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Prospective teachers' knowledge: Concept of division

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The purpose of this study was to determine prospective teachers' knowledge about the concept of division. One focus of interest was whether the prospective teachers were able to represent division of fractions. The participants were introduced to an alternative model for representation of fractions based on a rate or ratio model of division involving whole numbers. A second focus of interest was whether the prospective teachers would be able to apply this model to problems of division of fractions. The findings revealed that the prospective teachers' successfully represented division of whole numbers using models of fair sharing and, to a lesser extent, repeated subtraction. However, they had difficulty in successfully representing division of fractions. Some improvement was observed in participants' performance in attempts to represent division of fractions after introduction of the rate/ratio model. However the prospective teachers often used the rate or ratio model mistakenly where the situations were not appropriate for the model, which appeared to be associated with difficulty in multiplicative thinking.

Concept of division, multiplicative thinking, conceptual knowledge of mathematics, pedagogical content knowledge of mathematics

INTRODUCTION

Teachers' classroom practice and students' learning are highly affected by the knowledge which teachers possess (Haswesh, 1986; Shulman, 1987; Fennema & Franke, 1992; Dooren, Verschaffel & Onghena, 2002). Disappointingly, many mathematics teachers possess low levels of the content and pedagogical content knowledge required to teach the subject effectively (e.g. Tirosh, 2000; Ball, 1990; Brown, Cooney and Jones, 1990). The concept of division is one of the subject areas in mathematics where prospective and practising primary level teachers often seem to have insufficient knowledge. Though they are usually able to represent division problems involving whole numbers, they often cannot extend the representations to make sense of division of fractions such as $\frac{3}{4} \div \frac{1}{2}$ (Payne, 1976; Fischbein, Deri, Nello & Marino, 1985; Fendel, 1987; Ball, 1990; Tirosh, 2000; Flores, 2002; and Squire & Bryant, 2002). In the wake of this, division of fractions has been a concept isolated from the general concept of division in teachers' and students' representations of knowledge (Greer, 1992; Siebert, 2002). The concept is usually taught mechanically in elementary schools, with many teachers and students merely using an algorithm (like invert and multiply) and being unaware of the relationships between division algorithms and underlying concepts. Greer (1992) argued that though people were usually able to solve division sums involving fractions written mathematically, they were not able to use the concept to represent real life situations. He also suggested that for the teaching of concepts of multiplication and division beyond the domain of whole numbers there is a need to reorganise and

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reconceptualize the content of the curriculum to help students shift their thinking from additive to multiplicative thinking.

Use of models to teach the concept of division

One of the causes of the problem with division lies in the models which teachers use for teaching of division. Two models are commonly used. One model considers division as "fair sharing", and in this model a common representation of the expression 48 ÷ 4 might run as follows: *Share 48 lollies between 4 girls. How many lollies does each get?* (McSeveny, Conway & Wilke, 2000). The lollies are being divided or shared fairly among the girls. The fair sharing, or partitive model is a traditional teaching model for division of whole numbers, but it can act as a barrier in the representation of division of fractions. For example $48 \div \frac{1}{4}$ cannot be represented by the same model of fair sharing because it is senseless to share 48 lollies among a quarter of a girl.

Another model represents division as repeated subtraction. Using this model we would translate 48 ÷ 4 as: *How many times can 4 be subtracted from 48*? This model helps the learners to represent some division of fraction problems, but it also appears to be difficult for students to use this model to represent division situations when the divisor is bigger than the dividend. For example in representation of $\frac{1}{3} \div \frac{1}{2}$ it is confusing to ask how many times one half can be subtracted from one third, perplexing for the students of elementary classes who are not used to subtracting a bigger number from a smaller number. Many in the present generation of teachers seem to develop the understanding of the concept of division on the basis of either or both of the above models and find it difficult to extend their thinking about division beyond positive integers.

Connections between division and rate or ratio

Some prospective teachers are not able to connect the division of fractions with the concept of rate and ratio. They do not identify proportional relationships between the divisors and the dividends, and do not recognise the multiplicative relationship among the divisors, the dividends and the quotients that are important for representing the division of fractions effectively. That is why, in spite of solving division and multiplication problems successfully in middle school, many teachers and students still use additive reasoning (repeated addition or repeated subtraction) rather than multiplicative reasoning in which division is seen as a multiplicative comparison of two quantities, that is, a ratio (Flores 2002, p.243).

It is hypothesised that no matter what model of division learners use, to make progress with division of fraction problems they need to connect the concept of division with the concept of ratio/rate by understanding the multiplicative relationship between the divisor, the dividend and the quotient. Once this relationship is established, the learners should be able to make sense of the mathematical expressions involving division of fractions, to make sense of the algorithms used to solve the mathematical expressions involving division of fractions, and to translate real life situations into a form of division of fractions. This study is designed to provide information that can be used to examine these predictions.

Generally, in division and multiplication problems, multiplicative relationships between two sets of quantities are set up. In the information part of the problem statement, one instance of the relationship is given, and in the question part of the problem another instance of the same relationship is asked for. For example, 12 ÷ 4 can be interpreted as the value of the first quantity is 12 when the value of the second quantity is 4. This is the first instance of a relationship between the quantities that is given in the information part of the problem. To solve the problem means to establish another instance of the relationship where the value of the first quantity is to be found out if the value of the second quantity is 1. In each division question, a number has to be found out for “each one” or “every one”. For teaching purposes there is a need to establish a verbal analogy in a conditional, if – then, form:

If 12 is for 4	(Information part)
Then how many is for 1?	(Question part of the problem)

Vergnaud (1988) and Marshel, Barthuli, Brewer, and Rose, (1989) proposed schematic diagrams based on the ratio and rate concept of division for representation of different types of division problems. Table 1 shows how these problems can be represented using schematic diagrams. For instance in the first question in Table 1, one quantity is the number of children and other quantity is the number of oranges. So the name of each quantity is written in one of the columns of the schematic diagram. Then the first statement indicates that the value of one quantity is 12 when the value of the other quantity is 3. On the following line of the table a question in the second statement of the sum is generated, keeping the same multiplicative relationship between the quantities in the first line (as equal distribution is mentioned in the question). In the next step attention focuses on the first column, posing the question that if the value of one quantity changes from 3 to 1 (as it asks about each child), then what change will occur in the value of other quantity. That is why 1 is written under 3 and a question mark is put under 12.

Table 1: Schematic diagram of division representations

Division (sharing)	
Twelve oranges are being distributed equally among 3 children. How many oranges will each child receive?	
Children	Oranges
3	12
1	?
Division (repeated subtraction)	
Twelve oranges are being distributed equally among children. If each child receives four oranges, how many children are there?	
Children	Oranges
?	12
1	4

The rate or ratio model of division is based on establishing a multiplicative relationship between two similar quantities (ratio), or different quantities (rate). It considers the dividend and divisor as one pair of numbers from a set of infinite pairs of numbers that are related in the same proportion. If one of the quantities is unknown, knowing the multiplicative relationship between a pair of known quantities allows students to determine the unknown quantity.

Previous research has provided information about models of division and their limitations on representation of division problems (e.g. Bell, Swan & Taylor, 1981; Bell, Fischbein & Greer; 1984; Fischbein, Deri, Nello, & Marino, 1985). Tirosh (2000) reported that prospective teachers are often not aware of the difficulties caused by the models they use for representing division problems. It has been pointed out (Vergnaud, 1983) that the rate/ratio model is one of the models which can be used to understand and solve all types of division problems, especially in understanding and solving problems of division of fractions.

Sellke, Behr & Voelker (1991) showed that Year 7 students could improve performance using the Vergnaud system of representation. Use of the model with teachers has not been reported. We designed the current study to investigate the usefulness of the rate/ratio representation of division in a group of prospective primary level teachers who might be expected to show evidence of difficulty in understanding and representing situations involving the division of fractions. We were interested, first, to examine the prospective teachers’ understanding of division and of division in terms of rate/ratio. Then, if this group showed difficulty in representation of division, we were interested to see if their level of understanding could be improved through use of the Vergnaud approach. The following were the specific research questions:

- What procedural and conceptual understanding about the concept of division do prospective teachers possess?
- Do prospective teachers have the multiplicative understanding required to use the Vergnaud schematic diagram successfully?
- What changes occur in prospective teachers' understanding about the general concept of division in term of representing, solving and posing problems of division of fraction after being introduced with Vergnaud's schematic diagram based on the rate / ratio model of division?.

METHOD

Participants

Seventeen volunteer students from a cohort of primary/lower secondary Bachelor of Education (BEd) students at an Australian university participated in the study. There were three male participants. All the participants, except one who completed her schooling in Singapore, had their elementary and secondary education in Australia. For most of the students the study of mathematics had not been an area of strength in secondary school. All participants had completed compulsory mathematics courses in Year 11. Only four had also studied Applied Mathematics 1 in Year 11 and one had also studied Applied Mathematics 1 and 2 in Years 11 and 12 respectively. None of the students had studied Mathematics at university level. On a scale with a maximum of 7.0, the mean university GPA for the group was 5.24, with a standard deviation of 0.39. All participants had completed mathematics Curriculum Studies courses in their BEd.

Procedure

Phase 1 of the study addressed the first research question. Participants' existing knowledge of division was examined by asking them solve the problems on worksheet A (Appendix 1). The purpose of this phase was to identify the models being used to represent division and the difficulties they faced in representing the problems. In Phase 2 the prospective teachers' proportional reasoning was assessed on a set of ratio or rate problems along with their ability to represent rate and ratio problems as division problems given on worksheet B (Appendix 2).

In Phase 3 each participant individually went through a teaching session focused on the representation of division expressions in terms of rate and ratio between dividend and divisor. Participants were introduced to use of Vergnand's schematic diagram for solving division problems with whole numbers other than zero. The teaching was designed to help participants develop a connection between the concept of ratio/rate and the concept of division with whole numbers, to focus their attention on the multiplicative relationships among the divisors, the dividends and the quotients, and to see division expressions in terms of rate and ratio between the dividends and the divisors. In the overall teaching session, seven different division problems involving whole numbers were solved. First, the researcher solved two questions aloud giving detailed explanation at each step. One question was of a fair sharing type and the other was of repeated subtraction. The script for the teaching session has been fully reported in Rizvi (2004). After the teaching session the participants were again asked to solve problems on Worksheet A and Worksheet B and were asked to describe the changes which they would make in the schematic diagram to solve those problems which could not be solved by using the schematic diagram, and to provide reasons for these changes. The purpose of this phase was to collect evidence about how successfully the participants would be able to transfer the application of the schematic diagram to new problems

All the responses, explanations and "thinking aloud" which participants produced in each phase, were audio-taped and records of their working on paper were kept. Instructions provided for thinking aloud emphasised the need for participants to keep talking so that a full a report of processing activity was available (Ericsson & Simon, 1993). When participants were using pen and paper to solve the questions, the researcher asked probing questions to reveal their thinking behind their problem-solving moves.

Scoring procedures

In phase 1 the criteria on which participants' responses on each question would be evaluated were specified. The model used for each problem was identified using the criteria and agreed through discussion and consensus of the two researchers.

The focus of the analysis in Phase 2 was whether prospective teachers could identify which of the problems dealt with the additively related quantities (e.g. the age of one person related to the age of another person) and which problems involved multiplicatively related quantities (e.g. time and distance covered by a moving body). Additional points of interest were whether the participants recognised which problems involved directly proportional quantities (e.g. time and distance) and which problems involved inversely proportional quantities (e.g. time taken and speed of a moving body) and how they solved these problems. The participant's responses on each of the questions on worksheet B were assessed against steps of thinking that underlie the use of the ratio model shown in Table 2.

Table 2: Analysis Framework for Phase Two

Identification of types of variable quantities
- The same types (ratio), for example weight of sugar and flour in the recipe for making a cake.
- Different (rate), for example distance and time.

Identification of values associated with the variable quantities
- What values of the quantity are known and what are unknown.
- Whether a number describes the quantity or a unit of measurements is associated with numbers.
- Whether change of units was carried out.

Finding a relationship between quantities
- Recognition that by increasing one quantity another quantity increases or decreases?
- Recognition of additive or multiplicative relationships.

Generation of a correct solution.

Participants' performance against each criterion was scored on a 0 - 1 scale and the different types of difficulties which the participants faced in solving the problem were identified and categorised.

In Phase 3 participants solved the problems involving fractions given in Question 1 of worksheet A (Q1 ii, iii, iv, vi) using the rate and ratio model and to pose word problems for each of the expressions. Also of interest was whether they could solve word problems using the schematic diagram. They attempted Questions 2 - 5 of Worksheet A using the rate and ratio model and were asked to write mathematical expressions for each of the problems. Additionally, they were asked to identify the problems in Worksheet B that could be solved by the rate or ratio model and to describe the changes they would need to make in the rate or ratio model to solve those problems that could not be solved by the model.

RESULTS

Phase one

Representations for mathematical expressions involving division

We developed an account of the participants' conceptual knowledge in terms of three forms of representation by which the participants translated the mathematical expressions given in worksheet A:

a) Translation into words, which is labeled as semantic representation;

b) Translation into diagrams or actions, which is referred to as iconic representation; and

c) Representation of an expression as a word problem.

Generally the performance of the participants was quite satisfactory for the division expressions involving whole numbers. As shown in Table 3, the sharing model was predominant in their representation. The participants used repeated subtraction model for semantic representation and to some extent to for iconic representation. However, all the word problems that they posed were based on use of a fair sharing model.

Table 3: Representation of division involving whole numbers

Model	Percentage of total representations
Fair sharing	72.4
Repeated subtraction	24
other	3.4

Performance deteriorated on problems involving fractions. Eleven participants managed to represent division expressions where the divisor was a whole number and only the dividend was a fraction, since they could use a sharing model satisfactorily for this expression. Performance was poor for the mathematical expressions where the divisor was a fraction but the dividend was whole number and where the dividend and the divisor both were fractions. Five participants swapped to use of repeated subtraction model to represent the expressions where the divisors were fractions.

It appears that the participants' failure to represent expressions when the divisor was a fraction was associated with lack of familiarity with the repeated subtraction model. A correlation of r = 0.89 was found between use of the repeated subtraction model in division of whole numbers and successful representation of expressions having a fraction as a divisor and whole number as a dividend. Where the divisors were fractions the participants were not able to represent the expression in the form of word problems. This observation supported the findings of several researchers (e.g. Fischbein, Deri, Nello, & Marino, 1985) relating to the limitations of students' intuitive models for division of fractions.

Five participants were able to develop semantic and iconic representations for the mathematical expression where both the divisors and the dividends were fractions such as $\frac{1}{2} \div \frac{1}{4}$, using the repeated subtraction model. The problems $\frac{1}{4} \div \frac{1}{2}$ and $\frac{1}{5} \div \frac{1}{2}$ were the most difficult, with only five participants being able represent these expressions semantically and none being able to extend their representation beyond this point. Table 4 shows the group means score for the representation of division expressions on 3 – point scale.

Table 4: Group means score for the representation of division expressions (3 point scale)

Whole number	Dividend was fraction	Divisor was fraction	Fractions with dividend bigger than divisor	Fractions with divisor bigger than dividend
3	1.94	0.53	0.59	0.29

No participants were able to pose any sort of word problems for the expressions where the divisors were fractions. Participants posed only sharing type word problems for the expressions where the divisor was a whole number. Ten participants used the invert and multiply algorithms for the expressions involving division of fractions. Apart from this no other algorithm was used. No participants could explain the thinking that lies behind this algorithm, though four of them, who represented the expression pictorially, justified their answers by referring to the answer obtained by pictorial representations.

Participants' ability to solve word problems and to translate them into mathematical expressions

Performance in solving word problems and in translating these into mathematical expressions was assessed. No participants wrote division expressions for Q.2a, whereas they wrote a division expression for Q.2b. This observation reaffirmed Tirosh's (2000) views that people intuitively thought that division made things smaller. For the question where they knew that the answer would be bigger, participants did not select division as a correct operation for the problem.

The participants did not write division expressions for the problems based on the concept of ratio and proportion (Q.3). This suggests that participants were unable to identify or activate the connections between the concept of rate and ratio and the concept of division and could not think that similar proportional relations existed between divisor - dividend pairs. For the repeated subtraction type word problem (Q.7), only seven participants wrote division expressions. However, two did not figure out which number should appear as divisor and which as dividend in the mathematical expression. No participants made mistakes in writing division expression for a sharing type word problem (Q. 8).

The result of Phase 1 shows that to a great extent the participants relied on a fair sharing model of division to represent division expressions. They were able to represent division of a whole number by a whole number and division of a fraction by a whole number with the help of the model. A few used a repeated subtraction model at the level of semantic, and to a lesser extent iconic, forms to represent division of a whole number by a fraction, but none were able to pose word problems for the expressions. The participants' inability to pose word problems for such expressions indicates that they were not able to relate symbolic mathematical forms of knowledge to everyday life situations. Performance was very poor in representing division of a fraction by a fraction.

Phase Two

The participants' performance in Phase 1 reaffirmed that these prospective teachers did indeed face difficulties in developing successful representations of division of fractions, mainly because of their total reliance on sharing and repeated subtraction models. This suggested that it would be important to investigate how the participants could be assisted to develop a more adequate understanding of division, using the alternative rate and ratio model. As discussed above, the rate/ratio model could be understood only if the prospective teachers had a sound multiplicative understanding.

The prospective teachers' multiplicative understanding

The overall performance of the participants for each question of the Worksheet B is summarised in Table 5. The fourth column shows that for Question 4 only two participants identified the variable quantities involved in the problem. Similarly, two participants were able to identify the relations between variable quantities and were also able to solve the problem. The other 15 participants made mistakes. Ten of these considered that the variable quantities involved in the problem involved a directly proportional relationship. The other five did not realise that the quantities were related multiplicatively: rather they considered an additive relationship between them.

No participants solved all seven questions correctly. However, three participants solved six questions. On the other hand, there were three participants who could not solve more than two problems.

Participants who made errors in Question 2 failed to understand that the relationship between the ages of two persons could not be multiplicative. Another frequent error was made in Question 4 where the participants had to establish a relationship between the number of people who shared equally a certain amount of food and the number of days the food lasted. Fifteen participants failed to see the appropriate relationship. However, the frequency of considering the relationship as directly proportional was highest in this question, as ten of the fifteen participants solved the problem by establishing direct proportionality between the quantities.

Table 5: Summary of participants' overall performance for Worksheet B

	Criterion 1		**Criterion 2**		**Criterion 3**		**Difficulties faced by participants**
	Yes	No	Yes	No	Yes	No	
Q1	17	0	17	0	11	6	6 made computational errors.
Q2	17	0	7	10	7	10	10 considered additive relationships b/w as multiplicative.
Q3	7	10	7	10	7	10	4 did not identify directly proportional relationship. 6 were confused in using a trial and error method.
Q4	2	15	2	15	2	15	10 considered inversely proportional relationships as directly proportional relationships. 2 considered multiplicative relationship variable quantities as additive relationship 3 used trial and error method
Q5	17	0	17	0	17	0	No difficulty
Q6	11	6	11	6	11	6	6 considered multiplicative relationships as additive.
Q7	12	5	12	5	12	5	5 considered the additive relationship as multiplicative.

The results obtained from this phase suggest that the group of prospective teachers who were involved in this research made mistakes in recognising the particular relationship by which the quantities involved in a problem were linked with each other. That is why it was envisaged that the participants might have used the rate and ratio model for division inappropriately for those questions where quantities were linked additively, or where the quantities are linked in an inversely proportional relationship. To confirm this hypothesis, in the third and last phase, the participants, after having been introduced to the rate/ratio model of division, were asked to use this model to reconsider the questions used in Phase 1 and 2.

Phase Three

At the end of the teaching session, all the participants were able to represent and solve mathematical expressions and word problems involving division of whole numbers by setting up the rectangular array using the schematic diagram. They also found each of three relationships between the sets of the dividends and the divisors as described by the researcher in Question 1 of the teaching session. After the teaching session the participants were asked to solve problems on Worksheet A and Worksheet B again.

Ability to solve division of mathematical expressions using the rate and ratio model

In this phase all the participants were able to represent the mathematical expressions in the form of the rectangular array in Vergnand's schematic diagram. For example the participants' representation for the expression $4 \div \frac{1}{3}$ is shown below.

$$\begin{array}{cc} 4 & \frac{1}{3} \\ \\ ? & 1 \end{array}$$

They all acknowledged that solving the expressions meant to find out 'a number' which would have the same multiplicative relationship with '1' as the multiplicative relationship present between the dividends and the divisors.

For the expression $4 \div \frac{1}{3}$, nine participants found it difficult to find the unknown number because it was hard to figure out how many times four was bigger than $\frac{1}{3}$, or how many times $\frac{1}{3}$ is smaller than 4. However, they were successful in getting a correct answer by recognising other relationships in the rectangular array, such as 1 was three times $\frac{1}{3}$ and could establish the same relationship in the other column. One participant reasoned as follows:

> So four divided by a third, the relationship between a third and one would be increasing by a factor of three, so it would be four times three, which is twelve. (Participant D).

The remaining eight participants had developed another representation to cope with this situation. For example, one of the participants drew four circles and cut each circle into three equal parts. Then by counting all the parts, he figured out that there were twelve one-third portions in 4, so he said that 4 was 12 times bigger than one third. The same eight participants also successfully represented the expressions in Phase 1 with the use of a repeated subtraction model. So it is assumed that these eight participants had been able to extend their mental representation of division as repeated subtraction to the rate and ratio model of division. These participants also checked the multiplicative relationship between the numbers written in the columns of the array to verify the answer. All 17 participants also looked at the third relationship among the numbers in the rectangular array because they checked the answers by performing cross multiplication between numbers in the array.

Some participants were able to find the relationship between the invert and multiply algorithm and the rate of ratio model of division.

> Participant: So you see, we do the same thing like it's similar to four multiplied three-over-one. Now I'm thinking about the factor that it's changing to increase to that, to one, to the whole number. As most of the time, you did division by converting into multiplication, then three over one. But you did not have any proper reason for this. But now you have a reason
>
> Investigator: What is the reason?
>
> Participant: It's increasing by a factor of three, so it's looking at increasing the divisor by a factor of three to get to one whole. So another number will also increase by three times.

Ability to pose word problems for the mathematical expressions

Participants were asked to pose word problems for the expressions where the divisors were fractions such as $4 \div \frac{1}{3}$, $\frac{1}{3} \div \frac{1}{5}$, and $\frac{1}{4} \div \frac{1}{2}$. In Phase 1 none of the participants could pose word problems for these expressions, one noting *"For us they are symbols – just Maths symbols. We don't associate it with everyday life"*.

However, after going through the instructional intervention all participants managed to pose word problems for at least one problem. Eleven participants successfully posed problems for all the expressions, five posed problems for two, and one posed problems for one of the expressions.

Representation of rate/ratio type word problems in terms of division expression

Participants were asked to again solve Questions 2, 3, 4, and 5 of worksheet A using the rate and ratio model and to write mathematical expressions for each of the problems. For Question 3, nine participants used proportional reasoning before instruction. They had considered that in the situation the amount of flour would always remain double that of the sugar, so they multiplied the quantities by 2, as in $\frac{1}{2}$x2 =1 and $\frac{1}{4}$ x 2 = $\frac{1}{2}$. However, they did not think of this as a division question. After instruction they were able to see a connection between their representations of the problem and the concept of division. So, all 17 participants wrote an expression $\frac{1}{4} \div \frac{1}{2}$.

One participant noted:

> Because we want to find out what amount of sugar goes with I kg of flour, so we would write amount of flour as a divisor which is $\frac{1}{2}$ in the question (Participant G)

Other participants expressed that they had initially multiplied $\frac{1}{4}$ x 2 without realizing that it could also be considered $\frac{1}{4} \div \frac{1}{2}$.

> ..Now I understand why I multiplied it [amount of sugar] by 2. Because I was making this double [amount of flour], so I was making this double [amount of sugar]. Yep, it is like a quarter kg of sugar with for a half kg of flour.

Identifying where Vergnaud's schematic diagram could be used

Participants were asked to solve Questions 2, 3, 4 and 6 of Worksheet B a second time using Vergnand's schematic diagram. A significant change was noticed from Phase 2 to Phase 3 in the problem solving approach of some of the participants. In Phase 2 more participants performed operations on the numbers associated with word problems, without realising what the numbers stood for, and either misunderstood or did not try to recognise the relationship between the quantities. In Phase 3, some participants considered that the numbers were representing a particular relationship between two different quantities and were successful in identifying the correct relationships. The mean scores for performance on Questions 2 - 6 of Worksheet B are shown in Table 6.

Table 6: Performance for Q 2, 3, 4 and 6 of Worksheet B

	Mean	**Standard deviation**
Phase 2	1.59	1.23
Phase 3	2.35	0.93

A paired-samples t-test indicated that difference in group means was statistically significant ($t(16) = 7.67$, $p < .001$). The effect size for this comparison was 0.62, a moderately strong effect. This effect provides another form of evidence suggesting that the Phase 3 training effect was of practical significance. Other evidence of the benefit associated with this training is discussed below.

In Phase 2, 10 participants could not understand the relationship between the quantities in Q2 of worksheet A. However, in Phase 3, when they were asked to solve the question seven participants changed their understanding about the problem. One participant looked at the relationship as follows:

> But it's not reduced in the same way, because when you have two people aging, it's just a number, it's not the actual products they're increasing by.

However, as shown in Table 7, three participants were persistent in their misunderstanding about the question and they established a directly proportional relationship between the age of one person and the age of the other.

Table 7: Performance in the problem where the quantities had additive relationships

Recognised the relationship between quantities in Q2 in Phase 2	Recognised the relationship between quantities in Q2 in Phase 3	Frequency
Yes	No	0
No	Yes	7
Yes	Yes	7
No	No	3

In Phase 2, seven participants solved Question 3 of Worksheet B which was related to the number of triangles and number of sides in a pattern of adjoining triangles. These students discovered a formula to connect the two variables. In Phase 3, the same seven participants correctly argued that the problem could not be solved with the help of Vergnaud's schematic diagram. Six participants who used only a trial and error method in Phase 2 later established a directly proportional relationship between the number of triangles and number of sides of the triangle in the given pattern in Phase 3. As indicated in Table 8, the remaining four participants established a directly proportional relationship in both phases.

Table 8: Performance in the problem where one quantity was a function of the other but not directly proportional

Recognised the relationship between quantities in Q3 in Phase 2	Recognised the relationship between quantities in Q3 in Phase 3	Frequency
Yes	No	0
No	Yes	0
Yes	Yes	7
No	No	10

Table 9 shows the results for Question 4 of Worksheet B. Only two participants gave appropriate representations on both occasions, noting in Phase 3 that the rate or ratio model could be modified by increasing one number and decreasing the other number by the same factor. Eleven participants used Vergnand's schematic diagram without considering the relationship between the quantities. On this problem the use of the new representation produced a benefit for only four students.

Table 9: Performance in the problem where the quantities were inversely proportional

Recognised the relationship between quantities in Q4 in Phase 2	Recognised the relationship between quantities in 4 in Phase 2	Frequency
Yes	No	0
No	Yes	4
Yes	Yes	2
No	No	11

As indicated in Table 10, all participants used the rate and ratio model successfully for Question 6 of Worksheet B, including those six who made mistakes in solving the problem in Phase 2. However, we cannot be sure whether the participants understood that the quantities involved in the problem had a directly proportional relationship between them, since, it was possible that their use of the rate or ratio model could reflect a set effect (Anderson, 2000). As they had represented and solved several problems by using schematic diagrams they might have stuck with this model without actually realising whether it was appropriate or not.

Table 10: Performance in the problem where quantities were directly proportional

Recognised the relationship between quantities in Q3 in Phase 2	Recognised the relationship between quantities in Q3 in Phase 3	Frequency
Yes	No	0
No	Yes	6
Yes	Yes	11
No	No	0

DISCUSSION

After examining the participants' overall performance, it is clear that these prospective teachers needed to develop a better understanding of division of fractions. Their existing knowledge related to the topic was not adequate to allow them to either work on division problems for themselves, or to assist students to understand and solve such problems. The main reason for this low level of performance was reliance on the fair sharing and the repeated subtraction models. In the wake of the dependence on these models, the participants could not place division of fractions in their schema of the concept of division. The participants did not give any clue that showed that they found any relationship between the concept of division and the concept of rate or ratio. Although they satisfactorily used proportional reasoning to represent word problems based on the concept of rate or ratio, they did not represent those problems in a form of mathematical expression containing the sign of division. They translated a word problem based on fair sharing into a division expression, but could not do so with the word problem based on repeated subtraction.

The results of the teaching intervention in Phase 3 indicate that the schematic diagram is a promising scaffold that can help prospective teachers to develop more coherent schemas for the concept of division. The findings provide evidence of a practically significant improvement in the participants' performance to represent, solve, and pose problems related to division of fraction when they used the rate/ ratio model.

However, we cannot claim that the participants have embedded a strong, new representation of fractions in long term memory, one that they will transfer to future problem solving with division of fractions. Although the participants showed improvement in posing problems, the problems they posed and the problems which were used by the researcher in the teaching phase belonged to the same context. Mostly the participants used different numbers in the same situations. There is therefore a need to test other students' capability to use the rate or ratio model on several occasions across time in future research.

On several occasions during the research, the participants showed insufficient, partial and inaccurate knowledge of fractions. For such type of knowledge Perkins and Simmons (1988) used the term garbled knowledge. For instance, the participants sometimes did not assign correct names to the fractions. They had problems in deciding which fraction was bigger than the other. Often when they could not use a fair sharing model, they 'repaired the situation' (in Perkins and Simmons' words) by thinking of division as multiplication. On several occasions the participants handled problem of division by a fraction, such as $\frac{1}{4} \div \frac{1}{2}$, as if it was division by whole number such as $\frac{1}{4} \div 2$. Similar prospective teachers' behaviour was noticed by Ball (1990). These prospective teachers mentioned that they had learnt the topic of fractions and division a long time ago. Their mathematics content knowledge was not adequate to enable them to deal with the topics mathematics curriculum of upper primary school.

Since the participants made frequent errors in recognising relationship between variable quantities, it is inferred that the participants' multiplicative thinking was not sufficient to use the rate/ratio model effectively in their teaching. Previous researchers have also reported similar flaws in students' and teachers' multiplicative thinking. Dooren, Bock, Verschaffel and Janssens (2003) reported that their students had a tendency to incorrectly solve nonproportional problems in a proportional way. Thompson and Bush (2003) also found that teachers frequently used proportional reasoning in additive situations.

If teachers have limited understanding of multiplicative thinking, the risk will remain that the teachers or their students might use the rate or ratio model inappropriately. So before students can use the rate or ratio model there is a need to develop their multiplicative thinking. The results of the current study suggest that Vergnaud's schematics diagram could be a tool to develop multiplicative reasoning as well as to establish connections between concepts of multiplication and division to the concept of rate and ratio. The understanding of division which students would develop by recognising multiplicative relationship between the divisors, the dividends and quotient would help them to comprehend the situation of division of fractions and to justify mathematical expressions involving fractions.

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APPENDIX 1

Work Sheet A

Q1 (a) In how many ways can you interpret the following expressions? Write down your interpretations.

(i) $108 \div 4$ (ii) $\frac{1}{2} \div 3$ (iii) $\frac{1}{2} \div \frac{1}{4}$ (iv) $\frac{1}{3} \div \frac{1}{5}$ (v) $4 \div \frac{1}{3}$

(vi) $5 \div 7$ (vii) $\frac{1}{4} \div \frac{1}{2}$ (viii) $\frac{1}{5} \div \frac{1}{2}$

(b) Draw or describe diagrams to represent each of the division situations Q1a (i) to Q1 (viii).

(c) Solve Q1a (i) to Q1a(viii).

(d) Try to write down a word problem for each expression in Q 1(i) to Q1 (viii).

Q2 (a) If a person's income in half a month is \$ 400 how much can he earn in a month?

(b) A man earns \$ 240 in 3 weeks. How much is his weekly salary?

Q3 (a) Jessica and Shane had a recipe to make a cake. In this recipe $\frac{1}{2}$ kg of flour is mixed with $\frac{1}{4}$ kg of sugar. Shane wants to use 1 kg of flour. How much <u>sugar</u> should he use to have the same taste?

(b) Jessica also used the same recipe in which $\frac{1}{2}$ kg of flour is mixed with $\frac{1}{4}$ kg of sugar. Because she bought 1kg of sugar she wants to use all the sugar. How much <u>flour</u> should she use?

Q4 (a) A girl walks $\frac{1}{2}$ km in $\frac{1}{3}$ of an hour. How much distance can she cover in one hour if she keeps her speed constant?

(b) A girl walks $\frac{1}{2}$ km in $\frac{1}{3}$ of an hour. How much time is required to travel a distance of 1 km if she walks with the same speed?

Q5 (a) $8\frac{1}{4}$ cm is approximately equal to $3\frac{1}{2}$ inches. About how many cms are in an inch?

(b) $8\frac{1}{4}$ cm is approximately equal to $3\frac{1}{2}$ inches. What part of an inch is equal to one cm?

Q6 (a) A boat moved 4 metres in a second with constant speed. How much time will it take to move to 12 metres?

(b) A boat moves at a constant speed of 4 metres per second. How far does it move in 3 seconds?

(c) A boat moved 12 metres in 3 seconds at a constant speed. How far can it move in a second?

Q7 It takes $\frac{3}{8}$ of a bottle of milk to fill a large glass. How many of these glasses can be filled with 40 bottles of milk.

Q 8 An ant moved exactly the same distance each day. After five days it had moved a distance of $\frac{3}{4}$ km. How much did it move each day?

APPENDIX 2

Work Sheet B

Q 1: A machine caps bottles at a speed of one bottle every 2 seconds. How many bottles does the machine cap in 4 hours?

Q 2: A boy is 4 years old and his sister is three times as old as he is. When the boy was 1year old, how old would his sister be?

Q 3: Look at the following patterns made up of match sticks

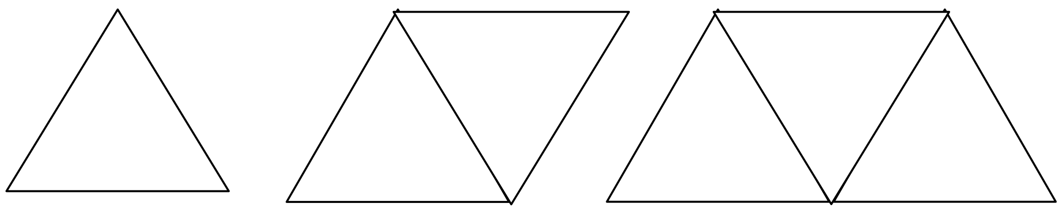

Fig 1 Fig 2 Fig3

Figure 3 has three triangles and seven sides. How many sides would a similar shape consisting of 30 triangles have?

Q 4: In a camp a certain amount of food lasts 32 people for 15 days. How many days would the same amount of food last if there were only 8 people. (Each person eats an equal amount of food.)

Q5: Ruth is 5 years old and John is 20 years. What will be the age of John when Ruth will be 10 years old?

Q6: There is a rectangle with sides 5 cm to 8 cm.

If you enlarge the rectangle so that its 5cm sides become 10 cm, what will be the length of the sides which were previously 8 cm?

Q7: Two men start walking on a track with the same speed but one after the other. Man A completes his 5 rounds when Man B completes 3 rounds. How many rounds will Man A have completed when Man B is finished his 6 rounds?

Q8: Jim is reading a book. He plans to read 5 pages per day to complete his reading in 13 days. If he wants to reduce his reading speed to 1 page per day, how many days will he take to complete his reading of the book?