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PROPERTY L AND W-* ALGEBRAS OF TYPE I[1]

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Abstract. Tpye I W-* algebras do not have property L.

Let $\mathscr{A}$ be a W-* algebra acting in separable Hilbert space h and let $\mathscr{U}(\mathscr{A})$ denote the unitary operators in $\mathscr{A}$. Corollary I.5.10 of [3] states that $\mathscr{A}$ has direct integral decomposition into factors given by $\mathscr{A} = \int_{\Lambda} \oplus \mathscr{A}(\lambda)\mu(d\lambda)$. This paper assumes the reader is familiar with [4] and Chapter I of [3].

DEFINITION. $\mathscr{A}$ has *property* L if there is a sequence $\{U_n\}$ contained in $\mathscr{U}(\mathscr{A})$ such that $\{U_n\} \to 0$ weakly and such that $\{U_n A U_n^*\} \to A$ strongly for each $A \in \mathscr{A}$.

Property L is a partial form of commutivity that was introduced by Pukánszky in [2]. We shall use direct integral theory to show that no type I W-* algebra has property L.

We establish some notation before proving two essential lemmas. $\mathscr{A}'$ denotes the commutant of $\mathscr{A}$ and is also a W-* algebra. By the center of $\mathscr{A}$, we mean the abelian W-* algebra $\mathscr{Z}(\mathscr{A}) = \mathscr{A} \cap \mathscr{A}'$. $\mathscr{A}_1$ represents the unit ball of $\mathscr{A}$ and h_∞ denotes the underlying Hilbert space of h, i.e., $h = \int_{\Lambda} \oplus h_\infty \mu(d\lambda)$ (cf. [3] Definition I.2.4).

LEMMA 1. *Let* $\mathscr{A} = \int_{\Lambda} \oplus \mathscr{A}(\lambda)\mu(d\lambda)$ *be a* W-* *algebra acting in* h *and let* S *denote* $B(h_\infty)_1$ *taken with the strong-* operator topology. Then if* N *is a Borel subset of* Λ, *the set* $F = \{(\lambda, T) \mid \lambda \in N, T \in \mathscr{A}(\lambda) \cap S\}$ *is a Borel subset of* $\Lambda \times S$.

PROOF. By [3] Lemma I.4.11, S is a complete separable metric space. Let d denote the metric which defines the topology on S. By [4] Lemma 1.5(a, c), there is a countable sequence of disjoint closed subsets e_i of Λ such that if $e = \Lambda - \bigcup_{i=1}^{\infty} e_i$, then $\mu(e) = 0$ and there is

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a countable sequence of operators $\{A_n\}$ contained in $\mathscr{A}$ such that $\{A_n(\lambda)\}$ is strong-* dense in $\mathscr{A}(\lambda)_1$ μ-a.e., and each $A_n(\lambda)$ is strong-* continuous on each set e_i.

Define subsets $F(i, j, m)$ of $\Lambda \times S$ as sets of all pairs (λ, T) satisfying the following conditions:

a) $\lambda \in N \cap e_i$,

b) $d(T, A_m(\lambda)) \leqq 1/j$.

Condition (a) defines a Borel set. Condition (b) defines a closed set. Thus $F(i, j, m)$ is a Borel subset of $\Lambda \times S$ and so is $F = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{m=1}^{\infty} F(i, j, m)$. q.e.d.

LEMMA 2. *Let $\mathscr{A}$ be a type* I *W-* algebra acting in h. Let $\{V_n\}$ be a sequence contained in $\mathscr{U}(\mathscr{A})$ such that $\{V_n^* A V_n - A\} \to 0$ strongly for each $A \in \mathscr{A}$. Then $\{V_n\}$ has a subsequence that converges strongly to a unitary $V \in \mathscr{Z}(\mathscr{A})$.*

PROOF. Let $\mathscr{A} = \int_\Lambda \oplus \mathscr{A}(\lambda)\mu(d\lambda)$ be the direct integral decomposition of $\mathscr{A}$ into factors. Since $\mathscr{A}$ is of type I, $\mathscr{A}(\lambda)$ is a type I factor μ-a.e. For each $x \in h$ and $A \in \mathscr{A}$, $|(AV_n - V_nA)x| = |V_n(V_n^*AV_n - A)x| \leqq |V_n||(V_n^*AV_n - A)x| = |(V_n^*AV_n - A)x| \to 0$ $(n \to \infty)$. Thus $\{AV_n - V_nA\} \to 0$ strongly for each $A \in \mathscr{A}$. By weak compactness of $B(h)_1$, $\{V_n\}$ has a subsequence, again called $\{V_n\}$, that converges weakly to some operator V. Thus $|V| \leqq 1$. Since $\mathscr{A}$ is weakly closed, $V \in \mathscr{A}$ and we may write $V = \int_\Lambda \oplus V(\lambda)\mu(d\lambda)$ by [3] Lemma I.5.2. Also $|V| = \mu -$ ess. sup. $|V(\lambda)|$ by [3] Lemma I.3.1. We shall show that $|V(\lambda)| \geqq 1$ μ-a.e. so that $|V| \geqq 1$ also, and it follows that $|V| = 1$.

To prove our assertion we argue as follows. Let $\{x_i\}$ be an orthonormal basis for h_∞ such that $\{x_1\}$ is a basis for h_1, $\{x_1, x_2\}$ is a basis for h_2, etc., where $\{h_i\}$ is an increasing sequence of finite dimensional Hilbert spaces generating h_∞ (cf. [3] Definition I.2.4).

Let S, e and the e_i be as in Lemma 1 and define subsets $E(i)$ of $\Lambda \times S$ as sets of all pairs (λ, T) satisfying the following conditions:

a) $\lambda \in e_i$,

b) $T \in \mathscr{A}(\lambda) \cap S$,

c) $Tx_1 = x_1$, $Tx_j = 0$ for $j > 1$.

Condition (a) defines a closed set. By Lemma 1, conditions (a) and (b) define a Borel set. Condition (c) defines a closed set and shows that T is an operator belonging to S. Thus $E(i)$ is a Borel subset of $\Lambda \times S$ and so is $E = \bigcup_{i=1}^{\infty} E(i)$. By [3] Lemma I.4.3, E is analytic.

If Π is the projection of $\Lambda \times S$ onto Λ, then $F = \Pi(E)$ is contained in $\Lambda - e$ and by [3] Lemmas I.4.4 and I.4.6, F is analytic and μ-measurable. Since $\mathscr{A}(\lambda)$ is a type I factor μ-a.e., we know that $T \in \mathscr{A}(\lambda)$ μ-a.e., and it follows that F differs from Λ by a μ-null set. By [3] Lemma I.4.7, there exists a Borel subset F_1 of F with positive measure and a μ-measurable mapping g of F_1 into S such that $(\lambda, g(\lambda)) \in E$ for each $\lambda \in F_1$. Put $g(\lambda) = 0$ for $\lambda \notin F_1$ and define μ-measurable operator valued function $B(\lambda)$ by $B(\lambda) = g(\lambda)$. Then by [3] Definition I.2.5, we may write $B = \int_\Lambda \oplus B(\lambda)\mu(d\lambda)$ and $B \in \mathscr{A}$ by [3] Lemma I.5.2. By hypothesis, $\{V_n^* B V_n\}$ converges strongly and hence weakly to B.

Since $\mathscr{A}$ is decomposable, V_n is decomposable for each n and we may write $V_n = \int_\Lambda \oplus V_n(\lambda)\mu(d\lambda)$. By [4] Lemma 1.7, $([V_n(\lambda)^* B(\lambda) V_n(\lambda) - B(\lambda)]x, y) \to 0$ in μ-measure for each $x, y \in h_\infty$ and in particular for $x = y = x_1$. Since $\{V_n\} \to V$ weakly, the same reasoning shows that $(V_n(\lambda)x_1, x_1) \to (V(\lambda)x_1, x_1)$ in μ-measure. Since μ is a finite measure it follows that $|(V_n(\lambda)x_1, x_1)|^2 - 1 \to |(V(\lambda)x_1, x_1)|^2 - 1$ in μ-measure also (cf. [1] Section 3.20).

We have

$$\begin{aligned} &([V_n(\lambda)^* B(\lambda) V_n(\lambda) - B(\lambda)]x_1, x_1) \\ &\quad = (V_n(\lambda)^* B(\lambda) V_n(\lambda)x_1, x_1) - (B(\lambda)x_1, x_1) \\ &\quad = (B(\lambda) V_n(\lambda)x_1, V_n(\lambda)x_1) - (x_1, x_1) \\ &\quad = ((V_n(\lambda)x_1, x_1)x_1, V_n(\lambda)x_1) - 1 \\ &\quad = (V_n(\lambda)x_1, x_1)(x_1, V_n(\lambda)x_1) - 1 \\ &\quad = (V_n(\lambda)x_1, x_1)\overline{(V_n(\lambda)x_1, x_1)} - 1 \\ &\quad = |(V_n(\lambda)x_1, x_1)|^2 - 1 . \end{aligned}$$

That $B(\lambda) V_n(\lambda)x_1 = (V_n(\lambda)x_1, x_1)x_1$ can be obtained as follows. Let $V_n(\lambda)x_1 = \sum_{i=1}^\infty c_i(\lambda)x_i$. Then $B(\lambda) V_n(\lambda)x_1 = c_1(\lambda)x_1 = (V_n(\lambda)x_1, x_1)x_1$. Thus $|(V_n(\lambda)x_1, x_1)|^2 - 1 \to 0$ in μ-measure and it follows that $|(V(\lambda)x_1, x_1)| = 1$ μ-a.e. (cf. [1] Section 3.20 Theorem 3). Now $|(V(\lambda)x_1, x_1)| \leqq |V(\lambda)||x_1|^2 = |V(\lambda)|$ by the Schwarz inequality; thus $1 \leqq |V(\lambda)|$ μ-a.e. Then by the last sentence of the first paragraph of the present proof, we have $|V| = 1$.

We shall show next that $V \in \mathscr{Z}(\mathscr{A})$ and that V is unitary. Since strong convergence implies weak convergence, we know that $\{AV_n - V_nA\} \to 0$ weakly for each $A \in \mathscr{A}$ and since $\{V_n\} \to V$ weakly, it follows that $\{AV_n - V_nA\} \to AV - VA$ weakly for all $A \in \mathscr{A}$. Thus $AV - VA = 0$ or, equivalently, $V \in \mathscr{A}'$ so that $V \in \mathscr{Z}(\mathscr{A})$. By [3] Theorem I.5.9, V is a diagonal operator. Thus for μ-a.a. λ, $V(\lambda)$ is a bounded

Borel measurable scalar valued function by [3] Definition I.2.5. Then if we apply [3] Lemma I.3.1 to VV^*, we have $VV^* = \int_\Lambda \oplus V(\lambda)V(\lambda)^*\mu(d\lambda) = \int_\Lambda \oplus V(\lambda)\overline{V(\lambda)}\mu(d\lambda) = \int_\Lambda \oplus |V(\lambda)|^2 I\mu(d\lambda) = \int_\Lambda \oplus I\mu(d\lambda) = I$ and we can show $V^*V = I$ similarly.

Finally, the strong convergence of $\{V_n\}$ to V is an immediate consequence of the weak convergence, the identity $|(V_n - V)x|^2 = ([V_n - V]x, [V_n - V]x) = (V_n x, V_n x) - (Vx, V_n x) - (V_n x, Vx) + (Vx, Vx)$ and the fact that $(V_n x, V_n x) = (x, x) = (Vx, Vx)$. q.e.d.

THEOREM 3. *Type* I *W-* algebras do not have property L.*

PROOF. If $\{U_n\}$ is a sequence of unitaries demonstrating property L, then by putting $V_n = U_n^*$ and applying Lemma 2, we arrive at a contradiction. q.e.d.

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