Properties of the Steiner Triple Systems of Order 19

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The research of P. R. J. Ö. was supported in part by the Academy of Finland.

Steiner Triple Systems

Definition

A **Steiner triple system** of order v, denoted by STS(v), is a pair $(X, \mathcal{B})$, where X is a finite set of v *points* and $\mathcal{B}$ is a collection of 3-subsets of points, called *blocks*, with the property that every 2-subset of points occurs in exactly one block.

Theorem

An $\mathrm{STS}(v)$ *exists iff* $v \equiv 1, 3 \pmod 6$.

Example

1	0	0	0	1	0	1
1	1	0	0	0	1	0
0	1	1	0	0	0	1
1	0	1	1	0	0	0
0	1	0	1	1	0	0
0	0	1	0	1	1	0
0	0	0	1	0	1	1

STS(7)

Classification of Steiner Triple Systems

Two STS(v) are **isomorphic** if there is a bijection between their point sets that maps blocks onto blocks. The number of isomorphism classes of STS(v) is denoted by $N(v)$.

v	$N(v)$
3	1
7	1
9	1
13	2
15	80
19	11 084 874 829

[KO] P. Kaski and P. R. J. Ö., The Steiner triple systems of order 19, *Math. Comp.* 73 (2004), 2075–2092.

Compressing Steiner Triple Systems

- The Steiner triple systems of order 19 have been compressed into about 39 gigabytes, or little more than 28 bits per system.
- Decompression of the entire collection of objects takes approximately 15 hours on a Linux PC with a 2-GHz CPU (5μs per isomorphism class), which is about 0.5% of the time required by a classification from scratch with our current implementation.
- If you want your own set of data, please contact `olli.pottonen@tkk.fi`

[KOPK] P. Kaski, P. R. J. Ö., O. Pottonen, and L. Kiviluoto, A catalogue of the Steiner triple systems of order 19, *Bull. Inst. Combin. Appl.*, to appear.

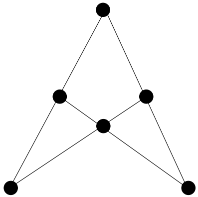
Properties of STSs

Mathon, Phelps and Rosa [MPR] recorded properties of STS(v) for $v \leq 15$. In this work we have tabulated data related to

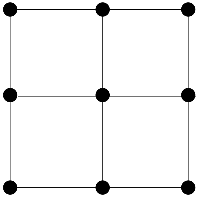
- *automorphism groups*,
- *subsystems and ranks*,
- small configurations,
- cycle structure and uniformity,
- almost parallel classes,
- the chromatic number,
- the chromatic index, and
- existential closure.

[MPR] R. A. Mathon, K. T. Phelps, and A. Rosa, Small Steiner triple systems and their properties, *Ars Combin.* 15 (1983), 3–110; and 16 (1983), 286.

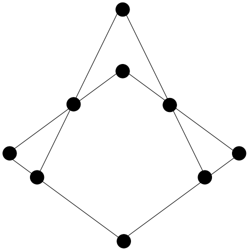
Small Configurations



Pasch

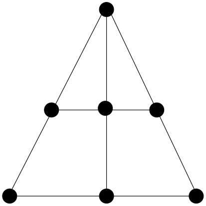


Grid

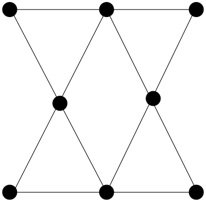


Prism

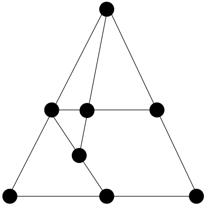
Small Configurations (cont.)



Mitre



Hexagon



Crown

Small Configurations: Results

There exists no STS(19) missing a grid and no STS(19) missing a prism. There are exactly four nonisomorphic anti-mitre STS(19). Moreover, there is a unique STS(19) with no hexagon and exactly four with no crown.

Theorem

The number of 4*-sparse,* 5*-sparse and* 6*-sparse* $\mathrm{STS}(19)$ *is* 2 591, 1*, and* 0*, respectively.*

Almost Parallel Classes

A partial parallel class with $\lfloor v/k \rfloor$ blocks is an **almost parallel class**.

APC	#	APC	#	APC	#	APC	#
0	2	79	764 738	110	526 902 725	141	43 290
36	1	80	1 224 282	111	495 595 995	142	25 609
40	1	81	1 924 007	112	458 547 878	143	14 838
48	5	82	2 974 055	113	417 254 801	144	8 604
50	1	83	4 513 033	114	373 408 256	145	4 827
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

[L] G. Lo Faro, On the size of partial parallel classes in Steiner systems STS(19) and STS(27), *Discrete Math.* 45 (1983), 307–312.

Chromatic Number

The **chromatic number** of an STS is the smallest number of colors needed to color the points so that there are no monochromatic blocks.

Theorem

Every $\mathrm{STS}(19)$ *is 3-chromatic and has 3-colorings with color class sizes*

1. $(7, 7, 5)$ *and*
2. $(8, 6, 5)$, *with two exceptions.*

Moreover, the combinations of 3-coloring patterns that can occur in an STS(19) were determined.

Chromatic Index

The **chromatic index** of an STS is the smallest number of colors needed to color the blocks so that no two intersecting blocks get the same color.

$\Rightarrow$ We have a partial parallel class for each color.

Theorem

The number of $\mathrm{STS}(19)$ *with chromatic index* 10, 11, *and* 12 *are* 11 084 870 752, 4 075 *and* 2, *respectively.*

Chromatic Index: Algorithms

Since most of the STS(19) have chromatic index $\lceil 57/6 \rceil = 10$, any algorithm that finds such colorings fast could be utilized in an initial step. Good (?) test cases for heuristic coloring algorithms!

Our approach:

1. Any STS(19) with chromatic index 10 must have at least 7 disjoint almost parallel classes ($6 \cdot 6 + 4 \cdot 5 = 56 < 57$).
2. For solutions of the first step, search for the final three parallel classes.
3. A slower brute-force algorithm is used for those instances that turn out to have chromatic index 11 in steps 1 and 2.

Computational Resources Needed

The most CPU intensive computations were those of determining

- small subconfigurations (10 CPU weeks),
- 3-existentially closed designs (9 CPU months),
- almost parallel classes (1.5 CPU years), and
- chromatic indices (8 CPU years).

Properties not Considered

1. Properties that do not have a large general interest. For example, various kinds of colorings.
2. Properties that cannot be presented in a compact manner. For example, properties that have been used as invariants (trains,...).
3. Properties that we were not able to compute. For example, intersection numbers, maximal sets of disjoint STSs, and the question whether all STS are derived.