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PROPERTIES OF THE OPERATOR DOMAINS OF THE FOURTH-ORDER LEGENDRE-TYPE DIFFERENTIAL EXPRESSIONS

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This paper is dedicated to the memory of Professor Peter Hess

Abstract. This paper is concerned with certain properties of the three fourth-order Legendre-type differential expressions. After normalization to the compact interval $[-1, 1]$ of the real line, there are five distinct such differential expressions. There is one of the second order (the classical Legendre differential expression), three expressions of the fourth order (discovered by H.L. Krall in 1938 and 1940), and one of the sixth order (discovered by Littlejohn in 1981). The three fourth-order expressions have a number of interesting properties when considered in the classical integrable-square space on $(-1, 1)$, and in the relevant measure integrable-square spaces on $[-1, 1]$. The paper discusses some of these properties and determines the smoothness conditions satisfied by elements of the maximal domains and the self-adjoint operator domains. These results are related to the orthogonal polynomials generated, firstly in the measure spaces and, secondly, by the fourth-order spectral differential equations linked to the Legendre-type differential expressions.

1. Introduction. The positive and non-negative integers are denoted by $\mathbb{N} = \{1, 2, 3, \ldots, \}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots, \}$, and the real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$.

With M and N real, non-negative parameters let the monotonic, non-decreasing function $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\hat{\mu}(x) = \begin{cases} -1 - M & (x \in (-\infty, -1]) \\ x & (x \in (-1, 1)) \\ 1 + N & (x \in [1, \infty)). \end{cases} \tag{1.1}$$

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Let μ be the regular, non-negative measure generated by $\hat{\mu}$ on the Borel sets of $\mathbb{R}$, and let $L^2([-1, 1]; \mu)$ denote the integrable-square Hilbert space of equivalence classes of Borel measurable functions with norm and inner product given, respectively, by

$$\|f\|_\mu^2 := \int_{[-1,1]} |f(x)|^2 \, d\mu(x) \tag{1.2}$$

$$\begin{aligned}(f, g)_\mu &:= \int_{[-1,1]} f(x)\bar{g}(x)\, d\mu(x) \\ &= Mf(-1)\bar{g}(-1) + \int_{-1}^{1} f(x)\bar{g}(x)\, dx + Nf(1)\bar{g}(1).\end{aligned} \tag{1.3}$$

The integral in (1.2) is a Lebesgue-Stieltjes integral and the integral in (1.3) is the standard Lebesgue integral.

The measure μ has finite moments in respect of the sequence of powers $\{x^n : n \in \mathbb{N}_0\}$; i.e.,

$$x \mapsto x^n \in L^2([-1, 1]; \mu) \text{ or } \int_{[-1,1]} |x^n|^2 \, d\mu(x) < \infty, \quad (n \in \mathbb{N}_0).$$

Furthermore, the set $\{x^n : n \in \mathbb{N}_0\}$ is linearly independent in $L^2([-1, 1]; \mu)$.

The Legendre-type polynomials are the orthogonal polynomial systems formed by applying the Gram-Schmidt orthogonalization process to the set $\{x^n : n \in \mathbb{N}_0\}$ in $L^2([-1, 1]; \mu)$. Five cases emerge from the measure μ:

$$\begin{array}{ll} \text{(i)} & M = N = 0 \\ \text{(ii)} & M > 0, \quad N = 0 \\ \text{(iii)} & M = 0, \quad N > 0 \\ \text{(iv)} & M = N > 0 \\ \text{(v)} & M > 0,\ N > 0, \quad M \neq N. \end{array} \tag{1.4}$$

Case (i) yields the classical Legendre polynomials; see Chihara [1, Chapter V] and Szegö [18, Chapter IV]. The Cases (ii), (iii) and (iv) were considered by H.L. Krall [11], [12], and A.M. Krall [10]. The final case (v) was developed by Littlejohn in his thesis [13].

The orthogonal polynomials in all these five cases (1.4) are special examples of the general Koornwinder polynomials considered in [9]; see in particular [9; Sections 1–4] with $\alpha = \beta = 0$. The general Koornwinder notation of $\{P_n^{\alpha,\beta,M,N}(x) : x \in [-1, 1]; n \in \mathbb{N}_0\}$ then reduces to

$$\{P_n^{0,0,M,N}(x) : x \in [-1, 1]; n \in \mathbb{N}_0\} \tag{1.5}$$

for the orthogonal polynomials considered in this paper.

Another significant unifying property of these five cases (1.4) of orthogonal polynomials is that each system is also generated by a formally symmetric spectral differential equation of the form

$$\sum_{r=0}^{s}(-1)^r(q_r(x)y^{(r)}(x))^{(r)} = \lambda y(x), \quad (x \in (-1,1)), \tag{1.6}$$

where $s \in \mathbb{N}$, the spectral parameter $\lambda \in \mathbb{C}$, and the coefficients

$$\{q_r : r = 0, 1, \dots, s\}$$

are real-valued polynomials on $\mathbb{R}$ with degree $q_r = 2r$ $(r = 0, 1, \dots, s)$. The best possible, (i.e., the smallest) integer s for which (1.6) is effective depends on the particular case determined by (1.4); the coefficients $\{q_r\}$ depend not only on the case in (1.4) but also on the parameters M and N, but not on the spectral parameter λ.

For case (i) of (1.4), we have $s = 1$, yielding the classical second-order Legendre differential equation (see [1] and [18]). For cases (ii), (iii), and (iv), the value of s is 2, yielding the fourth-order Legendre-type differential equations of H.L. Krall [11], [12]. For case (v), $s = 3$, yielding the sixth-order Legendre-type differential equation studied by Littlejohn [13]; see also the recent paper of Everitt, Littlejohn, and Loveland [7].

Later work on these Legendre-type differential equations was undertaken by Everitt, A.M. Krall, Littlejohn, Loveland, and Marić; see [3], [4], [5], [7], [8], and [10]. A detailed statement of some properties of all five Legendre-type differential equations (1.6) can be found in the research report of Everitt, Littlejohn, and Loveland [6, Sections 0, 1, and 2]. The spectral theory of the self-adjoint differential operators generated by the differential equations (1.6) in the Hilbert spaces $L^2([-1,1];\mu)$ is considered in detail in the thesis of Loveland [14]; see also the forthcoming papers [15] and [16].

In this paper we are concerned with properties of the operator domains arising in cases (ii), (iii), and (iv) of (1.4) for which the order of the respective Legendre-type differential equations is 4. We give below the explicit form of these differential equations quoting from [6, Section 1, (1.6)–(1.21)]. For this purpose it is convenient to introduce the positive numbers A and B defined by (see (1.4))

$$A = M^{-1} \text{ when } M > 0, \quad B = N^{-1} \text{ when } N > 0; \tag{1.7}$$

A and B are not defined and not required when $M = 0$ or $N = 0$, respectively.

The three fourth-order Legendre-type differential equations then take the form, hereby defining, in the notation of [6, (1.16), (1.18), and (1.20)] and in terms of the case numbers of (1.4), the three differential expressions

Case (ii) $(A = M^{-1},\ N = 0)$

$$\begin{aligned} ML_k^{(2)}[y](x) :&= ((1-x^2)^2y''(x))'' - (2(1-x)((2A+1)x \\ &\quad + 2A+3)y'(x))' + ky(x) = \lambda y(x), \quad (x \in (-1,1)) \end{aligned} \tag{1.8}$$

Case (iii) ($M = 0,\ B = N^{-1}$)

$$\begin{aligned} MR_k^{(2)}[y](x) := &((1-x^2)^2y''(x))'' - (2(1+x)((-2B-1)x \\ &+ 2B+3)y'(x))' + ky(x) = \lambda y(x), \quad (x \in (-1,1)) \end{aligned} \tag{1.9}$$

Case (iv) ($M = N > 0,\ B = A = M^{-1} = N^{-1}$)

$$\begin{aligned} ML_k^{(2)}[y](x) := &((1-x^2)^2y''(x))'' - ((8+4A(1-x^2))y'(x))' + ky(x) \\ &= \lambda y(x), \quad (x \in (-1,1)). \end{aligned} \tag{1.10}$$

In all three cases $k \geq 0$ is a translation parameter essential for the proof of certain spectral theoretic properties of the associated differential operators. (Note that L and R are used in cases (ii) and (iii) to indicate that the discontinuity in $\hat{\mu}$ is, respectively, at the endpoints -1 and $+1$; this use of L and R is continued below for a number of definitions concerning domains and operators).

If, in the notation of Koornwinder, see (1.5) above, the systems of orthogonal polynomials arising in cases (ii), (iii) and (iv) of (1.4) are denoted, respectively, by

$$\{P_n^{0,0,M,0} : n \in \mathbb{N}_0\}, \quad \{P_n^{0,0,0,N} : n \in \mathbb{N}_0\}, \quad \{P_n^{0,0,M,M} : n \in \mathbb{N}_0\}, \tag{1.11}$$

then, essentially, it was established by H.L. Krall [11, 12] that these polynomials are solutions, respectively, for the three differential equations (1.8), (1.9), (1.10) with

$$\lambda = \lambda L_n^{(2)}(A,k), \quad \lambda = \lambda R_n^{(2)}(B,k), \quad \lambda = \lambda_n^{(2)}(A,k),$$

for each $n \in \mathbb{N}_0$. Explicitly, these eigenvalues are given by, see [6, (1.17), (1.19), (1.21)]

$$\lambda L_n^{(2)}(A,k) = n(n+1)(n^2+n+4A)+k \qquad (n \in \mathbb{N}_0), \tag{1.12}$$

$$\lambda R_n^{(2)}(B,k) = n(n+1)(n^2+n+4B)+k \qquad (n \in \mathbb{N}_0), \tag{1.13}$$

$$\lambda_n^{(2)}(A,k) = n(n+1)(n^2+n+4A-2)+k \quad (n \in \mathbb{N}_0). \tag{1.14}$$

Furthermore, the explicit forms of the polynomials in (1.11) was obtained by H.L. Krall [11, 12] (see also A.M. Krall [10]) and are (recall (1.7))

$$P_n^{0,0,M,0}(x) = \sum_{r=0}^{n} \frac{(-1)^r(n+r)!(n^2+n+2A-r)}{(r!)^2(n-r)!(n^2+n+2A)}\left(\frac{1-x}{2}\right)^r, \tag{1.15}$$

$$P_n^{0,0,0,N}(x) = \sum_{r=0}^{n} \frac{(-1)^r(n+r)!(n^2+n+2B-r)}{(r!)^2(n-r)!(n^2+n+2B)}\left(\frac{1+x}{2}\right)^r, \tag{1.16}$$

$$P_n^{0,0,M,M}(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r(2n-2r)!(A+\frac{n(n-1)}{2}+2r)}{A2^n r!(n-r)!(n-2r)!}x^{n-2r}, \tag{1.17}$$

with normalization so that

$$P_0^{0,0,M,0}(1) = P_n^{0,0,0,N}(-1) = P_n^{0,0,M,M}(1) = 1.$$

In this paper it is convenient to take $k = 1$ and then, since, all three differential expressions are of the same order, to define

$$ML[\cdot] := ML_1^{(2)}[\cdot], \quad MR[\cdot] := MR_1^{(2)}[\cdot], \quad M[\cdot] := M_1^{(2)}[\cdot], \tag{1.18}$$

with, respectively, eigenvalues $\{\lambda L_n : n \in \mathbb{N}_0\}$, $\{\lambda R_n : n \in \mathbb{N}_0\}$, and $\{\lambda_n : n \in \mathbb{N}_0\}$.

All three differential expressions in (1.18) are Lagrange symmetric and of the general form considered in the now classic text of Naimark [17, Chapter V]. The common domain D of all three expressions is given by

$$D := \{f : (-1,1) \to \mathbb{C} : f \in AC_{\text{loc}}^{(r)}(-1,1),\ r = 0,1,2,3\},$$

and Green's formula takes the form, for all $f, g \in D$ and all compact $[\alpha,\beta] \subset (-1,1)$,

$$\int_\alpha^\beta \{\bar{g}(x)N[f](x) - f(x)\overline{N[g]}(x)\}\,dx = \{f,g\}(x)\Big|_\alpha^\beta, \tag{1.19}$$

where $N[\cdot]$ represents any one of $ML[\cdot]$, $MR[\cdot]$, $M[\cdot]$. Here, $\{\cdot,\cdot\}(\cdot) : D \times D \times (-1,1) \to \mathbb{C}$ is a general notation for a skew-symmetric bilinear form, which for the three cases of $N[\cdot]$ represented by (1.18), we write as, respectively,

$$L[f,g](\cdot), \quad R[f,g](\cdot), \quad [f,g](\cdot), \tag{1.20}$$

with $f, g \in D$.

As an example of the bilinear forms of (1.20) we give in detail the explicit formula for $L[f,g](\cdot)$, since this will be required in the following sections:

$$\begin{aligned} L[f,g] =& [((1-x^2)^2 f''(x))' - (1-x)p(x)f'(x)]\bar{g}(x) \\ & - [((1-x^2)^2\bar{g}''(x))' - (1-x)p(x)\bar{g}'(x)]f(x) \\ & - (1-x^2)^2(f''(x)\bar{g}'(x) - f'(x)\bar{g}''(x)) \quad (x \in (-1,1)); \end{aligned} \tag{1.21}$$

here, for convenience, we have defined

$$p(x) := 2((2A+1)x + 2A + 3) \quad (x \in (-1,1)). \tag{1.22}$$

The corresponding results for the bilinear forms $R[f,g](\cdot)$ and $[f,g](\cdot)$ are recorded in the thesis [14, Section 6.2], and [4, Section 2] or [14, Section 4.2].

The maximal domains of the three differential expressions of (1.18), in the classical space $L^2(-1,1)$, are represented by ΔL, ΔR, and Δ respectively. In the general notation of (1.19), they are defined as the linear manifold of $L^2(-1,1)$

$$\{f : (-1,1) \to \mathbb{C} : f \in D;\ f, N[f] \in L^2(-1,1)\}. \tag{1.23}$$

From (1.19), it follows that the limits

$$\lim_{x\to\pm1}\{f, g\}(x) := \{f, g\}(\pm1) \tag{1.24}$$

exist and are finite in $\mathbb{C}$, for all f, g in the relevant maximal domain.

The classical theory of the determination of all self-adjoint operators generated by $N[\cdot]$ in $L^2(-1, 1)$ is given in [17, Section 18]. In this space $L^2(-1, 1)$, the domains of all self-adjoint operators with *separated* boundary conditions applied at the endpoints ±1 are found by applying a well-determined number of boundary conditions to elements of the appropriate maximal domain of the form

$$\{f, w_-\}(-1) = 0, \quad \{f, w_+\}(1) = 0; \tag{1.25}$$

here w_- and w_+ are chosen in a prescribed way from the maximal domain. For details of this method, see [17, Section 18], and for the application in the cases of (1.18) see, respectively, [14, Sections 5.6, 6.6, 4.5].

In this paper we are concerned with the domains $D(TL)$, $D(TR)$ and $D(T)$ of the self-adjoint differential operators here denoted by TL, TR, and T in the Hilbert spaces $L^2([-1, 1]; \mu)$ with measure μ determined by cases (ii), (iii), and (iv), respectively, of (1.4) such that

(i) the spectrums of TL, TR, and T are discrete with eigenvalues $\{\lambda L_n : n \in \mathbb{N}_0\}$, $\{\lambda R_n : n \in \mathbb{N}_0\}$, and $\{\lambda_n : n \in \mathbb{N}_0\}$ (see (1.12), (1.13), (1.14), respectively);
(ii) the corresponding eigenvectors of TL, TR, and T are $\{P_n^{0,0,M,0} : n \in \mathbb{N}_0\}$, $\{P_n^{0,0,0,N} : n \in \mathbb{N}_0\}$, and $\{P_n^{0,0,M,M} : n \in \mathbb{N}_0\}$.

The operator T was first defined in [4] and [3], and later considered in [14, Chapter IV]. The operators TL and TR were first defined in [14, Chapters V and VI]. From these works and [6, p. 14], we make the following operator domain definitions, noting that the three functions 1, $1-x$, and $1+x$, ($x \in (-1, 1)$) all belong to the maximal domains ΔL and ΔR.

$$\text{(i)}\quad \begin{aligned} D(TL) &:= \{f \in \Delta L : L[f, 1](+1) = 0,\ L[f, 1-x](+1) = 0\} \\ D(TR) &:= \{f \in \Delta R : R[f, 1](-1) = 0,\ R[f, 1+x](-1) = 0\} \\ D(T) &:= \Delta \end{aligned} \tag{1.26}$$

$$\text{(ii)}\quad \begin{aligned} (TLf)(x) &:= \begin{cases} -8Af'(-1) + f(-1) & x = -1 \\ ML[f](x) & \text{almost all } x \in (-1, 1] \end{cases} \\ (TRf)(x) &:= \begin{cases} MR[f](x) & \text{almost all } x \in [-1, 1) \\ 8Bf'(1) + f(1) & x = 1 \end{cases} \\ (Tf)(x) &:= \begin{cases} -8Af'(-1) + f(-1) & x = -1 \\ M[f](x) & \text{almost all } x \in (-1, 1) \\ 8Af'(1) + f(1) & x = 1. \end{cases} \end{aligned} \tag{1.27}$$

These spectral properties of T as an operator in the Hilbert space $L^2([-1, 1]; \mu)$, with μ given by case (iv) of (1.4), were developed in [4] and [3], and then reported in [6, Chapter IV]. The corresponding results for TL and TR are discussed in [6, Chapters V and VI].

The definition of all three operators requires information on the elements of operator domains at the singular endpoints ± 1 of the interval $(-1, 1)$. The results in the following theorem justify the explicit definition of the operators in (1.27).

Theorem 1.1. *Let the Lagrange symmetric differential expressions* $ML[\cdot]$, $MR[\cdot]$, *and* $M[\cdot]$ *be defined by* (1.8), (1.9), (1.10) *and* (1.18); *let the maximal domains* ΔL, ΔR, *and* Δ *be defined within the Hilbert space* $L^2(-1, 1)$ *by* (1.23); *Let the operator domains* $D(TL)$, $D(TR)$ *and* $D(T)$ *be defined, within the designated Hilbert space* $L^2([-1, 1]; \mu)$, *by* (1.26). *Then the following properties hold:*

(i) *if* $f \in \Delta L$, *then* $f'' \in L^2(-1, 0]$, *and* $f, f' \in AC_{\text{loc}}[-1, 1)$ *and* $f(x) = O(|\ln(1 - x)|)$ $(x \to +1)$; *if* $f \in \Delta R$, *then* $f'' \in L^2[0, 1)$ *and* $f, f' \in AC_{\text{loc}}(-1, 1]$ *and* $f(x) = O(|\ln(1 + x)|)$ $(x \to -1)$; *if* $f \in \Delta$, *then* $f'' \in L^2(-1, 1)$ *and* $f, f' \in AC[-1, 1]$;

(ii) *if* $f \in D(TL)$ *or* $D(TR)$ *or* $D(T)$, *then* $f'' \in L^2(-1, 1)$ *and* $f, f' \in AC[-1, 1]$.

The results stated in (i) *are best possible in the following sense:*

(i)* *there exists* $g \in \Delta L$ *such that* $g(x) \sim \ln(1 - x)$ $(x \to +1)$;
there exists $g \in \Delta R$ *such that* $g(x) \sim \ln(1 + x)$ $(x \to -1)$;

(ii)* *for each domain* $D(TR)$, $D(TL)$, *and* $D(T)$, *there exists an element* g *such that* $g'' \notin L^p(-1, 1)$ *for any index* $p > 2$; *here each* g *is independent of* p.

Proof. The proof of statements (i) and (ii) for Δ and $D(T)$ is given in [4] and [3]. The proof of statements (i) and (ii) for ΔL, $D(TL)$, ΔR, and $D(TR)$ is given in the following sections of this paper. The proofs of (i)* and (ii)* are discussed below.

Remarks. 1. Even though the differential expression $M[\cdot]$ has singularities at the endpoints ± 1, in that the leading coefficient $x \mapsto (1 - x^2)^2$ has zeros of order 2 at both ± 1, nevertheless all functions in the maximal domain Δ have continuous first derivatives on $[-1, 1]$; this property does not extend to the expressions $ML[\cdot]$ and $MR[\cdot]$ since both maximal domains ΔL and ΔR have elements which are unbounded near either $+1$ or -1.

2. In comparison with the previous remark, the uniform properties of the operator domains $D(TL)$, $D(TR)$ and $D(T)$ are remarkable; the three domains have the property that all elements have continuous first derivatives on $[-1, 1]$.

3. The smoothness results of all elements of the operator domains $D(TL)$, $D(TR)$ and $D(T)$ justify the form of the definitions of the operators TL, TR and T given in (1.27).

4. The proof of Theorem 1.1 for the operator T is essentially given in the earlier papers of Everitt, A.M. Krall, and Littlejohn, [3] and [4]. However these results are included in the statement of Theorem 1.1 for completeness and also to show the remarkable uniformity in properties of the three operator domains given in (ii) of the

theorem. Also the previous results obtained for the operator T can be used to shorten the proof of Theorem 1.1 for the operators TL and TR.

The contents of the paper are as follows. Section two contains the statement of a technical lemma (due to Chisholm and Everitt [2]); the result of this lemma is essential to the proof of Theorem 1.1. Section three reviews the form of proof of Theorem 1.1. Section four covers the connection between the operators TL and TR, and their joint connection with the operator T. Section five contains the proof of (i) and (i)*, and Section six contains the proof of (ii) and (ii)* of the theorem.

2. A boundedness result in $L^2(-1,1)$. The following result is essential for our proof of Theorem 1.1. The proof of Theorem 2.1 may be found in [2, Section 2].

Theorem 2.1. (Chisholm-Everitt) *Let $[a,b]$ be a compact interval of $\mathbb{R}$ and suppose $\lambda,\ \nu : [a,b] \to \mathbb{C}$ satisfy*

$$\lambda \in L^2_{\mathrm{loc}}[a,b), \qquad \nu \in L^2_{\mathrm{loc}}(a,b].$$

Define the two operators $A,\ B : L^2(a,b) \to L^2_{\mathrm{loc}}(a,b)$ by

$$(Af)(x) := \nu(x)\int_a^x \lambda(t)f(t)\,dt, \quad (x \in (a,b))$$
$$(Bf)(x) := \lambda(x)\int_x^b \nu(t)f(t)\,dt, \quad (x \in (a,b)),$$

for all $f \in L^2(a,b)$. Then a necessary and sufficient condition for both A and B to map $L^2(a,b)$ into $L^2(a,b)$ is that there exists a positive number K such that

$$\int_a^x |\lambda(t)|^2\,dt \int_x^b |\nu(t)|^2\,dt \le K, \quad (x \in (a,b)).$$

3. Preliminaries. We remind the reader that details of the spectral analysis of the differential operators generated by the fourth-order differential expressions $MR[\cdot]$, $ML[\cdot]$ and $M[\cdot]$, in both the spaces $L^2(-1,1)$ and $L^2([-1,1];\mu)$, can be found in [3], [4] and the Loveland thesis [14, Chapters IV, V and VI]. The objective here is to give consideration to the proof of Theorem 1.1 as given in Section one above.

As pointed out in Section one, the results for the domains Δ and T stated in Theorem 1.1 are proved in [3] and [4].

In considering the proof of Theorem 1.1 for the domains ΔL, $D(TL)$ and ΔR, $D(TR)$, it is sufficient to prove the results for the L case, say. This follows from the similarity between the differential expressions $ML[\cdot]$ and $MR[\cdot]$; i.e., if x is replaced by $-x$ on $[-1,1]$ then, formally, $ML[\cdot]$ is mapped into $MR[\cdot]$, and vice versa. Thus in the following sections we give the proof of Theorem 1.1 for $ML[\cdot]$ only, i.e., for the domains ΔL and $D(TL)$.

4. The endpoint -1 for the differential expression $ML[\cdot]$. We note that the analytic properties of $ML[\cdot]$ at -1 are entirely similar to the properties of $M[\cdot]$ at the endpoint -1. Analytically, this similarity between these differential expressions at -1 is best seen in the properties of the differential equations, both of the fourth order:

$$ML[y] = 0 \quad \text{and } M[y] = 0 \quad \text{on } (-1, 1).$$

Both these equations have the same form of polynomial coefficients at the endpoint -1. This endpoint is a regular singularity of both equations; the Frobenius solutions have the same asymptotic behavior near this endpoint.

For details of these Frobenius solutions for $M[y] = 0$ at -1, see [4, Section 4, (4.2)–(4.5)], and in a more definitive form [14, Section 4.2, (4.2.2)–(4.2.5)]. For details of the Frobenius solutions of $ML[y] = 0$ at -1 see [14, Section 5.2, (5.2.12)–(5.2.15)]. The comparison confirms the analytical identification of the properties of the maximal domains Δ and ΔL, and the operator domains $D(T)$, $D(TL)$, in the neighborhood of this endpoint -1.

There is one further analytical similarity between these two differential expressions. In the proof of the required properties of the elements of $\Delta = D(T)$, see [4, Section 2, (2.11)] and [3, Section 2, (2.3)], use is made of an "imbedded" second-order linear differential equation. This technical device also extends to the proof of the properties of the domains ΔL and $D(TL)$ and the "imbedded" equations for $M[\cdot]$ and $ML[\cdot]$ both have a regular singularity at the endpoint -1, the same form of polynomial coefficients, and same form of Frobenius solutions.

Thus the previously obtained results for the differential expression $M[\cdot]$ imply the following results for $ML[\cdot]$ in the neighborhood of -1:

$$\begin{aligned} &\text{(i)} \qquad \text{if } f \in \Delta L \text{ then } f'' \in L^2(-1, 0] \text{ and } f, f' \in AC[-1, 0], \\ &\text{(ii)} \qquad \text{if } f \in D(TL) \text{ then } f'' \in L^2(-1, 0] \text{ and } f, f' \in AC[-1, 0]. \end{aligned} \tag{4.1}$$

Consequently, in proving Theorem 1.1 for ΔL and $D(TL)$, it suffices to show that the results required are valid in the neighborhood of the endpoint $+1$.

5. The maximal domain ΔL at endpoint $+1$. It suffices in this section to confine attention to the proof of the result

$$f \in \Delta L \quad \text{implies } f(x) = O(|\ln(1-x)|) \quad (x \to +1). \tag{5.1}$$

Without loss of generality, we can assume

(i) f to be real-valued on $(-1, 1)$;
(ii) f to be identically zero in the interval $[-1, \frac{1}{2}]$ by using the fundamental result in Naimark [17, Section 17.3, Lemma 2].

To summarize, we take $f \in \Delta L$ with the properties

$$f : (-1, 1) \to \mathbb{R} \qquad f(x) = 0 \qquad (x \in [-1, \tfrac{1}{2}]). \tag{5.2}$$

With $ML[\cdot]$ given by (1.18), integrate over $[0, x]$ with $x \in (\frac{1}{2}, 1)$ to obtain, and *hereby defining the mapping* $\Lambda : [0, 1] \times \Delta L \to \mathbb{R}$

$$\Lambda(x; f) := -((1-x^2)^2 f''(x))' + (1-x)p(x)f'(x) \tag{5.3}$$

$$= -\int_0^x \left\{ ML[f](t) - kf(t) \right\} dt, \tag{5.4}$$

where for convenience we have put, see (1.22),

$$p(x) := 2((2A+1)x+2A+3) \qquad (x \in [-1,1]), \tag{5.5}$$

and where $A = M^{-1} > 0$ is the parameter in the definition of the measure μ, in case (ii) of (1.4). We use either definition (5.3) or the equivalent equation (5.4) in our use of Λ. We note from (5.4) that

$$\Lambda'(\cdot; f) \in L^2[0,1] \quad \text{and} \quad \Lambda(\cdot; f) \in AC[0,1]. \tag{5.6}$$

Define the second-order, Lagrange symmetric differential expression $P[\cdot]$ by

$$P[g](x) := -((1-x^2)^2 g'(x))' + (1-x)p(x)g(x) \qquad (x \in [0,1]), \tag{5.7}$$

where $g : [0,1] \to \mathbb{R}$ and $g, g' \in AC_{\mathrm{loc}}[0,1)$. Now rewrite the definition (5.3) of Λ in the form

$$P[f'](x) = \Lambda(x; f) \qquad (x \in [0,1)). \tag{5.8}$$

This suggests we study the differential equation (the "imbedded" equation)

$$P[y](x) = \Lambda(x; f) \qquad (x \in [0,1)) \tag{5.9}$$

which requires consideration of the homogeneous equation

$$P[y](x) \equiv -((1-x^2)^2 y'(x))' + (1-x)p(x)y(x) = 0 \qquad (x \in [0,1)). \tag{5.10}$$

The Frobenius analysis of (5.10) for the regular singular endpoint at $+1$ gives indicial roots of 0 and -1. Thus we have one solution of the form

$$\varphi(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \qquad a_0 \neq 0 \tag{5.11}$$

with convergence for $|x-1| < 2$. Clearly for this solution φ, there exists $\xi \in [0,1)$ such that $\varphi(x) \neq 0$ for all $x \in [\xi, 1)$.

A second, independent solution of (5.10) is then given by

$$\psi(x) = \varphi(x) \int_{\xi}^{x} \frac{dt}{(1-t^2)^2 \varphi^2(t)} \qquad (x \in [\xi,1)) \tag{5.12}$$

and it then follows that the Wronskian $W(\varphi, \psi)$ satisfies

$$W(\varphi,\psi)(x) = (1-x^2)^2(\varphi(x)\psi'(x) - \varphi'(x)\psi(x)) = 1 \qquad (x \in [\xi,1)).$$

The asymptotic form of these solutions in the neighborhood of 1 can be shown to be, as $x \to 1$,

$$\varphi(x) = a_0 + O(|x-1|), \qquad \varphi'(x) = a_1 + O(|x-1|) \tag{5.13}$$

$$\begin{aligned} \psi(x) &= b_0(x-1)^{-1} + O(|\ln(1-x)|) \\ \psi'(x) &= -b_0(x-1)^{-2} + O(|x-1|^{-1}), \end{aligned} \tag{5.14}$$

where we note that $a_0 \neq 0$ and $b_0 \neq 0$.

Now define $\Psi : [\xi, 1) \times \Delta L \to \mathbb{R}$ by

$$\Psi(x; f) := \varphi(x) \int_{\xi}^{x} \psi(t)\Lambda(t; f)\, dt + \psi(x) \int_{x}^{1} \varphi(t)\Lambda(t; f)\, dt \tag{5.15}$$

using (5.6) and (5.13) to validate the definition.

By direct differentiation we obtain

$$P[\Psi(\cdot; f)](x) = \Lambda(x; f) \qquad (x \in [\xi, 1)) \tag{5.16}$$

and hence the general solution of (5.9) is of the form, with $\alpha, \beta \in \mathbb{R}$,

$$y(x) = \alpha\varphi(x) + \beta\psi(x) + \Psi(x; f) \qquad (x \in [\xi, 1)).$$

Returning to (5.8) we obtain the representation, for some unique $\alpha, \beta \in \mathbb{R}$, and functional identity

$$f'(x) = \alpha\varphi(x) + \beta\psi(x) + \Psi(x; f) \qquad (x \in [\xi, 1)). \tag{5.17}$$

From (5.14), (5.15) and use of (5.6), we obtain (here the symbol K represents a positive number, but not necessarily the same number from use to use, or line to line)

$$\begin{aligned} |\Psi(x; f)| &\leq K \int_{\xi}^{x} (1-t)^{-1}|\Lambda(t; f)|dt + K(1-x)^{-1} \int_{x}^{1} |\Lambda(t; f)|dt \\ &\leq K \int_{\xi}^{x} (1-t)^{-1}dt + K(1-x)^{-1} \int_{x}^{1} 1dt \leq K|\ln(1-x)| \quad (x \to 1). \end{aligned} \tag{5.18}$$

With these estimates used in (5.17) we obtain

$$|f'(x)| \leq K(1-x)^{-1} \qquad (x \in [\xi, 1)) \tag{5.19}$$

and on integrating

$$|f(x)| \leq K|\ln(1-x)|, \quad \text{i.e., } f(x) = O(|\ln(1-x)|) \qquad (x \to 1). \tag{5.20}$$

This completes the proof of (5.1) and (i) of Theorem 1.1 for ΔL.

To show that this result is best possible we appeal to the detailed Frobenius analysis of the solutions of the homogeneous differential equation $ML[y] = 0$ on $[0, 1)$ given in [14, Section 5, (5.2.8) to (5.2.11)]. It can be shown that this equation has a solution φ_0 with a series representation

$$\varphi_0(x) = \ln(1-x) \sum_{n=0}^{\infty} d_n(x-1)^n + \sum_{n=0}^{\infty} e_n(x-1)^n \quad \text{with } d_0 \neq 0,\, e_n \neq 0$$

valid for the interval $(-1, 1)$. Since $\varphi_0 \in L^2[0, 1)$ it follows that $\varphi_0 \in \Delta$ at least on $[0, 1)$; however

$$\varphi_0(x) \sim d_0 \ln(1-x) \qquad (x \to 1)$$

and this implies that (5.20) is best possible in general. This completes the proof of (i)* of Theorem 1.1 for ΔL.

6. The operator domain $D(TL)$ at endpoint $+1$. To complete the proof of (ii) of Theorem 1.1 for $ML[\cdot]$, we need to prove

$$f \in D(TL) \implies f'' \in L^2[0, 1) \text{ and } f, f' \in AC[0, 1]. \tag{6.1}$$

The sesquilinear form associated with the symmetric differential expression $ML[\cdot]$ is given by, see (1.21) above, for $f, g \in \Delta L$ and $x \in (-1, 1)$,

$$\begin{aligned} L[f, g](x) = {} & [((1-x^2)^2 f''(x))' - (1-x)p(x)f'(x)]\bar{g}(x) - [((1-x^2)^2\bar{g}''(x))' \\ & - (1-x)p(x)\bar{g}'(x)]f(x) - (1-x^2)^2(f''(x)\bar{g}'(x) - f'(x)\bar{g}''(x)). \end{aligned} \tag{6.2}$$

As before, see (1.24), we note

$$L[f, g](1) := \lim_{x\to 1} L[f, g](x) \ \textit{exists and is finite for all } f, g \in \Delta L. \tag{6.3}$$

A computation shows that both functions

$$x :\mapsto 1 \quad \text{and} \quad x \mapsto (1-x) \qquad (x \in [-1, 1]), \quad \in \Delta L. \tag{6.4}$$

We recall that, see (1.26)

$$D(TL) := \{f \in \Delta L : L[f, 1](1) = L[f, (1-x)](1) = 0\} \tag{6.5}$$

using the results (6.3) and (6.4).

If now $f \in \Delta L$ then from (6.2), (6.3) and (6.4) and a direct computation, also recalling the definition (5.3) and (5.6) of Λ, we obtain the connections

$$\begin{aligned} \Lambda(1; f) &= \lim_{x\to 1} \Lambda(x; f) \\ &= \lim_{x\to 1^-} (-((1-x^2)^2 f''(x))' + (1-x)p(x)f'(x)) = -L[f, 1](1). \end{aligned} \tag{6.6}$$

In the same way we obtain, using the result that $L[f, 1]$ is finite

$$\lim_{x\to 1} (-(1-x)p(x)f(x) + (1-x^2)^2 f''(x)) = L[f, (1-x)](1). \tag{6.7}$$

Using the estimates (5.20) for f, this last result (6.7) simplifies to

$$\lim_{x\to 1} (1-x^2)^2 f''(x) = L[f, (1-x)](1). \tag{6.8}$$

From (5.3) we obtain, for $x \in [0, 1)$,

$$\begin{aligned}|((1-x^2)^2 f''(x))'| &\le |\Lambda(x; f)| + |p(x)(1-x)f'(x)| \\ &\le \sup\{|\Lambda(x; f)| : x \in [0, 1]\} + \sup\{|p(x)(1-x)f'(x)| : x \in [0, 1)\} \\ &\le K_1 + K_2 \quad \text{(say)},\end{aligned} \tag{6.9}$$

where K_1 is finite from (5.6) or (6.6), and K_2 is finite from (5.5) and (5.19). Thus

$$((1-x^2)^2 f''(x))' \in L[0, 1]. \tag{6.10}$$

Now suppose that f satisfies the boundary condition, see (1.26),

$$L[f, (1-x)](1) = 0. \tag{6.11}$$

From (5.20) it follows that $\lim_{x\to 1} p(x)(1-x)f(x) = 0$ and then from (6.7) we obtain

$$\lim_{x\to 1}(1-x^2)^2 f''(x) = 0. \tag{6.12}$$

From (6.10) and (6.12) we deduce

$$(1-x^2)^2 f''(x) = -\int_x^1 ((1-t^2)^2 f''(t))' dt \qquad (x \in [0, 1))$$

which yields, using (6.9),

$$|(1-x^2)^2 f''(x)| \le \int_x^1 |((1-t^2)^2 f''(t))'| \, dt \le K(1-x) \quad (x \in [0, 1)).$$

This gives in turn

$$|f''(x)| \le K(1-x)^{-1} \qquad (x \in [0, 1))$$

and on integrating over $[0, 1)$

$$|f'(x)| \le K|\ln(1-x)| \qquad (x \in [0, 1)) \tag{6.13}$$

which should be compared with the earlier estimate, obtained without the condition (6.11), of (5.19). From (6.13) then we obtain

$$f \in \Delta L \ \text{ and } L[f, (1-x)](1) = 0 \ \text{ imply } f' \in L^2[0, 1). \tag{6.14}$$

Returning now to the representation (5.17) for f' we note that *if* (6.11) *holds* then $f' \in L^2[0, 1)$; clearly $\varphi \in L^2[0, 1)$; from the estimate (5.18) we obtain $\Psi(\cdot; f) \in$

$L^2[0, 1)$; from the asymptotic form (5.14) we see that $\psi \notin L^2[0, 1)$. For (5.17) to be consistent it follows then that $\beta = 0$ and we have the result

$$f \in \Delta L \text{ and } L[f, (1-x)](1) = 0 \text{ imply } f'(x) = \alpha\varphi(x) + \Psi(x; f) \quad (x \in [\xi, 1)). \tag{6.15}$$

The final stage in the proof is to suppose that the second boundary condition to determine $D(TL)$ is satisfied, i.e., from (1.26),

$$L[f, 1](1) = 0, \text{ i.e., } \Lambda(1; f) = 0, \tag{6.16}$$

from (6.6). From (5.6) and (6.16) we obtain

$$\Lambda(x; f) = -\int_x^1 \Lambda'(t; f)\, dt \qquad (x \in [0, 1)). \tag{6.17}$$

Returning to (6.15) and differentiating we find

$$f''(x) = \alpha\varphi'(x) + \Psi'(x; f) \qquad (x \in [\xi, 1)) \tag{6.18}$$

and

$$\Psi'(x; f) = \varphi'(x) \int_\xi^x \psi(t)\Lambda(t; f)\, dt + \psi'(x) \int_x^1 \varphi(t)\Lambda(t; f)\, dt. \tag{6.19}$$

The first term on the right-hand side of (6.19) is, from (5.6), (5.14), and (5.15), of the order $O(|\ln(1-x)|)$ $(x \to 1^-)$, and hence is in $L^2[\xi, 1)$. For the second term write, from (6.17),

$$\begin{aligned}\psi'(x) \int_x^1 \varphi(t)\Lambda(t; f)\, dt &= -\psi'(x) \int_x^1 \varphi(t)\Big(\int_t^1 \Lambda'(s; f)\, ds\Big)\, dt \\ &= -\psi'(x) \int_x^1 (t-1)\varphi(t)\Big(\frac{1}{t-1} \int_t^1 \Lambda'(s; f)\, ds\Big)\, dt;\end{aligned}$$

i.e.,

$$\Big|\psi'(x) \int_x^1 \varphi(t)\Lambda(t; f)\, dt\Big| \le \frac{K}{|x-1|^2} \int_x^1 |t-1| \Big\{\frac{1}{|t-1|} \int_t^1 |\Lambda'(s; f)|\, ds\Big\}\, dt \tag{6.20}$$

for all $x \in [\xi, 1)$.

We now make critical applications of Theorem 2.1. To the right-hand side of this last result we apply this theorem with

$$a = \xi, \quad b = 1, \quad \lambda(s) = (1-x)^{-1}, \quad \nu(x) = 1 \qquad (s \in [\xi, 1))$$

for which we have

$$\int_\xi^t |\lambda(s)|^2 ds \int_t^1 |\nu(s)|^2 ds \le K \qquad (t \in [\xi, 1)).$$

Since $\Lambda'(\cdot, f) \in L^2(\xi, 1)$, from (5.6), Theorem 2.1 gives

$$t \mapsto (1-t)^{-1} \int_t^1 |\Lambda'(s; f)|\, dx \qquad (t \in [\xi, 1), \quad \in L^2(\xi, 1). \tag{6.21}$$

This is followed by a second application of Theorem 2.1 with

$$a = \xi, \quad b = 1, \quad \lambda(t) = (1-t)^{-2}, \quad \nu(t) = 1 - t \quad (t \in [\xi, 1))$$

for which we have

$$\int_\xi^x |\lambda(t)|^2\, dx \int_x^1 |\nu(t)|^2\, dt \le K \quad (x \in [\xi, 1)).$$

This, together with (6.21), when applied to (6.20), shows that

$$x \mapsto \psi'(x) \int_x^1 \varphi(t)\Lambda(t; f)\, dt \quad (x \in [\xi, 1)), \quad \in L^2(\xi, 1).$$

Thus, returning to (6.19), it follows that $\Psi'(\cdot; f) \in L^2(\xi, 1)$ and hence to $L^2(0, 1)$. Finally, from (6.18), it follows that when both boundary conditions (6.11) and (6.16) are satisfied by $f \in \Delta L$, then this implies that $f'' \in L^2(0, 1)$. Thus provided we define $f^{(r)}(+1) = \lim_{x \to 1} f^{(r)}(x)$, for $r = 0, 1$, we obtain $f, f' \in AC[0, 1]$. This completes the proof of (ii) of Theorem 1.1 for $D(TL)$.

To show that this result is best possible we define a function $g : [-1, 1] \to \mathbb{R}$ by putting

$$g''(x) := ((1-x)^{1/2} \ln(1-x))^{-1} \quad (x \in [\tfrac{1}{2}, 1))$$

and then completing the definition on $[-1, \frac{1}{2}]$ by polynomial extension so that $g'' \in C^{(2)}[-1, 1)$. The function g itself is then defined by

$$g(x) := \int_0^x (x - t) g''(t)\, dt \quad (x \in [-1, 1]).$$

A computation shows that $g \in \Delta L$ and that g satisfies both boundary conditions (6.11) and (6.16). Hence $g \in D(T)$, $g'' \in L^2[-1, 1)$ and yet $g'' \notin L^p[-1, 1]$ for any $p > 2$. This completes the proof of (ii)* of Theorem 1.1 for $D(TL)$.

Taken together with all previous results and remarks, this completes the proof of Theorem 1.1.

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Norrie Everitt and Lance Littlejohn take this opportunity, with the agreement of Susan Loveland, to dedicate this paper to the memory of their friend and colleague, Peter Hess.

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