Properties of the Matrix Range of an Operator

FREDRIC M. POLLACK

Communicated by the Editors

All Hilbert spaces considered in this paper will be complex, and all operators will be bounded and linear. The algebra of all (bounded linear) operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{L}(\mathcal{H})$. For a positive integer n, $\mathcal{H}_n$ will designate n-dimensional Hilbert space. If B is a set of vectors in a linear topological space then $[B]$ and co (B) will denote respectively the closed linear span of the vectors in B and the convex hull of the vectors in B. The spectrum and numerical range of $T \in \mathcal{L}(\mathcal{H})$ will be denoted respectively by $\sigma(T)$ and $W(T)$. The complex numbers will be denoted by $\mathbf{C}$.

1. Introduction. In [2], W. B. Arveson defined, for any positive integer n, the n-dimensional matrix range of $T \in \mathcal{L}(\mathcal{H})$, denoted $W_n(T)$. Specifically, form $C^*(T)$ the C^*-algebra generated by T and the identity and denote by $CP(C^*(T), \mathcal{H}_n ; I)$ the set of all completely positive maps of $C^*(T)$ into $\mathcal{L}(\mathcal{H}_n)$ which preserve the identity (completely positive maps are discussed in [1] and [15]). We then define

$$W_n(T) = \{\varphi(T) : \varphi \in CP(C^*(T), \mathcal{H}_n ; I)\}.$$

If $n = 1$, then $W_1(T) = \overline{W(T)}$ and thus $W_n(T)$ is a generalization of the numerical range (refer to [4] or [1]). In [3] Arveson showed that if T_1 and T_2 are compact irreducible operators, then T_1 is unitarily equivalent to T_2 if and only if $W_n(T_1) = W_n(T_2)$, $n = 1, 2, 3, \cdots$.

The purpose of this paper is to uncover information about $W_n(T)$.

2. General results. The next lemma will be used repeatedly.

Lemma 2.1. *Let T and V be operators and assume T is not a multiple of the identity. Define a map by $\varphi : \alpha T + \beta I \rightarrow \alpha V + \beta I$ for all $\alpha, \beta \in \mathbf{C}$. If V is normal and φ is contractive ($\|\varphi\| \leqq 1$), then φ has a completely positive linear extension to any C^*-algebra containing T.*

Proof. Since T is not a multiple of the identity φ is well defined and linear

Indiana University Mathematics Journal, Vol. 22, No. 5 (1972)

and thus by Proposition 1.2.11 of [1] φ is completely contractive and hence by Theorem 1.2.9 of [1] the conclusion follows.

Now let $T \in \mathcal{L}(\mathcal{H})$ have norm one and denote by B_n the solid unit ball of $\mathcal{L}(\mathcal{H}_n)$. If $\varphi \in CP(C^*(T), \mathcal{H}_n ; I)$, then $\|\varphi\| = \|\varphi(I)\| = \|I\| = 1$ by [1]. Thus $W_n(T) \subset B_n$ for all n. The next theorem tells us when equality holds.

Theorem 2.1. *If $T \in \mathcal{L}(\mathcal{H})$ and $\|T\| = 1$, then the following conditions are equivalent:*

(i) $\|\alpha T + \beta I\| = |\alpha| + |\beta|$ *for all* $\alpha, \beta \in \mathbf{C}$,
(ii) $W_n(T) = B_n$ *for all* n,
(iii) $W_{n_0}(T) = B_{n_0}$ *for some* n_0 .

Proof. (i) $\Rightarrow$ (ii). To avoid trivialities, we assume T is not a multiple of the identity. Let U be an arbitrary unitary operator in B_n . The map φ defined by $\varphi : \alpha T + \beta I \rightarrow \alpha U + \beta I$ is contractive since by (i) we have $\|\varphi(\alpha T + \beta I)\| = \|\alpha U + \beta I\| \leqq |\alpha| + |\beta| = \|\alpha T + \beta I\|$. By Lemma 2.1 we conclude that there exists $\varphi_1 \in CP(C^*(T), \mathcal{H}_n ; I)$ such that $\varphi_1(T) = U$. Since U was arbitrary $W_n(T)$ contains all the unitaries in $\mathcal{L}(\mathcal{H}_n)$. Now $W_n(T)$ is always compact and convex and since the unitary operators are precisely the extreme points of B_n we must have $W_n(T) = B_n$. (ii) $\Rightarrow$ (iii) is clear. (iii) $\Rightarrow$ (i). Let $\{e_1 , e_2 , \cdots , e_{n_0}\}$ be an orthonormal basis for $\mathcal{H}_{n_0}$. Given $\gamma \in \mathbf{C}$, denote by S the operator defined by $Se_1 = (\gamma/|\gamma|)e_1$, $Se_i = 0$, $i \geqq 2$. Now $S \in B_{n_0}$ and thus there exists

$$\varphi \in CP(C^*(T), \mathcal{H}_{n_0} ; I)$$

such that $\varphi(T) = S$. A simple calculation shows that $\|S + \gamma I\| = 1 + |\gamma|$ and thus $1 + |\gamma| = \|S + \gamma I\| = \|\varphi(T + \gamma I\| \leqq \|T + \gamma I\| \leqq 1 + |\gamma|$ from which the conclusion easily follows.

The above theorem and the fact that $W_1(T) = \overline{W(T)}$ enable us to calculate the matrix range of the unilateral shift (see [10] for a definition and properties).

Corollary 2.1.1. *If U denotes the unilateral shift, then $W_n(U) = B_n$ for all n.*

Proof. Since $\|U\| = 1$ by Theorem 2.1 it suffices to show that $\overline{W(U)} = \{z : |z| \leqq 1\}$. However, $\sigma(U) = \{z : |z| \leqq 1\}$ and $\sigma(U) \subset \overline{W(U)}$ and so the conclusion follows.

Recall that the numerical radius of an operator, $|W(T)|$, is defined via $|W(T)| = \sup \{|\lambda| : \lambda \in W(T)\}$, and satisfies $\frac{1}{2}\|T\| \leqq |W(T)| \leqq \|T\|$ [9]. If we analogously define $|W_n(T)| = \sup \{\|S\| : S \in W_n(T)\}$ for $n \geqq 2$, then we have

Theorem 2.2. $|W_n(T)| = \|T\|$ *for* $n \geqq 2$.

Proof. Let π be a faithful representation of $C^*(T)$ on an infinite dimensional Hilbert space $\mathcal{K}$ [6]. Given $\epsilon > 0$, choose a unit vector x_1 such that

$$\|\pi(T)x_1\|^2 > \|\pi(T)\|^2 - \epsilon. \tag{2.1}$$

If x_1 is not an eigenvector of $\pi(T)$, then by the Gram–Schmidt process we can find a vector x_2 such that $\{x_1, x_2\}$ is an orthonormal set and $[x_1, x_2] = [x_1, \pi(T)x_1]$. Since $\mathcal{K}$ is infinite dimensional we can choose vectors $x_3, x_4, \cdots, x_n$ such that $\{x_1, x_2, x_3, \cdots, x_n\}$ is an orthonormal set. Let $\{e_i\}$ be an orthonormal basis for $\mathcal{K}_n$ and define an embedding $V : \mathcal{K}_n \to \mathcal{K}$ by $Ve_i = x_i$. It is easy to see that $V^*V = I$ (on $\mathcal{K}_n$) and VV^* is the projection onto $[x_1, x_2, \cdots, x_n]$. For A in $C^*(T)$ let $\varphi(A) = V^*\pi(A)V$. By Stinespring's Theorem [15] and our construction $\varphi \in CP(C^*(T), \mathcal{K}_n ; I)$ and hence $\varphi(T) \in W_n(T)$. Now $||\varphi(T)||^2 \geqq ||\varphi(T)e_1||^2 = (VV^*\pi(T)x_1, \pi(T)x_1) = ||\pi(T)x_1||^2 > ||\pi(T)||^2 - \epsilon = ||T||^2 - \epsilon$. The last equality results from (2.1) and the fact that a faithful representation of a C^*-algebra is isometric [6]. Since ϵ was arbitrary we are done.

If x_1 is an eigenvector for $\pi(T)$ we mimic the above proof, except that we do not need the Gram–Schmidt process.

3. Normality and the matrix range. Recall that for a normal operator $\overline{\text{co}}\ (\sigma(T)) = \overline{W(T)}$. The next theorem, proved in [3], shows that for $n \geqq 2$ $W_n(T)$ is actually the closed "matrix-valued" convex hull of the spectrum of T.

Theorem 3.1. (*Arveson*). *Let T be a normal operator and let n be a positive integer. Then*

$$W_n(T) = CL\left\{\sum_{i=1}^{r} \lambda_i K_i : r \geqq 1, \lambda_i \in \sigma(T), K_i \in \mathcal{L}(\mathcal{K}_n), K_i \geqq 0, \sum_{i=1}^{r} K_i = I\right\}$$

where "CL" denotes closure in the norm topology.

If T is self-adjoint, then since completely positive maps preserve adjoints, $W_n(T)$ consists entirely of self-adjoint operators. Is the converse true? When does $W_n(T)$ consist entirely of normal operators? If $W_n(T)$ consists of normal operators must T be normal? The next theorem resolves these questions. We will need the following simple lemma from [14]:

Lemma 3.1. *In $\mathcal{L}(\mathcal{K}_n)$, $n \geqq 2$, there exist positive self-adjoint operators K_1, K_2, K_3 with the following properties:*

(a) $K_1 + K_2 + K_3 = I$,

(b) *K_2K_3 and K_3K_2 are linearly independent.*

Theorem 3.2. *If $T \in \mathcal{L}(\mathcal{K})$, then the following conditions are equivalent:*

(i) *$W_n(T)$ consists entirely of normal operators for all n,*

(ii) *$W_{n_0}(T)$ consists entirely of normal operators for some $n_0 \geqq 2$,*

(iii) *T is normal and $\sigma(T)$ is contained in a line.*

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii). Assume to the contrary that T is not normal. Let π be a faithful representation of $C^*(T)$ on an infinite dimensional Hilbert space $\mathcal{K}$. Since T is not normal and π is faithful $\pi(T)$ is not normal and therefore we can find a unit vector x in $\mathcal{K}$ such that

$$||\pi(T)x||^2 \neq ||\pi(T)^*x||^2. \tag{3.1}$$

Suppose first that $\{x, \pi(T)x, \pi(T)^*x\}$ is a linearly independent set. Let $S_1 = [x, \pi(T)x]$, $S_2 = [x, \pi(T)^*x]$. By the Gram–Schmidt process we can find vectors y_1 and y_2 such that $S_1 = [x, y_1]$, $S_2 = [x, y_2]$ and $\{x, y_1\}$ and $\{x, y_2\}$ are both orthonormal sets. Since $\mathcal{H}$ is infinite dimensional we can find (if necessary) unit vectors $u_3, u_4, \cdots, u_{n_0}$ and $v_3, v_4, \cdots, v_{n_0}$ such that $\{x, y_1, u_3, \cdots, u_{n_0}\}$ and $\{x, y_2, v_3, v_4, \cdots, v_{n_0}\}$ are orthonormal sets. Let $\{e_1, e_2, \cdots, e_{n_0}\}$ be an orthonormal basis for $\mathcal{H}_{n_0}$ and define embeddings V_1, V_2 by

$$V_1 : e_1 \to x, \quad e_2 \to y_1, \quad e_3 \to u_3, \cdots, e_{n_0} \to u_{n_0}$$

$$V_2 : e_1 \to x, \quad e_2 \to y_2, \quad e_3 \to v_3, \cdots, e_{n_0} \to v_{n_0}.$$

If we set $\varphi_i(\cdot) = V_i^*\pi(\cdot)V_i$ $(i = 1, 2)$, then by Stinespring's theorem and our construction $\varphi_i \in CP(C^*(T), \mathcal{H}_{n_0}; I)$ and therefore $\varphi_i(T)$ is normal by hypothesis. Thus $\|\varphi_i(T)z\|^2 = \|\varphi_i(T)^*z\|^2$ for all z in $\mathcal{H}_{n_0}$. Taking $z = e_1$ we obtain

$$\|V_i^*\pi(T)V_ie_1\|^2 = \|V_i^*\pi(T)^*V_ie_1\|^2, \qquad i = 1, 2. \tag{3.2}$$

If we let $P_i = V_i^*V_i$, then P_1 is the projection on $[x, y_1, u_3, u_4, \cdots, u_{n_0}]$ and P_2 is the projection on $[x, y_2, v_3, v_4, \cdots, v_{n_0}]$. Evaluating first the left side and then the right side of (3.2) we obtain

$$\|V_1^*\pi(T)V_1e_1\|^2 = (P_1\pi(T)x, \pi(T)x) = \|\pi(T)x\|^2$$

and

$$\|V_1^*\pi(T)^*V_1e_1\|^2 = (P_1\pi(T)^*x, \pi(T)^*x) = \|P_1\pi(T)^*x\|.$$

Thus we have

$$\|\pi(T)x\|^2 = \|P_1\pi(T)^*x\|^2. \tag{3.3}$$

Letting $i = 2$ in (3.2), by a similar calculation we obtain

$$\|\pi(T)^*x\|^2 = \|P_2\pi(T)x\|^2. \tag{3.4}$$

From (3.1) we must have either $\|\pi(T)x\|^2 > \|\pi(T)^*x\|^2$ or $\|\pi(T)^*x\|^2 > \|\pi(T)x\|^2$. If the former holds, then from (3.3) we have $\|\pi(T)x\|^2 = \|P_1\pi(T)^*x\|_s = \|\pi(T)^*x\|^2 - \|(I - P_1)\pi(T)^*x\|^2$ and therefore $\|\pi(T)^*x\|^2 \geqq \|\pi(T)x\|^2$ which is a contradiction. If the latter holds, then from (3.4) we obtain $\|\pi(T)^*x\|^2 = \|P_2\pi(T)x\|^2 = \|\pi(T)x\|^2 - \|(I - P_2)\pi(T)x\|^2$ and thus $\|\pi(T)x\|^2 \geqq \|\pi(T)^*x\|^2$ which is also a contradiction. Thus T must be normal. If $\{x, \pi(T)x, \pi(T)^*x\}$ is a linearly dependent set, then the same proof is valid with just minor changes in notation. We now show that $\sigma(T)$ is contained in a line. Let $\lambda \in \sigma(T)$ and set $S = T - \lambda I$. S is normal and $0 \in \sigma(S)$. Let $K_1, K_2, K_3 \in \mathcal{L}(\mathcal{H}_{n_0})$ be as in Lemma 3.1. If λ_2 and λ_3 are arbitrary points of $\sigma(S)$, then by Theorem 3.1 $U = 0K_1 + \lambda_2K_2 + \lambda_3K_3 \in W_{n_0}(S)$. Since $C^*(S) = C^*(T)$ we have $W_{n_0}(S) = W_{n_0}(T) - \lambda I$ and thus $W_{n_0}(S)$ consists entirely of normal operators. Upon simplification the equation $UU^* - U^*U = 0$ becomes $(\lambda_2\bar{\lambda}_3 - \bar{\lambda}_2\lambda_3)K_2K_3 + (\lambda_3\bar{\lambda}_2 - \bar{\lambda}_3\lambda_2)K_3K_2 = 0$

and thus by condition (b) of Lemma 3.1 we must have $\lambda_2\bar{\lambda}_3 = \bar{\lambda}_2\lambda_3$ which means that $\lambda_2\bar{\lambda}_3$ is real. Since λ_2 and λ_3 were arbitrary elements of $\sigma(S)$, a little checking shows that $\sigma(S)$ must be contained in a line through the origin and thus $\sigma(T) = \sigma(S) + \lambda$ is contained in a line.

(iii) $\Rightarrow$ (i). Under the hypothesis T must be a linear function of a self-adjoint operator and this is sufficient to draw the conclusion.

Corollary 3.2.1. *If $W_n(T)$ consists entirely of self-adjoint operators, then T is self-adjoint.*

Proof. By Theorem 3.2 T is normal. If $\lambda \in \sigma(T)$, then by Theorem 3.1, $\lambda I \in W_n(T) \Rightarrow (\lambda I)^* = \lambda I \Rightarrow \lambda$ is real $\Rightarrow T$ is self-adjoint.

4. Applications. A theorem of S. Hildebrandt [12] states that for $T \in \mathcal{L}(\mathcal{H})$ we have

$$\overline{\text{co}}(\sigma(T)) = \bigcap_{B \in \mathcal{B}} \overline{W(BTB^{-1})} \tag{4.1}$$

where $\mathcal{B} = \{\text{all invertible operators in } \mathcal{L}(\mathcal{H})\}$. This theorem prompts the next definition.

Definition 4.1. If $T \in \mathcal{L}(\mathcal{H})$, then $V_n(T)$ is defined by

$$V_n(T) = \bigcap_{B \in \mathcal{B}} W_n(BTB^{-1}).$$

Recall that for an operator A the spectral radius of A, denoted by $r(A)$, satisfies $r(A) \leqq ||A||$ and that if A is normal equality holds.

Theorem 4.1. *If T is normal, then $V_n(T) = W_n(T)$ for $n = 1, 2, \cdots$.*

Proof. Since the inclusion $V_n(T) \subset W_n(T)$ always holds, it suffices to show that $W_n(T) \subset W_n(BTB^{-1})$ for any $B \in \mathcal{B}$. Define a linear map ψ from $[BTB^{-1}, I]$ by $\psi : \alpha(BTB^{-1}) + \beta I \rightarrow \alpha T + \beta I$ for all $\alpha, \beta \in \mathbf{C}$. If $\gamma \in \mathbf{C}$ we have $||T + \gamma I|| = r(T + \gamma I) = r(B(T + \gamma I)B^{-1}) \leqq ||BTB^{-1} + \gamma I||$ and hence ψ is contractive. By Lemma 2.1, ψ has a completely positive extension, ψ_1, to $C^*(BTB^{-1})$. If $S \in W_n(T)$, then there exists $\varphi \in CP(C^*(T), \mathcal{H}; I)$ such that $\varphi(T) = S$ and letting $\omega = \varphi \circ \psi_1$ we see that $S \in W_n(BTB^{-1})$. This completes the proof.

Corollary 4.1.1. *If T is normal, then $\overline{\text{co}}\ (\sigma(T)) = \overline{W(T)}$.*

Proof. By (4.1) and the above theorem we have

$$\overline{\text{co}}\ (\sigma(T)) = \bigcap_{B \in \mathcal{B}} \overline{W(BTB^{-1})} = V_1(T) = W_1(T) = \overline{W(T)}.$$

For an arbitrary $T \in \mathcal{L}(\mathcal{H})$ and a positive integer n, we define $S_n(T)$ by $S_n(T) = CL\{\sum_{i=1}^{r} \lambda_i K_i : r \geqq 1,\ \lambda_i \in \sigma(T),\ K_i \geqq 0,\ K_i \in \mathcal{L}(\mathcal{H}_n),\ \sum_{i=1}^{r} K_i = I\}$ where "CL" denotes closure in the norm topology. $S_n(T)$ is an n-dimensional analog of the convex hull of the spectrum of T. Since $\overline{\text{co}}\ (\sigma(T)) \subset \overline{W(T)}$ holds for any

$T \in \mathcal{L}(\mathcal{H})$, we can ask whether the n-dimensional analog of this inclusion is also valid.

Theorem 4.2. If $T \in \mathcal{L}(\mathcal{H})$, then $S_n(T) \subset W_n(T)$ for all n.

Proof. Let A be a normal operator such that $\sigma(A) = \sigma(T)$. If $\lambda \in \mathbf{C}$, then $\|A + \lambda I\| = r(A + \lambda I) = r(T + \lambda I) \leqq \|T + \lambda I\|$. Thus the linear map φ defined by $\varphi : \alpha T + \beta I \to \alpha A + \beta I$ is contractive and hence by Lemma 2.1 φ has a completely positive extension to $C^*(T)$. Arguing as in the last part of the proof of Theorem 4.1 we obtain $W_n(A) \subset W_n(T)$ for all n. But since A is normal and $\sigma(A) = \sigma(T)$ we have $S_n(T) = S_n(A) = W_n(A) \subset W_n(T)$ and the proof is complete.

Theorem 3.1 shows that $S_n(T) = W_n(T)$ if T is normal. We will show that equality also holds for other classes of operators.

5. Subnormal and Toeplitz operators. Recall that $T \in \mathcal{L}(\mathcal{H})$ is subnormal if there exists a normal operator A on a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ is a subspace of $\mathcal{K}$ which is invariant under A and the restriction of A to $\mathcal{H}$ coincides with T. In [10] it is shown that a subnormal operator has an essentially unique minimal normal extension which we denote by A and that $\sigma(A) = \sigma(T)$.

Theorem 5.1. If T is subnormal with minimal normal extension A, then we have $S_n(T) = W_n(T) = W_n(A)$ for all n.

Proof. Let P denote the projection of $\mathcal{K}$ onto $\mathcal{H}$. The map φ from $C^*(A)$ to $C^*(T)$ defined by $\varphi : B \to PBP$ is easily seen to completely positive and carries A to T. Arguing as in Theorem 4.1 we have $W_n(T) \subset W_n(A)$ for all n. Now by Theorems 4.2 and 3.1 and since A is normal we have

$$S_n(T) \subset W_n(T) \subset W_n(A) = S_n(A). \tag{5.1}$$

Since $\sigma(A) \subset \sigma(T)$ we have $S_n(A) \subset S_n(T)$ and thus all the inclusions in (5.1) become equalities and the conclusion follows.

Remark. An operator T satisfying $T^*T \geqq TT^*$ is said to be hyponormal. All subnormal operators are hyponormal but there are hyponormal operators which are not subnormal [10]. We conjecture that if T is hyponormal, then $W_n(T) = S_n(T)$ for all n. If T is a hyponormal weighted shift [10], then it is shown in [14] that $S_n(T) = W_n(T)$.

Let $D = \{z : |z| = 1\}$. If $\varphi \in \mathcal{L}_\infty(D)$, then φ induces an operator L_φ on $\mathcal{L}_2(D)$ defined by $L_\varphi : f \to \varphi \cdot f$. L_φ is called the Laurent operator induced by φ and it is clearly normal. Denote by $\mathcal{H}^2(D)$ the subspace of $\mathcal{L}_2(D)$ which consists of all functions whose negative Fourier coefficients vanish (where the Fourier coefficients are taken with respect to the standard orthonormal basis $e_n(z) = z^n$, $n = 0, \pm 1, \pm 2, \cdots$) and let P be the projection of $\mathcal{L}_2(D)$ onto $\mathcal{H}^2(D)$. If $\varphi \in \mathcal{L}_\infty(D)$, then φ induces an operator T_φ on $\mathcal{H}^2(D)$ defined by $T_\varphi : f \to P(\varphi \cdot f)$. T_φ is called the Toeplitz operator induced by φ (see [5] for further properties).

Theorem 5.2. *With the above notation we have*

$$W_n(T_\varphi) = W_n(L_\varphi) = S_n(T_\varphi).$$

Proof. The map ψ from $C^*(L_\varphi)$ to $C^*(T_\varphi)$ defined by $\psi : B \to PBP$ is easily seen to be completely positive and carries L_φ to T_φ. Arguing as in the proof of Theorem 4.1 we have $W_n(T_\varphi) \subset W_n(L_\varphi)$. By the Hartman–Wintner spectral inclusion theorem [11], $\sigma(L_\varphi) \subset \sigma(T_\varphi)$ and thus $S_n(L_\varphi) \subset S_n(T_\varphi)$. Thus $W_n(L_\varphi) = S_n(L_\varphi) \subset S_n(T_\varphi) \subset W_n(T_\varphi) \subset W_n(L_\varphi)$ and the theorem is proved.

6. Matrix-normal operators. Theorems 5.1 and 5.2 enable us to reduce the problem of finding $W_n(T)$ for the not necessarily normal operator T, to the more tractable problem of finding the n-dimensional matrix range of a normal operator for which we can use Theorem 3.1. Therefore, given a non-normal operator T we can ask when there exists a normal operator S such that $W_n(S) = W_n(T)$ for $n = 1, 2, \cdots$.

Definition 6.1. An operator T is matrix-normal if there exists a normal operator S such that $W_n(T) = W_n(S)$ for all n.

Given $T \in \mathcal{L}(\mathcal{H})$, recall that a nonempty closed proper subset of the plane K, which contains $\sigma(T)$, is called a spectral set for the operator T if whenever f is a rational function on K with no poles on K, then the inequality $\|f(T)\| \leqq \sup \{|f(z)| : z \in K\}$ holds. We will need a dilation theorem which was proved independently by C. Foias, C. Berger and A. Lebow [13].

Theorem 6.1. *If K is a convex spectral set for $T \in \mathcal{L}(\mathcal{H})$, then there exists a normal operator S defined on a larger Hilbert space $\mathcal{H}_1 \supset \mathcal{H}$ such that*

(i) $\sigma(S) \subset \partial K$,
(ii) $T^n x = PS^n x$ ($x \in \mathcal{H}$, $n = 0, 1, 2, \cdots$),
where P is the projection of $\mathcal{H}$ onto $\mathcal{H}_1$.

The next theorem gives a sufficient condition for an operator to be matrix-normal.

Theorem 6.2. *Let $T \in \mathcal{L}(\mathcal{H})$. If $W(T)$ is a spectral set for T, then there exists a normal operator S such that $W_n(S) = W_n(T)$ for all n. Moreover we can choose S so that $\sigma(S) \subset \partial \overline{W(T)}$.*

Proof. Since $\overline{W(T)}$ is compact and convex, we can apply Theorem 5.1 to $\overline{W(T)}$ to obtain a normal operator S with properties (i) and (ii). The map φ from $C^*(S)$ to $C^*(T)$ defined by $\varphi(B) = PBP$ is completely positive and by taking $n = 1$ in (ii) of Theorem 6.1 we see that $\varphi(S) = T$. Arguing as in the proof of Theorem 4.1 we have $W_n(T) \subset W_n(S)$ for all n. Since S is normal and $\sigma(S) \subset \partial(\overline{W(T)})$ we have $\overline{\text{co}}\ (\sigma(S)) = \overline{W(S)} \subset \overline{W(T)}$ from which we see that $\overline{W(S + \lambda I)} \subset \overline{W(T + \lambda I)}$ for $\lambda \in \mathbf{C}$ and hence $\|S + \lambda I\| = \sup \{|z + \lambda| : z \in \overline{W(S)}\} \leqq \sup \{|z + \lambda| : z \in \overline{W(T)}\} = \|T + \lambda I\|$. This last inequality shows that the map ψ defined by $\alpha T + \beta I \to \alpha S + \beta I$ is contractive and hence by

Lemma 2.1 ψ has a completely positive extension to $C^*(T)$. Arguing as in Theorem 4.1 we have $W_n(S) \subset W_n(T)$ for all n, and the proof is complete.

Is the converse of Theorem 6.2 true? Explicitly, if T is matrix-normal must $\overline{W(T)}$ be a spectral set? Our previous results have shown that subnormal and Toeplitz operators are matrix-normal and it is easy to see that $\overline{W(T)}$ is a spectral set if T is subnormal. However, for arbitrary Toeplitz operators the answer is not known. Although we cannot show that $\overline{W(T)}$ is a spectral set if T is matrix-normal, we can ask what conditions are imposed on an operator if it is matrix-normal.

Recall that the numerical radius, $|W(A)|$, of an operator A is defined by $|W(A)| = \sup \{|\lambda| : \lambda \in W(A)\}$ and that A is *normaloid* if $|W(A)| = ||A||$. Also A is *spectraloid* if $r(A) = |W(A)|$. Every normaloid operator is spectraloid [10]. We will need the next lemma which is proved in [14].

Lemma 6.1. *If B and C are compact convex nonempty plane sets and $B \not\subseteq C$, then there exists a complex number λ such that*

$$\sup \{|z + \lambda| : z \in B\} < \sup \{|z + \lambda| : z \in C\}. \tag{6.1}$$

Theorem 6.3. *If T is matrix-normal, then*

(i) *$T + \lambda I$ is normaloid for all $\lambda \in \mathbf{C}$,*

(ii) $\overline{\text{co}}\ (\sigma(T)) = \overline{W(T)}$,

(iii) *there exists a normal operator S_1 such that $W_n(S_1) = W_n(T)$ for all n and $\sigma(S_1) \subset \partial\overline{W(T)}$. Moreover there exists a representation π of $C^*(T)$ such that $\pi(T) = S_1$.*

Proof. Since T is matrix-normal there exists a normal operator S such that $W_n(T) = W_n(S)$ for all n and therefore for any $\lambda \in \mathbf{C}$ we have $W_n(T + \lambda I) = W_n(S + \lambda I)$ for all n. Thus by Theorem 2.2 we have $||T + \lambda I|| = ||S + \lambda I||$ for $\lambda \in \mathbf{C}$. Since $\overline{W(T)} = \overline{W(S)}$ we have

$$||T + \lambda I|| = ||S + \lambda I|| = |W(S + \lambda I)| = |W(T + \lambda I)| \quad \text{for} \quad \lambda \in \mathbf{C}, \tag{6.2}$$

and therefore $T + \lambda I$ is normaloid. Since a normaloid operator is spectraloid we obtain for $\lambda \in \mathbf{C}$

$$||T + \lambda I|| = r(T + \lambda I) = \sup \{|z + \lambda| : z \in \overline{\text{co}}\ (\sigma(T))\}. \tag{6.3}$$

However, from (6.2) we have

$$||T + \lambda I|| = ||W(S + \lambda I)|| = \sup \{|z + \lambda| : x \in \overline{W(S)}\}. \tag{6.4}$$

Thus for $\lambda \in \mathbf{C}$ we obtain

$$\sup \{|z + \lambda| : z \in \overline{\text{co}}\ (\sigma(T))\} = \sup \{|z + \lambda| : z \in \overline{W(S)}\}. \tag{6.5}$$

We know that $\overline{\text{co}}\ (\sigma(T)) = \overline{W(T)} = \overline{W(S)}$ and thus by Lemma 6.1 we must have $\overline{\text{co}}\ (\sigma(T)) = \overline{W(S)} = \overline{W(T)}$, otherwise (5.5) would not hold for all $\lambda \in \mathbf{C}$.

Letting S_1 be a diagonal (and therefore normal) operator such that $\sigma(S_1) = EX(\overline{W(S)})$ (the extreme points of $\overline{W(S)}$) we obtain for $\lambda \varepsilon \mathbf{C}$,

$$(6.6) \quad \begin{aligned} \|S_1 + \lambda I\| &= \sup \{|z + \lambda| : z \varepsilon EX(\overline{W(S)})\} \\ &= \sup \{|z + \lambda| : z \varepsilon \overline{\text{co}}\,(EX(\overline{W(S)})) = \overline{W(S)}\} = \|S + \lambda I\|. \end{aligned}$$

Since S_1 and S are both normal (6.6) is easily seen to imply that $W_n(S_1) = W_n(S)$ for all n and thus $W_n(T) = W_n(S_1)$ for all n. Also, $\sigma(S_1) = EX(\overline{W(S)}) = EX(\overline{W(T)}) \subset \partial(\overline{W(T)})$.

If [1] Arveson proved that if $\lambda \varepsilon \sigma(T) \cap \partial W(T)$, then there exists a representation π of $C^*(T)$ such that $\pi(T) = \lambda$. Now $\overline{\text{co}}\,(\sigma(T)) = \overline{W(T)} = \overline{W(S)}$ and therefore by Milman's Lemma [8] $EX(\overline{W(S)}) \subset \sigma(T)$ and thus $\sigma(S_1) = EX(\overline{W(S)}) \subset \sigma(T) \cap \partial W(T)$. If $\lambda \varepsilon \sigma(S_1)$, denote by π_λ the representation of $C^*(T)$ such that $\pi_\lambda(T) = \lambda$ and let $\pi = \sum_{\lambda \varepsilon \sigma(S_1)} \oplus \pi_\lambda$. It is clear from our construction of S_1 that $\pi(T) = S_1$.

References

1. W. B. Arveson, *Subalgebras of C*-algebras*, Acta Math **123** (1969), 141–224.
2. ————, *Unitary invariants for compact operators*, Bull. Amer. Math. Soc. **76** (1970), 88–91.
3. ————, *Subalgebras of C*-algebras II*, (to appear).
4. S. H. Berberian & G. H. Orland, *On the closure of the numerical range of an operator*, Proc. Amer. Math. Soc. **18** (1967), 499–503.
5. A. Brown & P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **231** (1963), 89–102.
6. J. Dixmier, *Les C*-algebras et leurs representations*, Gauthier-Villars, Paris, 1964.
7. W. F. Donoghue, *On the numerical range of a bounded operator*, Mich. Math. J. **4** (1957), 261–263.
8. N. Dunford & J. T. Schwartz, *Linear operators, part I: general theory*, Interscience, New York, 1958.
9. P. R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity*, Chelsea, New York, 1951.
10. ————, *A Hilbert space problem book*, Van Nostrand, Princeton, 1967.
11. P. Hartman & A. Wintner, *On the spectra of Toeplitz's matrices*, Amer. J. Math. **72** (1950), 359–366.
12. A. Hildebrandt, *Über den numerischen Wertebereich eines Operators*, Math. Ann. **163** (1966), 230–247.
13. A. Lebow, *On von Neumann's theory of spectral sets*, J. Math. Anal. Appl. **7** (1963), 64–90.
14. F. M. Pollack, Ph.D. dissertation, University of California, Los Angeles, 1971
15. W. T. Stinespring, *Positive functions on C^n-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211–216.

This research was partially supported by NSF Grant no. GP-6727.

University of California, Los Angeles

Date communicated: April 28, 1972