the three values a_f, b_f, and c_f in D.

Then the family $\mathcal{F}$ is normal in D.

Proof. We assume that for each $f \in \mathcal{F}$ there exist three values $a_f, b_f, c_f \in \mathbb{C}_\infty$ so that (i) and (ii) are satisfied. Now for each f in $\mathcal{F}$, define the Möbius transformation g_f by

$$g_f(0) = a_f, \quad g_f(1) = b_f, \quad g_f(\infty) = c_f.$$

By the preceeding lemma, (ii) implies that the family $\mathcal{G} = \{g_f : f \in \mathcal{F}\}$ satisfies the uniform Lipschitz condition on $\widehat{\mathbb{C}}$,

$$\sigma(g_f(z), g_f(w)) \leq (\pi/m^3)\sigma(z, w).$$

For each $f \in F$ define $h_f = g_f^{-1} \circ f$. So each analytic function in the family $\mathcal{H} = \{h_f : f \in \mathcal{F}\}$ omits the three values 0, 1, and ∞ in D. By Theorem 2.2, the family $\mathcal{H}$ is normal in D, and hence equicontinuous there.

Now let $\varepsilon > 0$ be given, and let $z_0 \in D$ be an arbitrary point of D. We know that for the positive number $\frac{m^3}{\pi}\varepsilon$ there is $\delta > 0$ such that

$$\sigma(h_f(z), h_f(z_0)) < \frac{m^3}{\pi}\varepsilon \quad \text{whenever } \sigma(z, z_0) < \delta \quad \text{for all } h_f \in \mathcal{H}.$$

We combine to conclude that

$$\sigma(f(z), f(z_0)) = \sigma(g_f \circ h_f(z), g_f \circ h_f(z_0)) \leq \frac{\pi}{m^3}\sigma(h_f(z), h_f(z_0)) < \varepsilon$$

whenever $\sigma(z, z_0) < \delta$ for all $f \in \mathcal{F}$. □

3. Properties of the Julia Set

In this section we develop several properties of the Julia set of a rational map. The proofs of these results will exploit Theorem 2.4, highlighting its importance to our study.

However, we must begin with a rather out of place theorem.

Theorem 3.1. *If a sequence $\{R_n\}$ of rational functions converges uniformly on the entire complex sphere to a function R, then R is rational and for all sufficiently large integers n, $deg(R_n) = deg(R)$.*

Proof. Suppose the sequence $\{R_n\}$ of rational functions converges uniformly on the entire complex sphere to the function $R(z)$. It can be shown using Cauchy's integral formula that the uniform limit of analytic functions is analytic. Hence R is analytic on the entire complex sphere, and so it is rational.

Next we prove the other half of the statement. Without loss of generality, we assume that $R(\infty) \neq 0$, since otherwise we could consider the sequence $\{1/R_n\}$. Since the case where $R(z)$ is constant is trivial, we only consider the case where $R(z)$ is non-constant. With these assumptions, $R(z)$ has distinct zeros, say $z_1, \ldots, z_j$ and these all lie in $\mathbb{C}$. Suppose $R(z)$ has a zero of order M at z_0. We know there exists $r > 0$ so that z_0 is the only zero of $R(z)$ in the disk of radius r and center at z_0, $D(z_0, r)$. Applying Rouche's theorem, we know that there exists $\rho > 0$ so that for all sufficiently large integers n, $R_n(z)$ has M zeros in the disk $D(z_0, \rho) = \{z \in \widehat{\mathbb{C}} : \sigma(z, z_0) < \rho\}$. We set $D_0 = D(z_0, r) \cap D(z_0, \rho)$, and consruct a D_j for each of the zeros z_j of $R(z)$. Now let $K = \left[\bigcup D_j\right]^c$ be the complement of the union of the D_j in $\widehat{\mathbb{C}}$. Since K is compact, $R(z)$ is bounded away from zero on K. Hence for n large, R_n is bounded away from zero on K as well. This shows that for all sufficiently large integers n, R_n and R have the same number of zeros. $\square$

Again this result did not fit well in any section of the paper, but we include it here as it is needed to show a fundamental fact about the Julia set of a rational map of degree at least two.

For simplicity we give our definition of an exceptional point in terms of its backward orbit. So we state

The *backward orbit of* $z \in \widehat{\mathbb{C}}$ is denoted by $O^-(z)$ and defined by

$$\begin{aligned} O^-(z) :&= \{w \in \widehat{\mathbb{C}} : \exists n \geq 0, R^n(w) = z\} \\ &= \bigcup_{n \geq 0} R^{-n}(\{z\}). \end{aligned}$$

Definition 5. *A point $z \in \widehat{\mathbb{C}}$ is said to be* ***exceptional*** *for the rational map R when $O^-(z)$ is finite, and we denote the set of such points by $E(R)$.*

Notice that the terminology suits such points, as a rational map of degree at least two can have at most two exceptional points, by Theorem 1.4. Another important fact that we would like to make clear is that $E(R)$ is completely invariant under R.

With $E(R)$ defined we may now state

Lemma 3.2. *Let R be rational map. If $deg(R) \geq 2$, then $E(R) \subset F(R)$.*

We omit the proof of this result, but there is an argument given on p.66 of [1].

Theorem 3.3. *Let R be a non-constant rational map. If $deg(R) \geq 2$, then $J(R)$ is infinite.*

Proof. We first show that $J(R)$ is nonempty. Assume $J(R)$ is empty, we shall reach a contradiction. Since $J(R)$ is empty, the family $\{R^n\}$ is normal on the entire complex sphere. So there exists a subsequence which we denote by $\{R^{k(n)}\}$ that converges uniformly on $\widehat{\mathbb{C}}$. By the preceeding theorem, it follows that eventually the iterates in this subsequence have the same degree. Hence there exist two integers i, j with $i > j$ and $deg(R^i) = deg(R^j)$. However, $deg(R^n) = [deg(R)]^n$ and this would imply that R is constant or has degree one. This is a contradiction, and hence $J(R) \neq \emptyset$.

Now let $z_0 \in J(R)$. We know that $J(R)$ is completely invariant, so if $J(R)$ is finite then z_0 is exceptional. This cannot be the case, as the exceptional points of R lie in $F(R)$. Therefore $J(R)$ is infinite. $\square$

Theorem 3.4. *Let R be a rational map of degree at least two, and let U be any non-empty open set such that $U \cap J(R) \neq \emptyset$. Then:*
(i) $\bigcup_{n=0}^{\infty} R^n(U) \supset \widehat{\mathbb{C}} \setminus E(R)$; and
(ii) For all sufficiently large integers n, $R^n(U) \supset J(R)$.

Proof. Set $\mathcal{U} = \bigcup_{n=0}^{\infty} R^n(U)$ and let $K = \widehat{\mathbb{C}} \setminus \mathcal{U}$. Assume that K contains three distinct points. So each iterate R^n does not take on three distinct values in U. By Theorem 2.4, the family $\{R^n\}$ is then normal in U, and so $U \subseteq F(R)$. Since this is a contradiction, we have that $\mathcal{U}$ contains every point of $\widehat{\mathbb{C}}$ with the exception of at most two points.

Now consider a point z in the complement of $E(R)$. Since z is not exceptional its backward orbit $O^-(z)$ is infinite, by definition. From our preceding remark it then follows that $\mathcal{U} \cap O^-(z) \neq \emptyset$. Hence there exists a positive integer n and a point w so that $R^n(w) = z$. For this same point w there also exists a positive integer m such that $R^m(u) = w$ for some $u \in U$. We combine to obtain that $R^{n+m}(u) = z$, and so $z \in \mathcal{U}$. Therefore

$$\bigcup_{n=0}^{\infty} R^n(U) \supset \widehat{\mathbb{C}} \setminus E(R).$$

Next let U_1, U_2, and U_3 be three pairwise disjoint nonempty open subsets of U, each of which meets $J(R)$. We show that for each $k = 1, 2, 3$ there are two positive integers q and n such that

$$R^n(U_k) \supseteq U_q.$$

Supposing to the contrary, we assume that there is some k so that $R^n(U_k)$ fails to contain any U_q for $q = 1, 2, 3$ and all $n \in \mathbb{N}$. By Theorem 2.4, $\{R^n\}$ is then normal in U_k, and so $U_k \subseteq F(R)$. This is a contradiction.

So given $k \in \{1, 2, 3\}$ there exist a pair of integers which we denote by $(n(k), q(k))$ so that $R^{n(k)}(U_k) \supseteq U_{q(k)}$. Clearly for some $k \in \{1, 2, 3\}$ one of the following must hold true

$$R^{n(k)+n(j)+n(i)}(U_k) \supseteq U_k, \ R^{n(k)+n(j)}(U_k) \supseteq U_k, \ \text{or} \ R^{n(k)}(U_k) \supseteq U_k.$$

So for some positive integer L and some $k \in \{1, 2, 3\}$ we have $R^L(U_k) \supseteq U_k$.

We set $S = R^L$. Notice the above gives us that the sets in the family $\{S^m(U_k)\}_{m=0}^{\infty}$ are nested

$$U_k \subseteq S(U_k) \subseteq S^2(U_k) \subseteq \cdots \subseteq S^m(U_k) \subseteq S^{m+1}(U_k) \subseteq \cdots .$$

Now we apply (i) to obtain that the family $\{S^m(U_k)\}_{m=0}^{\infty}$ is a covering of J. By compactness, a finite subcollection $\{S^m(U_k)\}_{m=0}^{M}$ covers J, and so $S^M(U_k) \supset J$. We set $N = M \cdot L$ and combine to conclude that there is a positive integer N so that

$$J \subset R^N(U_k) \subseteq R^N(U).$$

By Theorem 1.3,

$$J = R(J) \subset R^{N+1}(U),$$

and so induction gives us that $J \subset R^n(U)$ for all $n \geq N$. $\square$

Theorem 3.5. *Let R be a non-constant rational map. If $deg(R) \geq 2$, then $J(R)$ is perfect and either the interior of $J(R)$ is empty or $J(R) = \widehat{\mathbb{C}}$.*

Proof. We shall consider the two possible cases.
Case 1. $J \neq \widehat{\mathbb{C}}$. Let J' be the derived set,

$$J' = \{z \in \widehat{\mathbb{C}} : \forall r > 0,\ D(z,r) \cap J \setminus \{z\} \neq \emptyset\}.$$

Since J is closed, we already have that $J' \subseteq J$. We first show that J' is an infinite, closed, completely invariant subset of J, and given this, we will then be able to show that $J \subseteq J'$. Let $r > 0$ be given. We know that the infinite collection of discs of radius r about each point z of J is a covering of J,

$$\bigcup_{z \in J} D(z,r) \supset J.$$

By compactness, there is a finite subcollection $\{D(z_n, r)\}_{n=0}^{N}$ which covers J. This shows that J' is nonempty, as one of the $D(z_n, r)$ contains infinitely many points of J.

Suppose $z_0 \notin R^{-1}(J')$. Let $w_0 = R(z_0)$, so $w_0 \notin J'$. By definition, there is $\rho > 0$ so that $D(w_0, \rho) \cap J \setminus \{w_0\} = \emptyset$. For this positive number ρ there exists $\delta > 0$ so that $\sigma(R(z), w_0) < \rho$ whenever $\sigma(z, z_0) < \delta$. Let D denote the set

$$D = \{z \in \widehat{\mathbb{C}} : \sigma(z, z_0) < \delta\}.$$

So we have that $R(D) \subseteq D(w_0, \rho)$, and hence $R(D) \cap J \setminus \{w_0\} = \emptyset$. Now since R has finite degree, we know that $R^{-1}(\{w_0\})$ will have finitely many distinct elements. Since the case where $R^{-1}(\{w_0\}) = \{z_0\}$ is trivial, we assume that the preimage of $\{w_0\}$ contains elements distinct from z_0, say $\zeta_1, \ldots, \zeta_d$. For $\zeta_d \in R^{-1}(\{w_0\}) \setminus \{z_0\}$, let $\sigma(z_0, \zeta_d) = r_d$. Set $r = min\{\delta, \frac{r_1}{2}, \ldots, \frac{r_d}{2}\}$. Since $r > 0$ and $D(z_0, r) \cap J \setminus \{z_0\} = \emptyset$, we have that $z_0 \notin J'$, by definition of the derived set. Therefore $J' \subseteq R^{-1}(J')$.

Next suppose $z_0 \in R^{-1}(J')$. Let $w_0 = R(z_0)$, so $w_0 \in J'$. Let $r > 0$ be given, and let $D = \{z \in \widehat{\mathbb{C}} : \sigma(z, z_0) < r\}$. Since R is an open map we know that $R(D)$ is an open neighborhood of w_0. By definition of the derived set, we know that $R(D) \cap J \setminus \{w_0\} \neq \emptyset$. So let $\omega \in R(D) \cap J \setminus \{w_0\}$. We know there is ζ in D so that $\omega = R(\zeta)$. Cleary $\zeta \neq z_0$ and $\zeta \in J$, as J is completely invariant. This shows that $z_0 \in J'$, and thus $J' = R^{-1}(J')$. By surjectivity, we conclude that J' is completely invariant.

Since J' is a closed, completely invariant subset of $\widehat{\mathbb{C}}$, we know that J' is infinite. Indeed if not, then J' would contain an exceptional point. This could not be true, since J is closed and hence contains all of its limit points.

Let G be the complement of J' in $\widehat{\mathbb{C}}$. We know that G is nonempty, as $J' \subseteq J \neq \widehat{\mathbb{C}}$. Also since J' is completely invariant, so is its complement G. We now choose three distinct points a_f, b_f, and c_f to be three given points in J'. By Theorem 2.4, the family of iterates $\{R^n\}$ is normal in G, and so $G \subseteq F$. It follows that $J \subseteq J'$, and hence $J = J'$.

Case 2. We assume J has nonempty interior. Let z_0 be an interior point of J. So there is some open neighborhood U of z_0 completely contained in J. We now apply Theorem 3.4 to the open set U to get that for all sufficienty large integers n, $R^n(U) \supset J$. Let $N \in \mathbb{N}$ be such that $R^N(U) \supset J$. Since $U \subset J$ and since J is completely invariant, it follows that $R^n(U) \subset J$ for all $n \in \mathbb{N}$. Thus $R^N(U) = J$, and so $R^N(U)$ is compact. However, $R^N(U)$ is open, as U is open. Since the only nonempty subset of $\widehat{\mathbb{C}}$ that is open and compact is $\widehat{\mathbb{C}}$ itself, it follows that $R^N(U) = \widehat{\mathbb{C}}$. By transitivity, $J = \widehat{\mathbb{C}}$. $\square$

In 1918 a French mathematician named Samuel Lattés discovered the rational function
$$R(z) = \frac{(z^2+1)^2}{4z(z^2-1)},$$
this is the first known example of a rational function for which the Julia set is the entire complex sphere.

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