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PROPERTIES OF THE MODULUS OF CONTINUITY FOR MONOTONOUS CONVEX FUNCTIONS AND APPLICATIONS

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ABSTRACT. For a monotone convex function $f \in C[a,b]$ we prove that the modulus of continuity $\omega(f;h)$ is concave on $[a,b]$ as function of h. Applications to approximation theory are obtained.

KEY WORDS AND PHRASES. Concave modulus of continuity, approximation by positive linear operators, Jackson estimate in Korneichuk's form.

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1. INTRODUCTION.

In a recent paper, Gal [1] the modulus of continuity for convex functions is exactly calculated, in the following way.

THEOREM 1. (see [1]) Let $f \in C[a,b]$ be monotone and convex on $[a,b]$. For any $h \in [0,b-a]$ we have:

(i) $\omega(f;h) = f(b) - f(b-h)$, if f is increasing on $[a,b]$,

(ii) $\omega(f;h) = f(a) - f(a+h)$, if f is decreasing on $[a,b]$,

where $\omega(f;h)$ denotes the classical modulus of continuity.

Denote

$KM[a,b] = \{f \in C[a,b];\ f$ is monotonous convex or monotonous concave on $[a,b]\}$

The purpose of the present paper is to prove that for $f \in KM[a,b]$ the modulus of continuity $\omega(f;h)$ is concave as function of $h \in [0,b-a]$ and to apply this result to approximation by positive linear operators and to Jackson estimates in Korneichuk's form.

2. MAIN RESULTS AND APPLICATIONS.

A first main result is the following

THEOREM 2. For all $f \in KM[a,b]$, the modulus of continuity $\omega(f;h)$ is concave as function of $h \in [0,b-a]$.

PROOF. Let firstly suppose that f is increasing and convex on $[a,b]$. If f is increasing on $[a,b]$, by Theorem 1, (i), we have $\omega(f;h) = f(b) - f(b-h)$. Hence

$$\alpha\omega(f;h_1) + (1-\alpha)\omega(f;h_2) = f(b) - \alpha f(b-h_1) - (1-\alpha)f(b-h_2) \tag{1.1}$$

and

$$\omega(f;\alpha h_1 + (1-\alpha)h_2) = f(b) - f(b - \alpha h_1 - (1-\alpha)h_2) \tag{1.2}$$

for all $\alpha \in [0,1]$ and all $h_1, h_2 \in [0,b-a]$.

Since f is convex on $[a,b]$ we get

$$f(b-\alpha h_1-(1-\alpha)h_2)\leq \alpha f(b-h_1)+(1-\alpha)f(b-h_2),$$

wherefrom taking into account (1.1) and (1.2) too, we get

$$\alpha\omega(f;h_1)+(1-\alpha)\omega(f;h_2)\leq \omega(f;\alpha h_1+(1-a)h_2) \tag{1.3}$$

Now, if f is decreasing on $[a,b]$, since by Theorem 1, (ii), we have $\omega(f;h)=f(a)-f(a+h)$, we immediately get

$$\alpha\omega(f;h_1)+(1-\alpha)\omega(f;h_2)=f(a)-\alpha f(a+h_1)-(1-\alpha)f(a+h_2) \tag{1.4}$$

and

$$\omega(f;\alpha h_1+(1-\alpha)h_2)=f(a)-f(a+\alpha h_1+(1-\alpha)h_2) \tag{1.5}$$

for all $\alpha\in[0,1]$ and all $h_1,h_2\in[0,b-a]$.

Since f is convex on $[a,b]$ we have

$$f(a+\alpha h_1+(1-\alpha)h_2)\leq \alpha f(a+h_1)+(1-\alpha)f(a+h_2),$$

which together with (1.4) and (1.5) gives again (1.3).

In the following we need the

DEFINITION 1. (see e.g. [2]) Let $f\in C[a,b]$ be. If $\omega(f;h)=sup\{\,|f(x)-f(y)|\,;\,|x-y|\leq h\}$ is the usual modulus of continuity, the least concave majorant of $\omega(f;h)$ is given by

$$\widetilde{\omega}\,(f;h)=sup\left\{\frac{(\delta-\alpha)\omega(f;\beta)+(\beta-\delta)\omega(f;\alpha)}{\beta-\alpha};\ 0\leq\alpha\leq\delta\leq\beta\leq b-a\right\}.$$

An immediate consequence of Definition 1 is the

COROLLARY 1. For any $f\in KM[a,b]$ we have

$$\widetilde{\omega}\,(f;h)=\omega(f;h)$$

PROOF. Putting $\alpha=\delta$ in Definition 1 we get

$$\omega(f;h)\leq\widetilde{\omega}\,(f;h).$$

Then, taking into account Theorem 2, for $0\leq\alpha\leq\delta\leq\beta\leq b-a$ we have

$$\frac{(\delta-\alpha)\omega(f;\beta)+(b-\delta)\omega(f;\alpha)}{\beta-\alpha}\leq\omega\left(f;\frac{\beta(\delta-\alpha)}{\beta-\alpha}+\frac{\alpha(\beta-\delta)}{\beta-\alpha}\right)=\omega(f;\delta)$$

wherefrom passing to supremum, we immediately get

$$\widetilde{\omega}\,(f;\delta)\leq\omega(f;\delta),$$

which proves the corollary.

REMARK. It is easy to see that Corollary 1 remains valid for all $f\in C[a,b]$ having a concave modulus of continuity $\omega(f;h)$.

Now, firstly we will apply the previous results to approximation by positive linear operators.

Thus, investigating the sequence of Lehnhoff polynomials in [3], $L_n(f)(x)$, defined for $f\in C[-1,1]$, H.H. Gonska [2] proves that

$$|\,L_n(f)(x)-f(x)\,|\leq\sqrt{\frac{30}{11}}\,\widetilde{\omega}\left(f;\frac{\sqrt{1-x^2}}{n}+\frac{|\,x\,|}{n^2}\right)$$

Taking now into account Corollary 1 we immediately get the

COROLLARY 2. If $f \in KM[-1,1]$ then for all $x \in [-1,1]$, $n \in N$ we have

$$|L_n(f)(x) - f(x)| \leq \sqrt{\frac{30}{11}}\, \omega\left(f; \frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2}\right)$$

In the same paper, for $f \in C[0,1]$, H.H. Gonska obtains estimates in terms of the modulus $\widetilde{\omega}(f;h)$ in the approximation by the so-called Shepard operator, $S_n^{\mu}(f), 1 \leq \mu \leq 2$. Then by Corollary 1 and by Theorem 4.3 in [2] we immediately get the

COROLLARY 3. For all $f \in KM[0,1]$ and all $n \in N$ we have

$$\| S_n^1(f) - f \| \leq \frac{n+1}{n}\, \omega\left(f; \frac{1}{\ln(2n+2)}\right)$$

$$\| S_n^{\mu}(f) - f \| \leq \frac{14}{2-\mu}\, \omega\left(f; \frac{1}{(n+1)^{\mu-1}}\right),\ 1 < \mu < 2$$

$$\| S_n^2(f) - f \| \leq 19\omega\left(f; \frac{\ln(n+1)}{n+1}\right).$$

Finally, we will apply our results to the following so-called Jackson estimate in Korneichuk's form.

THEOREM 3. (see e.g. [4], p. 147) For any $f \in C[-1,1]$ we have

$$E_{n-1}(f) \leq \omega\left(f; \frac{\pi}{n}\right),\ n = 1, 2, \cdots,$$

where $E_k(f)$ denotes the best approximation by polynomials of degree $\leq k$.

Now, we will prove the

THEOREM 4. If $f \in C[-1,1]$ has a concave modulus of continuity $\omega(f;h), h \in [0,2]$, then we have

$$E_{n-1}(f) \leq \frac{1}{2}\, \omega\left(f; \frac{\pi}{n}\right)$$

PROOF. Extending ω to $[0,\pi]$ by taking $\omega(f;h) = \omega(f;2), h \in [2,\pi]$, obviously ω remains concave on $[0,\pi]$.

Denote $\omega(h) = \omega(f;h), h \in [0,\pi]$ and

$$\Lambda_{\omega} = \{g \in C[-1,1]; \omega(g;h) \leq \omega(h), \forall h \in [0,\pi]\}.$$

Obviously $f \in \Lambda_{\omega}$. Then by [5, Theorem 8 and Lemma 2, p. 122-123], as in the proof of Theorem 9, p. 123 in [5], there is $g \in Lip_M 1$ such that

$$\| f - g \| \leq \frac{1}{2}\, \omega\left(f; \frac{\pi}{n}\right) - \frac{\pi M}{2n}.$$

Now by Theorem V, (ii), in [4, p. 147], there is P_{n-1} polynomial of degree $\leq n-1$ such that

$$\| g - P_{n-1} \| \leq \frac{\pi M}{2n}.$$

Hence we get

$$\| f - P_{n-1} \| \leq \| f - g \| + \| g - P_{n-1} \| \leq \frac{1}{2}\, \omega\left(f; \frac{\pi}{n}\right)$$

which proves the theorem.

REMARK. For $f \in KM[-1,1]$, Theorem 4 remains valid.

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