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Citation: *Journal of Mathematical Physics* **56**, 063502 (2015); doi: 10.1063/1.4922018

View online: <http://dx.doi.org/10.1063/1.4922018>

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Properties of the Katugampola fractional derivative with potential application in quantum mechanics

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(Received 1 March 2015; accepted 22 May 2015; published online 5 June 2015)

Katugampola [e-print [arXiv:1410.6535](https://arxiv.org/abs/1410.6535)] recently introduced a limit based fractional derivative, D^α (referred to in this work as the Katugampola fractional derivative) that maintains many of the familiar properties of standard derivatives such as the product, quotient, and chain rules. Typically, fractional derivatives are handled using an integral representation and, as such, are non-local in character. The current work starts with a key property of the Katugampola fractional derivative, $D^\alpha[y] = t^{1-\alpha} \frac{dy}{dt}$, and the associated differential operator, $D^\alpha = t^{1-\alpha} D^1$. These operators, their inverses, commutators, anti-commutators, and several important differential equations are studied. The anti-commutator serves as a basis for the development of a self-adjoint operator which could potentially be useful in quantum mechanics. A Hamiltonian is constructed from this operator and applied to the particle in a box model. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4922018>]

I. INTRODUCTION

The concept of a fractional derivative, $D^\alpha \equiv \frac{d^\alpha}{dt^\alpha}$, where α is not an integer has been discussed for a long time.¹⁻⁴ Evidently, Leibniz wondered about the case when $\alpha = \frac{1}{2}$ shortly after the advent of calculus itself.^{1,3,4} Over history, fractional calculus, and in particular fractional differential equations, has been applied in a variety of fields.³⁻⁸ For several recent surveys of these applications, see Refs. 3, 4, and 6. Additionally, a fractional version of calculus of variations is receiving much attention.⁹⁻¹³ Until very recently,^{1,13,14} fractional derivatives have been cast as fractional integrals.^{2,5} This class of fractional derivatives is unsatisfying in some respects. First, the integral definition introduces non-local effects.¹ Also, many of the properties familiar to the standard (α an integer) derivative like the product, quotient, and chain rules do not always hold.¹ In 2014, Khalil *et al.* introduced a limit based definition analogous to that for standard derivatives.¹⁴ This was quickly generalized by Katugampola,¹ whose definition forms the basis for this work and is referred to here as the Katugampola fractional derivative. (D^α will henceforth be referring to the Katugampola fractional derivative.) This definition has a number of useful properties which are reviewed below.

A key property of the Katugampola fractional derivative is $D^\alpha[f(t)] = t^{1-\alpha} \frac{df(t)}{dt}$ for differentiable functions, f . (Note: operators of a very similar form, $t^\alpha D^1$, have been applied in combinatorial theory.¹⁵) Using this as a starting point, a variety of properties are explored in this work from a (fractional) differential operator perspective. Several important differential equations are investigated along with commutator and anti-commutator properties. A new self-adjoint operator is defined and its properties are explored. This operator is suggested as potentially being of value in quantum mechanics. A plausible self-adjoint Hamiltonian is constructed which differs from that traditionally used in the Laskin formulation of fractional quantum mechanics.¹⁷⁻²⁰ The fractional particle in a box problem is used as a concrete example.

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II. THE KATUGAMPOLA DERIVATIVE

Throughout this work, we are considering the Katugampola derivative formulation of fractional calculus and specifically the operator, D^α , defined as

$$D^\alpha[f(t)] = \lim_{\epsilon \rightarrow 0} \frac{f(te^{\epsilon t^{-\alpha}}) - f(t)}{\epsilon}, \quad t > 0, \quad (1)$$

$$D^\alpha[f(0)] = \lim_{t \rightarrow 0^+} D^\alpha[f(t)],$$

where $0 < \alpha \leq 1$ and $t \geq 0$. This definition yields the following results (from Theorem 2.3 of Katugampola¹):

- $D^\alpha[af + bg] = aD^\alpha[f] + bD^\alpha[g]$ (linearity).
- $D^\alpha[fg] = fD^\alpha[g] + gD^\alpha[f]$ (product rule).
- $D^\alpha[f(g)] = \frac{df}{dg} D^\alpha[g]$ (chain rule).
- $D^\alpha[f] = t^{1-\alpha} f'$, where $f' = \frac{df}{dt}$.

For a discussion of the chain rule for these types of derivatives, see Refs. 1, 13, and 14.

A. The D^α operator

The last of the bulleted items is the starting point for studying the properties of the D^α operator,

$$D^\alpha[y] = t^{1-\alpha} y'. \quad (2)$$

So, in operator form,

$$D^\alpha = t^{1-\alpha} D^1. \quad (3)$$

Currently $0 < \alpha \leq 1$. If that range is extended, then¹

$$D^\alpha[y] = D^{\alpha-1} D^1[y], \quad (4)$$

for $1 < \alpha \leq 2$.

One can use Eq. (2) to construct a number of important derivatives.¹ The exponential function is listed specifically here since it will be used in developing the momentum operators for quantum mechanics,

$$D^\alpha[e^{ct}] = ct^{1-\alpha} e^{ct}. \quad (5)$$

Note the special case of

$$D^\alpha[e^{\frac{t^\alpha}{\alpha}}] = \frac{1}{\alpha} t^{1-\alpha} \alpha t^{\alpha-1} e^{\frac{t^\alpha}{\alpha}} = e^{\frac{t^\alpha}{\alpha}}, \quad (6)$$

that is, $e^{\frac{t^\alpha}{\alpha}}$ is an eigenfunction of D^α with an eigenvalue of 1.

One can also consider the iterated operator $D^\beta D^\alpha$. Note that $D^\beta D^\alpha \neq D^\gamma$ for $\gamma = \alpha + \beta$. We see this by repeated differentiation of a function, y ,

$$\begin{aligned} D^\beta [D^\alpha[y]] &= t^{1-\beta} (t^{1-\alpha} y'' + (1-\alpha) t^{-\alpha} y') \\ &= t^{2-(\alpha+\beta)} y'' + (1-\alpha) t^{1-(\alpha+\beta)} y' \\ &= t^{2-\gamma} y'' + (1-\alpha) t^{1-\gamma} y'. \end{aligned} \quad (7)$$

In comparison, this does not equal

$$\begin{aligned} D^\gamma[y] &= t^{1-\gamma} y' \quad 0 < \gamma \leq 1 \\ &= t^{2-\gamma} y'' \quad 1 < \gamma \leq 2. \end{aligned} \quad (8)$$

The sum $\alpha + \beta = \gamma$ will appear often. If $0 < \alpha, \beta \leq 1$, then $0 < \gamma \leq 2$.

B. The anti-derivative operator, I^α

One can define the inverse of the D^α operator as a fractional integral,

$$(D^\alpha)^{-1} \equiv D^{-\alpha} \equiv I^\alpha = \int dx \frac{(\cdot)}{x^{1-\alpha}}, \quad (9)$$

where the (\cdot) symbol is serving as place holder for the function to be operated upon. One verifies

$$I^\alpha [D^\alpha[y]] = \int dx \frac{x^{1-\alpha} y'}{x^{1-\alpha}} = \int dx y' = y \quad (10)$$

(where one takes y to vanish at the lower limit) and

$$\begin{aligned} D^\alpha [I^\alpha[y]] &= D^\alpha \left[\int dx \frac{y}{x^{1-\alpha}} \right] \\ &= t^{1-\alpha} \left(\int dx \frac{y}{x^{1-\alpha}} \right)' \\ &= t^{1-\alpha} \frac{y}{t^{1-\alpha}} = y. \end{aligned} \quad (11)$$

One can consider a mixed case

$$\begin{aligned} D^\beta [I^\alpha[y]] &= D^\beta \left[\int dx \frac{y}{x^{1-\alpha}} \right] = t^{1-\beta} \left(\int dx \frac{y}{x^{1-\alpha}} \right)' \\ &= t^{1-\beta} \frac{y}{t^{1-\alpha}} = t^\delta y, \end{aligned} \quad (12)$$

where $\delta = \alpha - \beta$. Note when $\beta = \alpha$, this equals y as expected from Eq. (11).

We can also consider

$$\begin{aligned} I^\alpha [D^\beta[y]] &= \int dx \frac{x^{1-\beta} y'}{x^{1-\alpha}} = \int x^{\alpha-\beta} y' dx \\ &= \int x^\delta y' dx. \end{aligned} \quad (13)$$

Using the same assumption as for Eq. (10), integration by parts gives

$$\begin{aligned} I^\alpha [D^\beta[y]] &= t^\delta y - \delta \int x^{\delta-1} y dx \\ &= t^\delta y - \delta \int \frac{y}{x^{1-\delta}} dx. \end{aligned} \quad (14)$$

If $\delta \geq 0$ (i.e., $\alpha \geq \beta$), then

$$I^\alpha [D^\beta[y]] = t^\delta y - \delta I^\delta[y]. \quad (15)$$

Note,

$$\lim_{\beta \rightarrow \alpha} I^\alpha [D^\beta[y]] = I^\alpha [D^\alpha[y]] = t^0 y - 0 I^0[y] = y. \quad (16)$$

C. The identity and the D^1 operators

With the inverse operations, one has a natural way to incorporate the identity operator E via

$$I^\alpha D^\alpha = D^\alpha I^\alpha = E, \quad (17)$$

for functions, y , that are taken to vanish at the lower limit of the integral operator defined in Eq. (9). One now has the set $\{E, D^\alpha, I^\alpha\}$ for $0 < \alpha \leq 1$ which can be extended using Eq. (4). Consider now the limit $\alpha \rightarrow 1$. One sees that

$$\lim_{\alpha \rightarrow 1} D^\alpha = D^1 = \frac{d}{dt} \quad (18)$$

and

$$\lim_{\alpha \rightarrow 1} I^\alpha = I^1 = \int^t (\cdot) dx. \tag{19}$$

The standard derivative and integral are recovered. In general, at the other extreme, there is a discontinuity. When $\alpha \rightarrow 0$,

$$\lim_{\alpha \rightarrow 0} D^\alpha = D^0 \neq E. \tag{20}$$

We can see this explicitly by

$$D^0[y] = t^{1-0}y' = ty' \neq y, \tag{21}$$

except for the special case of $y = t$. Also,

$$\lim_{\alpha \rightarrow 0} I^\alpha = I^0 \neq E \tag{22}$$

seen explicitly as

$$\int^t dx \frac{y}{x^{1-0}} = \int^t \frac{y}{x} dx \neq y, \tag{23}$$

except for the special case of $y = t$. So the operators D^α and I^α are not continuously connected to the identity via the parameter α . They are continuously connected to the integer order derivatives and integrals.

D. Parity

It is interesting to suspend, for the moment, the requirement that $t \geq 0$ in Eq. (2) and consider the parity (i.e., $t \rightarrow -t$) of D^α by applying the operator \hat{P} and using Eq. (3),

$$\begin{aligned} \hat{P}D^\alpha &= \hat{P}(t^{1-\alpha}D^1) = (-t)^{1-\alpha}(-D^1) \\ &= (-1)^{2-\alpha}t^{1-\alpha}D^1 = (-1)^{2-\alpha}D^\alpha \\ &= (e^{i\pi})^{2-\alpha}D^\alpha = e^{-\alpha i\pi}D^\alpha. \end{aligned} \tag{24}$$

This may be a way to operationalize an extension to negative values of t . Considering limits, this recovers $\hat{P}D^1 = -1D^1$ as $\alpha \rightarrow 1$. If the range of α is extended up to 2, then it also recovers $\hat{P}D^2 = D^2$ for $\alpha \rightarrow 2$. The half-odd-integers are interesting,

$$\hat{P}D^{\frac{1}{2}} = -iD^{\frac{1}{2}} \tag{25}$$

and

$$\hat{P}D^{\frac{3}{2}} = iD^{\frac{3}{2}}. \tag{26}$$

E. Differential equations

Several important fractional differential equations can be considered ($0 < \alpha, \beta \leq 1$ throughout this section),

$$D^\alpha[y] + \lambda y = 0, \tag{27}$$

$$y = Ae^{\frac{-\lambda t^\alpha}{\alpha}}, \tag{28}$$

and

$$D^\beta [D^\alpha[y]] = 0, \tag{29}$$

$$t^{2-(\alpha+\beta)}y'' + (1-\alpha)t^{1-(\alpha+\beta)}y' = 0,$$

$$y = A\frac{t^\alpha}{\alpha} + B. \tag{30}$$

A curious feature of this solution is that it is independent from β . This can be shown (but not done here) by a substitution, $v = y'$, and noting the cancellation of the $(\alpha + \beta)$ term in the exponent of t .

Finally,

$$D^\beta [D^\alpha [y]] + \lambda y = 0, \tag{31}$$

$$t^{2-(\alpha+\beta)} y'' + (1 - \alpha) t^{1-(\alpha+\beta)} y' + \lambda y = 0,$$

$$y_1 = A \gamma^{-\frac{\alpha}{\gamma}} \lambda^{\frac{\alpha}{2\gamma}} (t^\gamma)^{\frac{\alpha}{2\gamma}} \Gamma\left(\frac{\beta}{\gamma}\right) J_{-\frac{\alpha}{\gamma}}\left(\frac{2\sqrt{\lambda t^\gamma}}{\gamma}\right), \tag{32}$$

$$y_2 = B (\gamma^2)^{-\frac{\alpha}{\gamma}} \lambda^{\frac{\alpha}{2\gamma}} (t^\gamma)^{\frac{\alpha}{2\gamma}} \Gamma\left(\frac{\gamma + \alpha}{\gamma}\right) J_{\frac{\alpha}{\gamma}}\left(\frac{2\sqrt{\lambda t^\gamma}}{\gamma}\right), \tag{33}$$

where $\gamma = \alpha + \beta$, Γ is the gamma function, and J is Bessel's function. The solution was obtained using MATHEMATICA.

III. COMMUTATOR PROPERTIES

The previous two differential equations arose from sequential operation of two fractional differential operators. The operators do not commute and, further, $D^\alpha D^\beta \neq D^{\alpha+\beta}$. Although this deviates from the behavior of standard (α an integer) derivatives, non-commutativity offers a richness that is interesting to explore. Consider, first, the commutator properties. Repeating from Eq. (29),

$$D^\beta [D^\alpha [y]] = D^\beta [t^{1-\alpha} y'] \tag{34}$$

$$= t^{2-(\alpha+\beta)} y'' + (1 - \alpha) t^{1-(\alpha+\beta)} y'.$$

And similarly for the reverse order of operation,

$$D^\alpha [D^\beta [y]] = t^{2-(\alpha+\beta)} y'' + (1 - \beta) t^{1-(\alpha+\beta)} y'. \tag{35}$$

The commutator is readily evaluated to be

$$[D^\alpha, D^\beta] y = D^\alpha [D^\beta [y]] - D^\beta [D^\alpha [y]] \tag{36}$$

$$= t^{2-(\alpha+\beta)} y'' + (1 - \beta) t^{1-(\alpha+\beta)} y' - t^{2-(\alpha+\beta)} y'' - (1 - \alpha) t^{1-(\alpha+\beta)} y'$$

$$= (\alpha - \beta) t^{1-(\alpha+\beta)} y'.$$

Letting $\gamma = (\alpha + \beta)$ and $\delta = (\alpha - \beta)$ gives

$$[D^\alpha, D^\beta] y = \delta t^{1-\gamma} y' = \delta D^\gamma [y], \tag{37}$$

and thus

$$[D^\alpha, D^\beta] = \delta D^\gamma. \tag{38}$$

One can readily verify several properties,

$$[D^\alpha, D^\alpha] = 0 \tag{39}$$

because $\delta = (\alpha - \alpha)$ and

$$[D^\alpha, D^\beta] = - [D^\beta, D^\alpha] \tag{40}$$

because $-\delta = (\beta - \alpha)$. Further,

$$[D^a, [D^b, D^c]] = [D^a, (b - c) D^{b+c}] \tag{41}$$

$$= (b - c)(a - b - c) D^{a+b+c}$$

$$= (ab - b^2 - ac + c^2) D^{a+b+c},$$

so it can readily be shown that

$$[D^a, [D^b, D^c]] + [D^b, [D^c, D^a]] + [D^c, [D^a, D^b]] = 0. \tag{42}$$

Thus, the Jacobi identity holds.

Now consider

$$[D^\alpha, t]. \quad (43)$$

Acting on a general y with the commutator gives

$$\begin{aligned} [D^\alpha, t] y &= D^\alpha[ty] - tD^\alpha[y] \\ &= t(t^{1-\alpha}y') + t^{1-\alpha}y - t(t^{1-\alpha}y') \\ &= t^{1-\alpha}y. \end{aligned} \quad (44)$$

Thus,

$$[D^\alpha, t] = t^{1-\alpha}. \quad (45)$$

It is interesting to consider

$$[D^\alpha, t] y = t^{1-\alpha}y = D^\alpha[Y], \quad (46)$$

where $Y = \int^t y dt'$ or

$$D^\alpha y = [D^\alpha, t] y'. \quad (47)$$

This can be extended for an arbitrary differentiable function of t ,

$$[D^\alpha, f(t)] = t^{1-\alpha}f'. \quad (48)$$

So,

$$\begin{aligned} D^\alpha f &= fD^\alpha + [D^\alpha, f(t)] \\ &= fD^\alpha + t^{1-\alpha}f'. \end{aligned} \quad (49)$$

Finally, one can consider $[I^\alpha, D^\beta]$ and in doing so make use of Eqs. (12) and (14),

$$\begin{aligned} [I^\alpha, D^\beta] y &= t^\delta y + \delta \int^t \frac{y}{x^{1-\delta}} dx - t^\delta y \\ &= \delta \int^t \frac{y}{x^{1-\delta}} dx. \end{aligned} \quad (50)$$

So,

$$[I^\alpha, D^\beta] = \delta \int^t dx \frac{(\cdot)}{x^{1-\delta}}. \quad (51)$$

IV. ANTI-COMMUTATOR PROPERTIES

The anti-commutator is now considered by first defining the operator

$$\hat{C}_\gamma \equiv \frac{1}{2} \{D^\alpha, D^\beta\} = \frac{1}{2} (D^\alpha D^\beta + D^\beta D^\alpha). \quad (52)$$

Operating on y with \hat{C}_γ and using Eqs. (34) and (35) give

$$\begin{aligned} \hat{C}_\gamma y &= \frac{1}{2} (t^{2-(\alpha+\beta)} y'' + (1-\alpha)t^{1-(\alpha+\beta)} y') \\ &\quad + t^{2-(\alpha+\beta)} y'' + (1-\beta)t^{1-(\alpha+\beta)} y') \\ &= \frac{1}{2} (2t^{2-(\alpha+\beta)} y'' + (2-(\alpha+\beta)t^{1-(\alpha+\beta)} y') \\ &= \frac{1}{2} (2t^{2-\gamma} y'' + (2-\gamma)t^{1-\gamma} y'). \end{aligned} \quad (53)$$

An interesting and important feature of \hat{C}_γ is that it only depends on $\alpha + \beta$ and hence one can consider $\alpha = \beta = \gamma/2$. Thus,

$$\hat{C}_{2\alpha} = \frac{1}{2} \{D^\alpha, D^\alpha\} = (D^\alpha)^2 \quad (54)$$

(n.b., $(D^\alpha)^2 \neq D^{2\alpha}$). Consider the cases, $\alpha = \frac{1}{2}$,

$$\hat{C}_1 y = t y'' + \frac{1}{2} y' \tag{55}$$

and $\alpha = 1$,

$$\hat{C}_2 y = y''. \tag{56}$$

The parity of \hat{C}_γ is

$$\begin{aligned} \hat{P} \hat{C}_\gamma &= \hat{P} \frac{1}{2} (D^\alpha D^\beta + D^\beta D^\alpha) \\ &= \frac{1}{2} (\hat{P} (D^\alpha D^\beta) + \hat{P} (D^\beta D^\alpha)), \end{aligned} \tag{57}$$

so use of Eq. (24) yields

$$\begin{aligned} \hat{P} \hat{C}_\gamma &= \frac{1}{2} (e^{-\alpha i \pi} D^\alpha e^{-\beta i \pi} D^\beta + e^{-\beta i \pi} D^\beta e^{-\alpha i \pi} D^\alpha) \\ &= e^{-(\alpha+\beta)i\pi} \frac{1}{2} (D^\alpha D^\beta + D^\beta D^\alpha) \\ &= e^{-\gamma i \pi} \hat{C}_\gamma. \end{aligned} \tag{58}$$

A. Operator equations for \hat{C}_γ

Since it was shown above that only the quantity $\alpha + \beta$ is important, $\hat{C}_\gamma = \hat{C}_{2\alpha}$ will be used exclusively for the remainder of the paper; doing so will cast many of the expressions in a more simple form. Let us now consider some simple but important operator equations. First, the homogeneous equation,

$$\hat{C}_{2\alpha} y = 0, \tag{59}$$

which becomes the differential equation

$$t^{2-2\alpha} y'' + (1 - \alpha)t^{1-2\alpha} y' = 0 \tag{60}$$

and has the solution

$$y = A \frac{1}{\alpha} t^\alpha + B, \tag{61}$$

as expected from Eq. (30). Next, consider the constant equation

$$\hat{C}_{2\alpha} y = \Lambda, \tag{62}$$

which becomes the differential equation

$$t^{2-2\alpha} y'' + (1 - 1\alpha)t^{1-2\alpha} y' = \Lambda \tag{63}$$

and has solutions

$$y = \frac{\Lambda}{2\alpha^2} t^{2\alpha} + A \frac{1}{\alpha} t^\alpha + B \tag{64}$$

(it is insightful to view $t^{2\alpha}$ as $(t^\alpha)^2$). Finally, consider the eigenvalue equation

$$\hat{C}_{2\alpha} y = -\Lambda y, \tag{65}$$

which becomes the differential equation

$$t^{2-2\alpha} y'' + (1 - \alpha)t^{1-2\alpha} y' + \Lambda y = 0. \tag{66}$$

This has solutions

$$y = A \cos \left[\frac{\sqrt{\Lambda}}{\alpha} t^\alpha \right] + B \sin \left[\frac{\sqrt{\Lambda}}{\alpha} t^\alpha \right]. \tag{67}$$

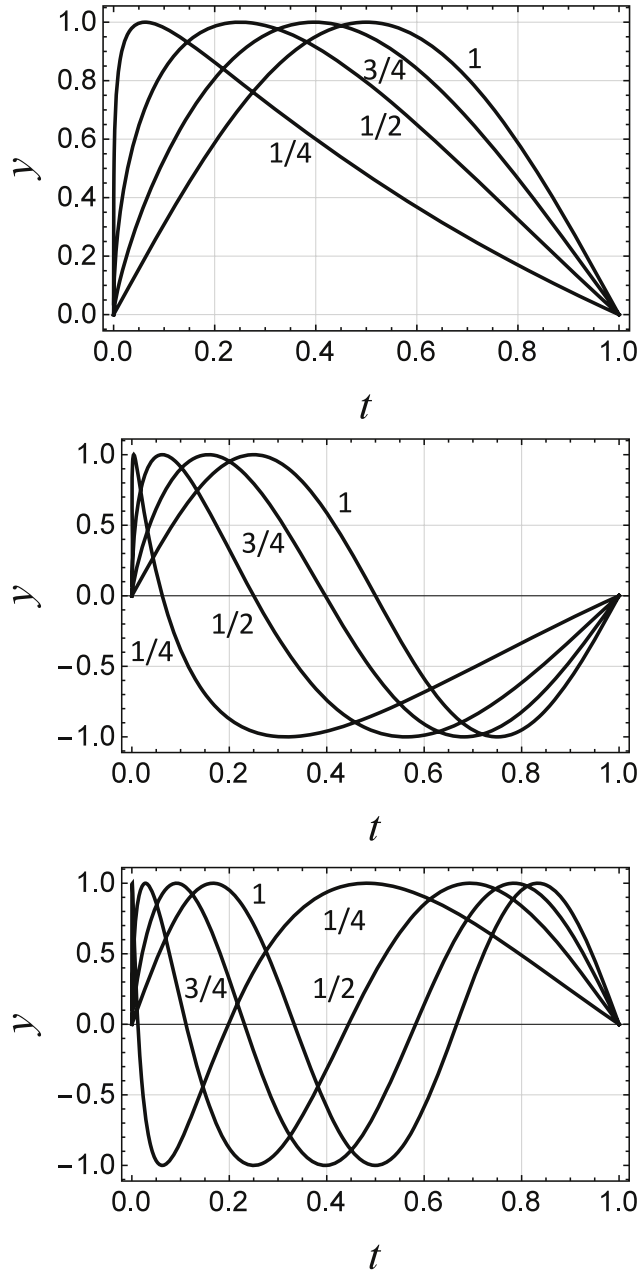


FIG. 1. The first three eigenstates for $\hat{C}_{2\alpha}y = -\Lambda y$ with boundary conditions $y(0) = y(1) = 0$ ordered from top to bottom. Each graph shows the curve for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ with the number next to the curve indicating the value of α . The value of $\alpha = 1$ corresponds to $y'' = -\Lambda y$ and shows the well-known sine function. As α decreases from 1, the peak of the curve shifts to smaller values of t and a pronounced asymmetry about $t = \frac{1}{2}$ develops.

The plots in Fig. 1 show the solutions for boundary conditions $y(0) = y(1) = 0$ for the first three eigenvalues and for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

These results show that the solutions are the same as those for $D^2 (D^2[y] = y'')$ except with t replaced by $\frac{1}{\alpha}t^\alpha$. That is,

$$y'' = 0 \implies y = At + B, \tag{68}$$

$$y'' = \Lambda \implies y = \frac{\Lambda}{2}t^2 + At + B, \tag{69}$$

and

$$y'' = -\Lambda y \implies y = A \cos [\sqrt{\Lambda}t] + B \sin [\sqrt{\Lambda}t]. \tag{70}$$

These can be compared with Eqs. (61), (64), and (67), respectively.

B. Self-adjoint variant of $\hat{C}_{2\alpha}$

The operator $\hat{C}_{2\alpha}$ is not self-adjoint but it can be made so by multiplying by an integrating factor h , where

$$\begin{aligned} h(t) &= \frac{1}{t^{2-2\alpha}} e^{\int t \frac{(1-\alpha)x^{1-2\alpha}}{x^{2-2\alpha}} dx} \\ &= \frac{1}{t^{2-2\alpha}} e^{(1-\alpha)\int t^{\frac{1}{x}} dx} \\ &= \frac{1}{t^{2-2\alpha}} t^{1-\alpha} = t^{\alpha-1}. \end{aligned} \tag{71}$$

So, one then defines

$$\begin{aligned} \hat{A}_{2\alpha} &\equiv h\hat{C}_{2\alpha} = t^{1-\alpha} \frac{d^2}{dt^2} + (1-\alpha)t^{-\alpha} \frac{d}{dt} \\ &= \frac{d}{dt} \left[t^{1-\alpha} \frac{d}{dt} \right]. \end{aligned} \tag{72}$$

One can consider several operator equations as was done for $\hat{C}_{2\alpha}$. All of these differential equations will be in standard Sturm-Liouville form.

First, the homogeneous equation,

$$\hat{A}_{2\alpha} y = 0, \tag{73}$$

as expected, yields the same results as for $\hat{C}_{2\alpha}$ in Eq. (61),

$$y = A \frac{1}{\alpha} t^\alpha + B. \tag{74}$$

The constant equation,

$$\hat{A}_{2\alpha} y = \Lambda, \tag{75}$$

has solution

$$y = \frac{\Lambda}{1+\alpha} t^{\alpha+1} + A \frac{1}{\alpha} t^\alpha + B. \tag{76}$$

The eigenvalue equation

$$\hat{A}_{2\alpha} y = -\Lambda y \tag{77}$$

has solutions

$$\begin{aligned} y &= A\Lambda^{\frac{\eta}{2}} \sqrt{t^\alpha} J_\eta \left(\frac{2\eta}{\alpha} \sqrt{\Lambda} (t^\alpha)^{\frac{1}{2\eta}} \right) \\ &+ B\Lambda^{\frac{\eta}{2}} \sqrt{t^\alpha} J_{-\eta} \left(\frac{2\eta}{\alpha} \sqrt{\Lambda} (t^\alpha)^{\frac{1}{2\eta}} \right), \end{aligned} \tag{78}$$

where $\eta = \frac{\alpha}{1+\alpha}$. Figure 2 shows y for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ where $\Lambda = A = B = 1$.

Figure 3 shows the first three eigenvalue solutions for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. The boundary conditions are $y(0) = y(1) = 0$. This forces $B = 0$ and leads to $\sqrt{\Lambda} = \frac{\alpha}{2\eta} n_\eta$, where n_η is the n^{th} zero of J_η . Thus,

$$y = A\sqrt{t^\alpha} J_\eta \left(n_\eta (t^\alpha)^{\frac{1}{2\eta}} \right), \tag{79}$$

where the constant A has now incorporated the constant $\Lambda^{\frac{\eta}{2}}$ here for simplicity of notation.

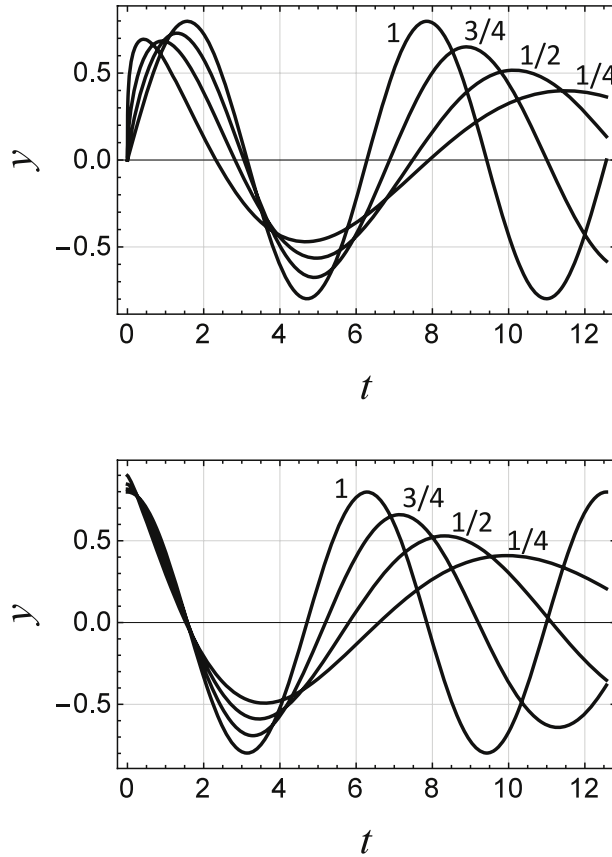


FIG. 2. The top (bottom) graph is for the first (second) term in Eq. (78) for $A = B = \Lambda = 1$ and $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ with the number next to the curve indicating the value of α . For $\alpha = 1$, one recovers the sine and cosine functions expected from $y'' = -\Lambda y$. These graphs illustrate a general characteristic in that decreasing α elongates the functions, i.e., for $\alpha < 1$, the period of oscillation from positive to negative increases as α decreases. The period is, of course, constant for $\alpha = 1$.

Note when $\alpha = 1$ ($\eta = \frac{1}{2}$),

$$y = A\sqrt{t}J_{\frac{1}{2}}\left(n_{\frac{1}{2}}t\right). \tag{80}$$

Using Ref. 16, $\sqrt{x}J_{\frac{1}{2}}(kx) \propto \sin[kx]$,

$$y = A \sin[n\pi t], \tag{81}$$

where $n_{\frac{1}{2}} = n\pi$ is interpreted as the n^{th} zero of sine.

The eigenvalues can be determined from $\sqrt{\Lambda} = \frac{\alpha}{2\eta}n_{\eta}$,

$$\Lambda = \frac{1}{4}\left(\frac{\alpha}{\eta}\right)^2 n_{\eta}^2. \tag{82}$$

Figure 4(a) shows the first 5 values of Λ as a function of α . The spectrum for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ is also shown in Fig. 4(b). Note, all spectra are linearly increasing.

When $\alpha = 1$, one recovers

$$\Lambda = n_{\frac{1}{2}}^2 = n^2\pi^2. \tag{83}$$

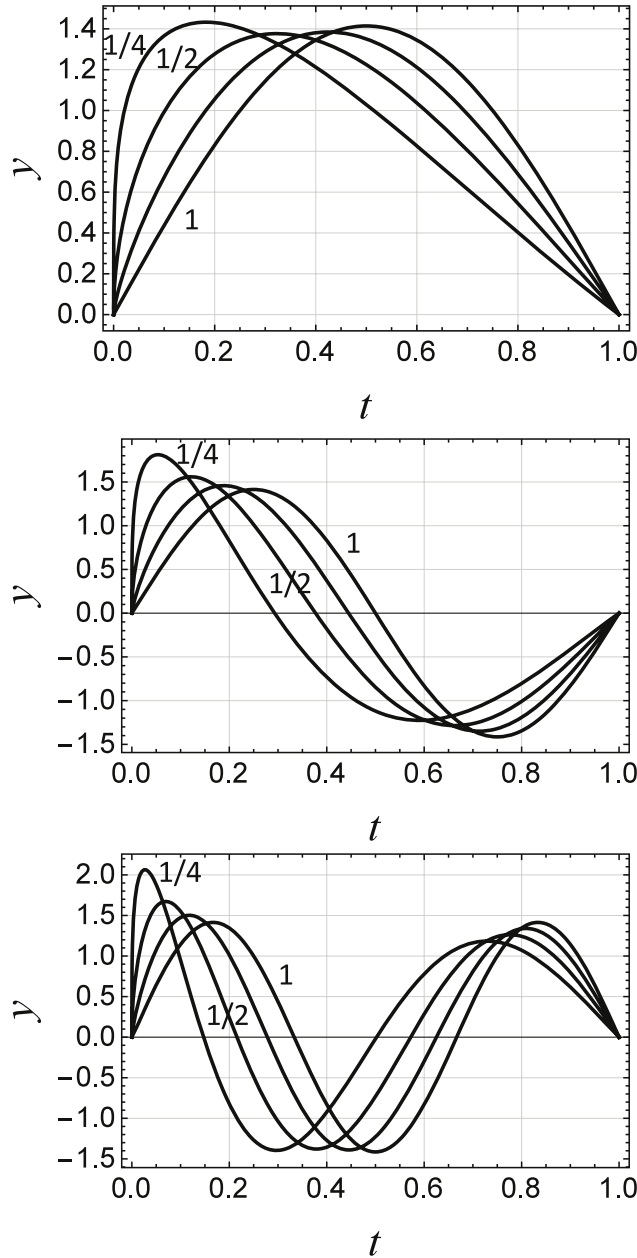


FIG. 3. The first three eigenstates for $\hat{A}_{2\alpha}y = -\Lambda y$ with boundary conditions $y(0) = y(1) = 0$ ordered from top to bottom. Each graph shows the curve for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ with the number next to the curve indicating the value of α (to ease the crowding, the $\alpha = \frac{3}{4}$ curve is not labeled). The value of $\alpha = 1$ corresponds to $y'' = -\Lambda y$ and shows the well-known sine function. Comparison with Fig. 1 shows a softening of the asymmetry about $t = \frac{1}{2}$ for the $\hat{A}_{2\alpha}$ compared with the $\hat{C}_{2\alpha}$. In an application to quantum mechanics developed in Sec. V, these graphs are precisely the wavefunctions for a fractional particle in a box.

For other types of boundary value problems, it will be important to know the fractional derivative of y in Eq. (78). This expression is quite complicated so it is not shown explicitly here but it is easily obtainable using MATHEMATICA. Figure 5 shows the behavior of the fractional derivative for that case of Fig. 3. The fractional derivative tends to zero as $\alpha \rightarrow 0$ which is quite the opposite of the behavior of the first derivative as can be gleaned from inspection of Fig. 3.

Figure 6 shows the results for the first three eigenvalue solutions for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. The boundary conditions are $D^\alpha[y(0)] = y'(1) = 0$.

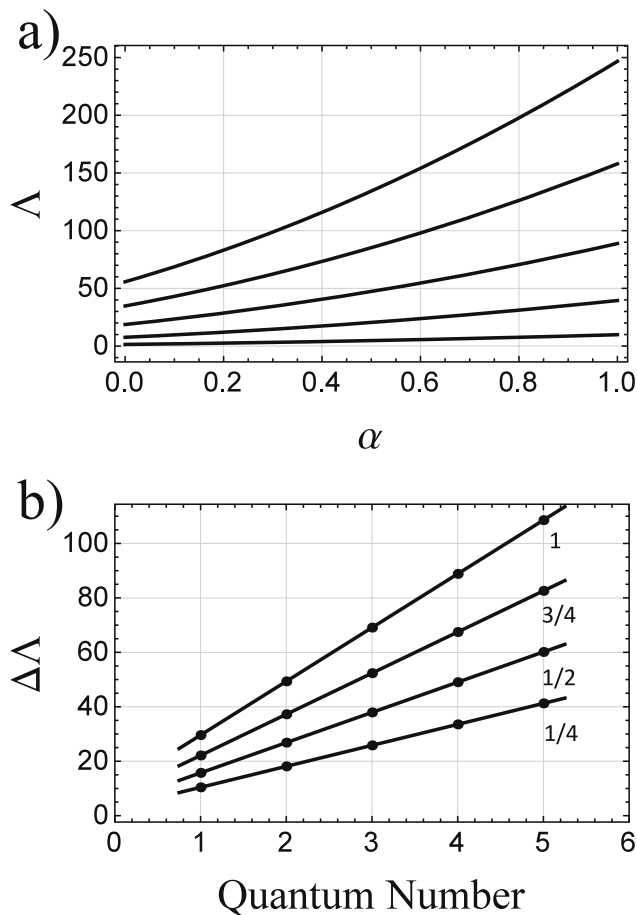


FIG. 4. (a) The values of the first four eigenstates of $\hat{A}_{2\alpha} y = -\Lambda y$ with boundary conditions $y(0) = y(1) = 0$ as a function of α . (b) The spectrum, $\Delta\Lambda$, for the first 5 quantum numbers (\bullet). The effect of α is to reduce the value of Λ as $\alpha \rightarrow 0$. However, Λ goes as the square of the quantum number for all values of α as can be seen from the linear behavior of the spectrum (solid lines are best linear fit results.)

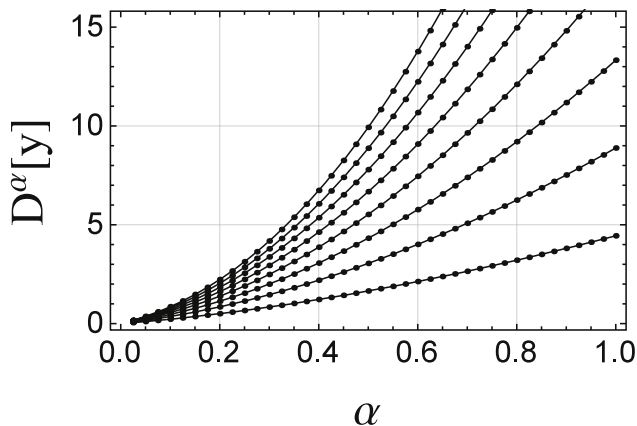


FIG. 5. Numerically calculated limiting values of the fractional derivative for the first 8 eigenfunctions for $\hat{A}_{2\alpha} y = -\Lambda y$ with boundary conditions $y(0) = y(1) = 0$. The curves increase in quantum number from bottom to top. The (\bullet) are the calculated values and the lines are smooth fits through those dots to guide the eye.

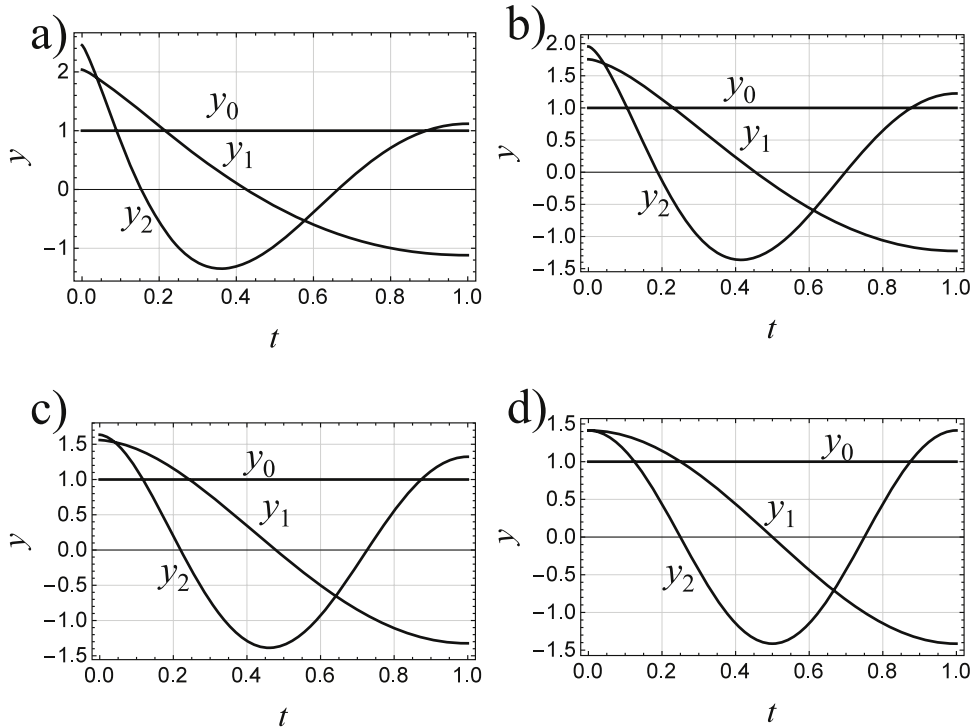


FIG. 6. The first three eigenvalue solutions for $\alpha = \frac{1}{4}$ (a), $\frac{1}{2}$ (b), $\frac{3}{4}$ (c), 1 (d). The boundary conditions are $D^\alpha[y(0)] = y'(1) = 0$. Note $\Lambda = 0$ is an eigenvalue for these conditions; $y_0 = 1$ is the eigenfunction associated with $\Lambda = 0$ for all α . The curves are labelled as y_0, y_1, y_2 in increasing value of the eigenvalue.

V. APPLICATIONS TO QUANTUM MECHANICS

In the early 2000s, Laskin developed a formulation of fractional quantum mechanics based on the path integral over Lèvy trajectories.¹⁷⁻²⁰ Since that time numerous applications of fractional quantum mechanics have appeared; a small sampling includes Refs. 21–25. A key aspect of that formulation is the presence of the fractional Laplacian which in the one dimensional case would be of the form

$$\left(-\frac{d^2}{dx^2}\right)^{a/2}, \tag{84}$$

where $1 < a \leq 2$ (note, in this section, x is used as the independent variable throughout rather than t). The operator is not Hermitian but has nonetheless proven to be very useful. Interpretations of non-Hermitian Hamiltonians in general (not necessarily in fractional quantum mechanics) have been given.²⁶ Further, the fractional derivative requires an integral representation and this is non-local.²³

The approach taken here is different and hence does not coincide with the Laskin formulation. Nevertheless, we offer it as a potential application for the $\hat{A}_{2\alpha}$ operator developed in this work. This approach carries some nice features in which the resultant Hamiltonian is self-adjoint and it is constructed as a function of Katugampola derivatives so it is local in character.

The development is based on two key ideas. First, analogous to the standard (integer derivatives) formulation of quantum mechanics, we let $e^{\pm ik \frac{x^\alpha}{\alpha}}$ play the role of the eigenstates of the fractional momentum operator. That is, the fractional momentum operator is defined as the operator whose eigenvectors are $e^{\pm ik \frac{x^\alpha}{\alpha}}$ and whose eigenvalues are $\pm k$. Before doing so, a Hermitian fractional momentum operator is discussed. Second, we use a self-adjoint Hamiltonian built from $\hat{A}_{2\alpha}$.

A. The self-adjoint Hamiltonian operator

Starting with the premise of working with a Hermitian operator representation of fractional momentum and then from there building up a Hamiltonian analogous to normal quantum mechanics (\hbar and m suppressed),

$$\hat{p} = -iD^1 \longrightarrow \hat{H} = (-iD^1)(-iD^1) + V(\hat{x}) = -D^2 + V(\hat{x}). \tag{85}$$

Taking that the core of the fractional momentum operator should be D^α and using $D^\alpha = x^{1-\alpha}D^1$, one is able to produce a self-adjoint operator of the form²⁷

$$ir(x)D^1 + s(x) + \frac{i}{2}r'(x), \tag{86}$$

where r and s are any real, differentiable functions. The natural choice is $r = -x^{1-\alpha}$ (so $r' = -(1-\alpha)x^{-\alpha}$) and $s = 0$. This results in a definition for the fractional momentum,

$$\hat{p}^\alpha = -iD^\alpha - \frac{i(1-\alpha)}{2}x^{-\alpha}. \tag{87}$$

To construct the Hamiltonian, one can consider (adding additional generality)

$$\begin{aligned} \hat{p}^\alpha \hat{p}^\beta &= \left(-iD^\alpha - \frac{i(1-\alpha)}{2}x^{-\alpha}\right) \left(-iD^\beta - \frac{i(1-\beta)}{2}x^{-\beta}\right) \\ &= -D^\alpha D^\beta - \frac{(1-\alpha)x^{-\alpha}D^\beta}{2} - \frac{(1-\beta)D^\alpha x^{-\beta}}{2} \\ &\quad - \frac{(1-\alpha)(1-\beta)x^{-\alpha}x^{-\beta}}{4}. \end{aligned} \tag{88}$$

One must use the commutator from Eq. (49) to change the order of

$$\begin{aligned} D^\alpha x^{-\beta} &= x^{-\beta}D^\alpha + x^{1-\alpha}(-\beta)x^{-\beta-1} \\ &\quad x^{-\beta}D^\alpha - \beta x^{-\alpha}x^{-\beta}. \end{aligned} \tag{89}$$

So,

$$\begin{aligned} \hat{p}^\alpha \hat{p}^\beta &= -D^\alpha D^\beta - \frac{(1-\alpha)x^{-\alpha}D^\beta}{2} - \frac{(1-\beta)x^{-\beta}D^\alpha}{2} \\ &\quad + \frac{(1-\beta)\beta x^{-\alpha}x^{-\beta}}{2} - \frac{(1-\alpha)(1-\beta)x^{-\alpha}x^{-\beta}}{4}, \end{aligned} \tag{90}$$

which is rather complicated. Even simplifying to $\alpha = \beta$, one is left with

$$\begin{aligned} \hat{p}^\alpha \hat{p}^\alpha &= -D^\alpha D^\alpha - (1-\alpha)x^{-\alpha}D^\alpha + (1-\alpha) \left[\frac{\alpha x^{-2\alpha}}{2} - \frac{(1-\alpha)x^{-2\alpha}}{4} \right] \\ &= -D^\alpha D^\alpha - (1-\alpha)x^{-\alpha}D^\alpha + (1-\alpha) \left(\frac{3\alpha}{4} - \frac{1}{4} \right) x^{-2\alpha} \\ &= -\hat{C}_{2\alpha} - (1-\alpha)x^{-\alpha}D^\alpha - \frac{1}{4}(3\alpha^2 - 4\alpha + 1)x^{-2\alpha}, \end{aligned} \tag{91}$$

which is also complicated. Such a construct is the most rigorous approach if one insists upon a Hermitian operator representation, but the resultant Hamiltonian will be unwieldy from an analytic perspective. If one gives ground on Hermiticity for \hat{p}^α , one can truncate the operator in Eq. (87) by dropping the residual $\frac{i}{2}r'(x)$ such that a starting point for a suitable (albeit non-Hermitian) definition of a fractional momentum (suppressing \hbar) is

$$\hat{p}^\alpha = -iD^\alpha. \tag{92}$$

With this definition, $\psi = e^{ik\frac{x^\alpha}{\alpha}}$ are eigenfunctions,

$$\hat{p}^\alpha e^{ik\frac{x^\alpha}{\alpha}} = k e^{ik\frac{x^\alpha}{\alpha}} \tag{93}$$

from Eq. (6). The definition of momentum given in Eq. (92) is not equivalent to that used in Laskin fractional quantum mechanics. Aside from being defined using an integral representation-based fractional derivative, that treatment also incorporates a $(-i)^\alpha$ as opposed to simply a $-i$ in Eq. (92). One might consider constructing a general fractional Hamiltonian based on this as (suppressing the $\frac{1}{2m}$)

$$\hat{H}^{\alpha\beta} = \hat{p}^\alpha \hat{p}^\beta + V(\hat{x}), \tag{94}$$

which would include

$$\hat{H}^\alpha = (\hat{p}^\alpha)^2 + V(\hat{x}) \tag{95}$$

as a special case. The issue with $\hat{H}^{\alpha\beta}$ is that it is not self-adjoint. It is perhaps not unreasonable to replace $\hat{p}^\alpha \hat{p}^\beta$ with a function of $\hat{A}_{2\alpha}$ and define

$$\hat{H} = -\hat{A}_{2\alpha} + V(\hat{x}). \tag{96}$$

Like momentum, the definition of the Hamiltonian used here is not that used in Laskin fractional quantum mechanics. There,

$$\hat{H} = \mathcal{D}_b \left(-\frac{d^2}{dx^2} \right)^{b/2}, \tag{97}$$

where \mathcal{D}_b is a scaling factor, $1 < b \leq 2$. Again, this derivative ultimately needs to be defined via an integral representation.²⁴ Also, the $(-1)^{b/2}$ appears Eq. (97) as opposed to -1 in Eq. (96).

B. The particle in a box model

As a concrete example, the particle in a box model is considered. Even this simple model can be confusing in fractional quantum mechanics due to the non-local nature of the theory.²³ It will be seen that the results arising from the definition given in Eq. (96) are quite different from those seen in more traditional fractional quantum mechanics.^{17,23,24}

The particle in a box model falls out immediately from the work in Sec. IV B. Here,

$$\hat{H}\psi_n = E_n\psi \tag{98}$$

and $\hat{H} = -\hat{A}_{2\alpha}$ within the box, which is taken to be $0 < x < 1$. So,

$$\hat{A}_{2\alpha}\psi_n(x) = -E_n\psi_n(x), \tag{99}$$

which is just Eq. (77). With boundary conditions $\psi(0) = \psi(1) = 0$, the wavefunction becomes

$$\psi_n(x) = N_n \sqrt{x^\alpha} J_\eta \left(n_\eta (x^\alpha)^{\frac{1}{2\eta}} \right), \tag{100}$$

where n_η is the n^{th} zero of $J_\eta(x)$ for $\eta = \frac{\alpha}{1+\alpha}$. The normalization constant, N_n , can be determined to be

$$N_n = \frac{1}{\sqrt{(\eta - 1) J_{\eta-1}(n_\eta) J_{\eta+1}(n_\eta)}}. \tag{101}$$

These wavefunctions are purely real. The energy levels are

$$E_n = \frac{(1 + \alpha)^2 n_\eta^2}{4}. \tag{102}$$

The square of the wavefunctions for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ is shown in Fig. 7.

The average value $\langle x \rangle$ and the variance $\delta x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ for the ground state are plotted in Fig. 8. The average value increases toward $x = 1/2$ as can be inferred from the graphs whereas the variance, although not strongly changing, does peak and fall again as α goes from 0 to 1.

There are some important differences between the results obtained here and those from previous work.²⁵ In those works, the box is usually taken to be $-1 < x < 1$ but the basic result is that

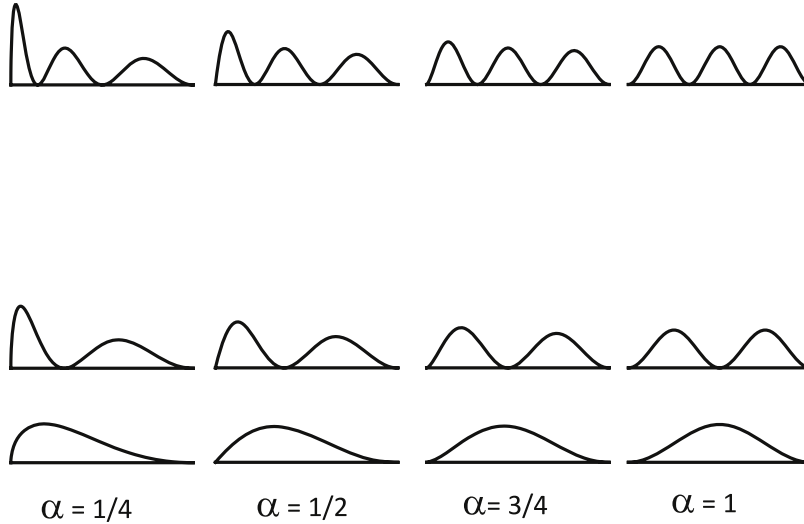


FIG. 7. The mod-square of ψ (although ψ is real) for the fractional particle in a box model for the first three quantum levels. The wavefunctions themselves are identical to the functions plotted in Fig. 3. The energy level spacing is artificial for ease of plotting. Fig. 4 shows how the energy levels behave as $E_n = \Lambda$.

the wavefunctions would translate to a $0 < x < L$ box in the form $\psi_n = A \sin\left(\frac{n\pi x}{L}\right)$, where L is the length of the box. This is identical to that for standard (integer derivative) quantum mechanics. The energy levels are found to be

$$E_n = \mathcal{D}_\alpha \left(\frac{\pi n}{L} \right)^\alpha \tag{103}$$

(suppressing the \hbar). The energy states increase with fractional power. This differs from the results captured in Eq. (102) and Figs. 5 and 6.

Casting fractional quantum mechanics in the treatment above offers some desirable characteristics such as a self-adjoint Hamiltonian, no burden of the non-local character, and automatic inclusion of a generalization to include momentum of differing fractional characters \hat{p}^α and \hat{p}^β . There are, however, some undesirable characteristics. One of which arises from the definition of the Katugampola fractional derivative which is defined only for $x \geq 0$. This is not a major difficulty in a

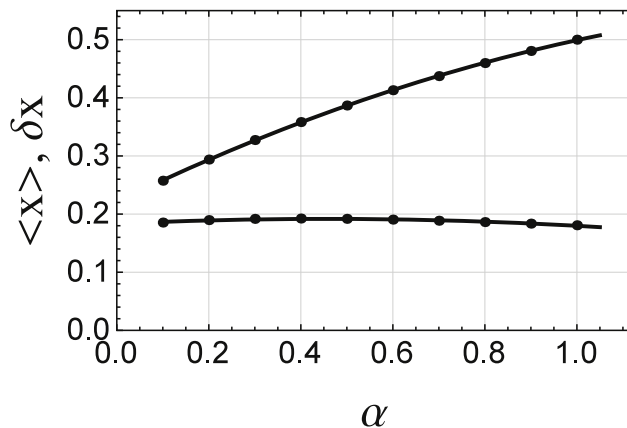


FIG. 8. The average value $\langle x \rangle$ and the variance $\delta x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ for the ground state of the fractional particle in a box. Although MATHEMATICA provides an analytic expression for both $\langle x \rangle$ and δx , it is very complicated, so numerical answers were obtained for a number of values of γ (\bullet). These were then fit to a parabolic curve. $\langle x \rangle$ increases monotonically with increasing α but δx has a mild peak at around $\alpha = \frac{1}{2}$. (Based on the parabolic curve fit, the peak is at $\gamma = 0.455$.)

physical setting because it is usually straightforward based on the physical nature of the problem as to how to make a change of variables $x \rightarrow -x$. Another undesirable characteristic is the singularity at $x = 0$ which carries through to the $\hat{A}_{2\alpha}$ operator. Although the standard (integer) derivative is not defined at $x = 0$, the Katugampola fractional derivative is. So, the singularity is eliminated at the boundary when using the fractional derivative.

An intuitive understanding of the fractional derivative is elusive; consequently physical intuition about how fractional quantum mechanics should behave is also difficult. The asymmetry in the wavefunctions shown in Fig. 7 is not intuitive to the authors. Some insight might possibly be gleaned from the results collected in Eqs. (61), (64), and (67) where the solutions are the same as those for D^2 ($D^2[y] = y''$) except with x replaced by $\frac{1}{\alpha}x^\alpha$. For $\alpha < 1$, this effectively acts to stretch out the domain of the function emphasizing small values of x and de-emphasizing large values. In terms of the quantum probability amplitude, it is “pushed” to lower values of x . In a sense, this introduces a “phantom” potential energy term. Although speculative, this idea is consistent with a similar feature found in the fractional version of the classical undamped harmonic oscillator in which the fractional derivative introduces a “phantom” damping term.²⁸

Finally, a question of when to insist on self-adjointness arises. In this treatment, Hermiticity was relinquished at the fractional momentum operator level but was maintained at the product of fractional momentum level (at the fractional kinetic energy level) via $\hat{A}_{2\alpha}$. One could wait to insist on Hermiticity until the Schrödinger equation level. In this case, one could use $\hat{C}_{2\alpha}$, Eq. (54), in place of the products of momentum and then carry out making the Schrödinger equation a self-adjoint differential equation. This would result in a weight function multiplying the eigenvalues and orthonormality being defined over that weight function. If this were to be done for the particle in a box, for example, then the wavefunctions would be equivalent to the functions plotted in Fig. 1.

VI. CONCLUSION

The properties of the recently developed Katugampola fractional derivative were explored. The advantage of the Katugampola fractional derivative is that it is limit based rather than defined via a fractional integral. This, among other things, does not introduce the non-local nature of earlier integral representations. A key property of the Katugampola derivative is $D^\alpha[y] = t^{1-\alpha}y'$. From this, a differential operator, $D^\alpha = t^{1-\alpha}D^1$, is self-evident. The focus of this work centered on exploring the properties of these differential operators. The commutators, anti-commutators, and inverse operators were studied. The anti-commutator served as the basis for defining a self-adjoint operator which could potentially have some application in quantum mechanics.

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