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# Proximality in Banach space valued Musielak-Orlicz spaces

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## Abstract

Let  $(A, \mathcal{A}, \mu)$  be a  $\sigma$ -finite complete measure space and let  $Y$  be a subspace of a Banach space  $X$ . Let  $\varphi$  be a generalized  $\Phi$ -function on  $(A, \mathcal{A}, \mu)$ . Denote by  $L^\varphi(A, Y)$  and  $L^\varphi(A, X)$  the Musielak-Orlicz spaces whose functions take values in  $Y$  and  $X$ , respectively. Firstly, let  $f \in L^\varphi(A, X)$ , we characterize the distance of  $f$  from  $L^\varphi(A, Y)$ . Then, if  $Y$  is weakly  $\mathcal{K}$ -analytic and proximal in  $X$ , we show that  $L^\varphi(A, Y)$  is proximal in  $L^\varphi(A, X)$ . Finally, we give the connection between the proximality of  $L^\varphi(A, Y)$  in  $L^\varphi(A, X)$  and the proximality of  $L^1(A, Y)$  in  $L^1(A, X)$ .

**Keywords:** proximality; Musielak-Orlicz space; best approximation; weakly  $\mathcal{K}$ -analytic

## 1 Introduction

It is well known that Musielak-Orlicz spaces include many spaces as special spaces, such as Lebesgue spaces, weighted Lebesgue spaces, variable Lebesgue spaces and Orlicz spaces; see [1]. Especially, in recent decades, variable exponent function spaces, such as Lebesgue, Sobolev, Besov, Triebel-Lizorkin, Hardy, Morrey, and Herz spaces with variable exponents, have attracted much attention; see [2–16] and references therein. Cheng and the author discussed geometric properties of Banach space valued Bochner-Lebesgue and Bochner-Sobolev spaces with a variable exponent in [17]. Very recently, Musielak-Orlicz-Hardy spaces have been systemically developed; see, for example, [18–22]. These spaces have many applications in various fields such as PDE, electrorheological fluids, and image restoration; see [6, 23–25].

In recent years, proximality in Banach space valued Bochner-Lebesgue spaces with constant exponent have been extensively studied; see [26–33]. Proximality in Banach space valued Bochner-Lebesgue spaces with variable exponent was discussed by the author in [34]. In fact, we generalized those results in [29, 31] to Banach space valued Bochner-Lebesgue spaces with a variable exponent. Khandaqji, Khalil and Hussein considered proximality in Orlicz-Bochner function spaces on the unit interval in [35], and Al-Minawi and Ayesh consider the same problem on finite measures in [36]. The best simultaneous approximation in Banach space valued Orlicz spaces was discussed in [37, 38]. Micherda discussed proximality of subspaces of vector-valued Musielak-Orlicz spaces via modular in [39]. However, as usual, one considers the best approximation via the norm, so in this paper, we will discuss proximality of subspaces of vector-valued Musielak-Orlicz spaces via the norm. To proceed, we need to recall some definitions. Our results will be given in the next section.

In what follows,  $(A, \mathcal{A}, \mu)$  will be a  $\sigma$ -finite complete measure space. Suppose  $D$  is a subset of  $A$ , let  $\chi_D$  be the indicator function on  $D$ . Let  $(X, \|\cdot\|)$  be a Banach space. The dual space of  $X$  is the vector space  $X^*$  of all continuous linear mappings from  $X$  to  $\mathbb{R}$  or  $\mathbb{C}$ . To avoid a double definition we let  $\mathbb{K}$  be either  $\mathbb{R}$  or  $\mathbb{C}$ .

**Definition 1** A convex, left-continuous function  $\varphi : [0, \infty) \rightarrow [0, \infty]$  with  $\varphi(0) = 0$ ,  $\lim_{t \rightarrow 0^+} \varphi(t) = 0$ ,  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$  is called a  $\Phi$ -function. It is called positive if  $\varphi(t) > 0$  for all  $t > 0$ .

It is easy to see that if  $\varphi$  is a  $\Phi$ -function, then it is nondecreasing on  $[0, \infty)$ .

**Definition 2** Let  $(A, \mathcal{A}, \mu)$  be a  $\sigma$ -finite complete measure space. A real function  $\varphi : A \times [0, \infty) \rightarrow [0, \infty]$  is called a generalized  $\Phi$ -function on  $(A, \mathcal{A}, \mu)$  if

- (a)  $\varphi(y, \cdot)$  is a  $\Phi$ -function for all  $y \in A$ ,
- (b)  $y \mapsto \varphi(y, t)$  is measurable for all  $t \geq 0$ .

If  $\varphi$  is a generalized  $\Phi$ -function on  $(A, \mathcal{A}, \mu)$ , we write  $\varphi \in \Phi(A, \mu)$ .

**Definition 3** Let  $\varphi \in \Phi(A, \mu)$ . Define

$$\varrho_\varphi(f) := \int_A \varphi(y, \|f(y)\|) \, d\mu(y)$$

for strongly  $\mu$ -measurable functions  $f : A \rightarrow X$ . Then the Bochner-Musielak-Orlicz space  $L^\varphi(A, X)$  is the collection of all strongly  $\mu$ -measurable functions  $f : A \rightarrow X$  endowed with the norm:

$$\|f\|_{L^\varphi(A, X)} := \inf\{\lambda > 0, \varrho_\varphi(f/\lambda) \leq 1\}.$$

Let

$$E^\varphi(A, X) := \{f \in L^\varphi(A, X) : \varrho_\varphi(\lambda f) < \infty \text{ for all } \lambda > 0\}.$$

**Definition 4** Let  $\varphi \in \Phi(A, \mu)$ . The function  $\varphi$  is said to obey the  $\Delta_2$ -condition if there exists  $K \geq 2$  such that

$$\varphi(s, 2t) \leq K\varphi(s, t)$$

for all  $s \in A$  and all  $t \geq 0$ .

When  $X$  is  $\mathbb{R}$  or  $\mathbb{C}$ , we simply denote  $L^\varphi(A, X)$  by  $L^\varphi(A)$ , and  $E^\varphi(A, X)$  by  $E^\varphi(A)$ . Usually,  $E^\varphi(A, X)$  is a proper subspace of  $L^\varphi(A, X)$ . But when the  $\varphi$  satisfies the  $\Delta_2$ -condition, they are the same. It is easy to see that  $E^\varphi(A, X) = L^\varphi(A, X)$  is equivalent to  $E^\varphi(A) = L^\varphi(A)$ , this means that the equality depends only on  $\varphi$ .

We remark that  $\rho_\varphi$  is a semimodular on the space of all  $X$ -valued strongly  $\mu$ -measurable functions on  $A$ . For a semimodular, we recommend the reader reference [6]. Let  $\rho$  be a semimodular on vector space  $E$ ,  $E_\rho = \{x \in E : \rho(x/\lambda) < \infty \text{ for some } \lambda > 0\}$ ,  $\|x\|_\rho = \inf\{\lambda > 0 : \rho(x/\lambda) \leq 1\}$ . We will use the following elementary result for a semimodular, which is Corollary 2.1.15 in [6].

**Lemma 1** *Let  $\rho$  be a semimodular on  $E$ ,  $x \in E_\rho$ .*

- (i) *If  $\|x\|_\rho \leq 1$ , then  $\rho(x) \leq \|x\|_\rho$ .*
- (ii) *If  $1 < \|x\|_\rho$ , then  $\|x\|_\rho \leq \rho(x)$ .*

Let  $X$  be a Banach space and let  $Y$  be a closed subspace of  $X$ . Then  $Y$  is called proximal in  $X$  if for any  $x \in X$  there exists  $y \in Y$  such that

$$\|x - y\| = \text{dist}(x, Y) = \inf\{\|x - u\| : u \in Y\}.$$

In this case  $y$  is called a best approximation of  $x$  in  $Y$ . If this best approximation is unique for any  $x \in X$ , then  $Y$  is said to be Chebyshev.

For simplicity, we denote  $\|\cdot\|_{L^\varphi(A, X)}$  or  $\|\cdot\|_{L^\varphi(A)}$  by  $\|\cdot\|_\varphi$ . For  $f \in L^\varphi(A, X)$ ,  $Y \subset X$ , let

$$\text{dist}_\varphi(f, L^\varphi(A, Y)) := \inf\{\|f - g\|_\varphi : g \in L^\varphi(A, Y)\}.$$

## 2 Main results

Firstly, we estimate  $\text{dist}_\varphi(f, L^\varphi(A, Y))$ .

**Theorem 1** *Let  $Y$  be a subspace of Banach space  $X$ . Suppose  $\varphi \in \Phi(A, \mu)$ . For  $f \in L^\varphi(A, X)$ , define  $\phi : A \rightarrow \mathbb{R}$  by  $\phi(s) := \text{dist}(f(s), Y)$ . Then*

- (i)  *$\phi \in L^\varphi(A)$  and  $\text{dist}_\varphi(f, L^\varphi(A, Y)) \geq \|\phi\|_\varphi$ ;*
- (ii)  *$\text{dist}_\varphi(f, L^\varphi(A, Y)) = \|\phi\|_\varphi$  for  $f \in E^\varphi(A, X)$ .*

*Proof* (i) Given  $f \in L^\varphi(A, X)$ , we see that there exists a sequence of simple functions  $\{f_n\}$  which converges to  $f$  almost everywhere and in  $L^\varphi(A, X)$ . Since the distance function  $d(x, Y)$  is a continuous function of  $x \in X$ ,  $\|f_n(s) - f(s)\| \rightarrow 0$  implies that  $|\text{dist}(f_n(s), Y) - \text{dist}(f(s), Y)| \rightarrow 0$ . Moreover, each function  $\phi_n : A \rightarrow \mathbb{R}$  defined by  $\phi_n(s) := \text{dist}(f_n(s), Y)$  is a simple function; therefore we conclude that  $\phi$  is measurable. Now, for any  $g \in L^\varphi(A, Y)$  and any  $\lambda > 0$ ,

$$\begin{aligned} \rho_\varphi(\lambda(f - g)) &= \int_A \varphi(s, \lambda\|f(s) - g(s)\|) \, d\mu(s) \\ &\geq \int_A \varphi(s, \lambda \text{dist}(f(s), Y)) \, d\mu(s) \\ &= \rho_\varphi(\lambda\phi). \end{aligned}$$

Thus, we have

$$\|f - g\|_\varphi \geq \|\phi\|_\varphi.$$

This implies  $\phi \in L^\varphi(A)$  and, by taking an infimum on  $g \in L^\varphi(A, Y)$ , we have

$$\text{dist}_\varphi(f, L^\varphi(A, Y)) \geq \|\phi\|_\varphi.$$

(ii) We first assume that  $f$  is a simple function. Let  $f(s) := \sum_{i=1}^m \chi_{A_i} x_i$  where  $\{A_i\}_{i=1}^m$  are disjoint measurable subsets in  $A$  such that  $0 < \mu(A_i) < \infty$  and  $0 \neq x_i \in X$  for  $i \in \{1, \dots, m\}$ . Without loss of generality, we suppose that  $\text{dist}_\varphi(f, L^\varphi(A, Y)) = 1$ . Let  $0 < \epsilon < 1$ . Since  $\phi(s) \leq$

$\|f(s)\|$ , we have  $\rho_\varphi(\lambda\phi) \leq \rho_\varphi(\lambda f) < \infty$  for any  $\lambda > 0$ . Then, by the dominated convergence theorem, we find that there exists  $\delta > 0$  such that

$$\int_{A_i} \varphi(s, \text{dist}(x_i, Y) + \delta) \, d\mu(s) \leq \int_{A_i} \varphi(s, \text{dist}(x_i, Y)) \, d\mu(s) + \frac{\epsilon}{m}, \quad \forall i \in \{1, \dots, m\}.$$

Now take  $y_i \in Y$  such that  $\|x_i - y_i\| < \text{dist}(x_i, Y) + \delta$  for  $i \in \{1, \dots, m\}$ . Let  $g(s) = \sum_{i=1}^m \chi_{A_i} y_i$ . Therefore  $\|f - g\|_\varphi \geq \text{dist}_\varphi(f, L^\varphi(A, Y)) = 1$ . By Lemma 1, we see that

$$\begin{aligned} 1 \leq \|f - g\|_\varphi &\leq \rho_\varphi(f - g) = \sum_{i=1}^m \int_{A_i} \varphi(s, \|x_i - y_i\|) \, d\mu(s) \\ &\leq \sum_{i=1}^m \int_{A_i} \varphi(s, \text{dist}(x_i, Y) + \delta) \, d\mu(s) \\ &\leq \sum_{i=1}^m \left( \int_{A_i} \varphi(s, \text{dist}(x_i, Y)) \, d\mu(s) + \frac{\epsilon}{m} \right) \\ &= \int_A \varphi(s, \text{dist}(f(s), Y)) \, d\mu(s) + \epsilon. \end{aligned}$$

Thus,  $\rho_\varphi(\phi) \geq 1 - \epsilon$ . Since  $\epsilon$  is arbitrary, we have  $\rho_\varphi(\phi) \geq 1$ . By Lemma 1 again, we have  $\|\phi\|_\varphi \geq 1$ . This means that  $\|\text{dist}(f(\cdot), Y)\|_\varphi \geq \text{dist}_\varphi(f, L^\varphi(A, Y))$ . Therefore, we have proved that  $\|\text{dist}(f(\cdot), Y)\|_\varphi = \text{dist}_\varphi(f, L^\varphi(A, Y))$  for simple functions.

Finally, let  $f \in E^\varphi(A, X)$ , there exists a sequence of simple functions  $\{g_n\}_{n \in \mathbb{N}}$  convergent to  $f$   $\mu$ -almost everywhere,  $\|g_n(s)\| \leq \|f(s)\|$   $\mu$ -almost everywhere and  $\|f - g_n\|_\varphi \rightarrow 0$  as  $n$  tends to  $\infty$ . Let  $\phi_n(s) = \text{dist}(g_n(s), Y)$ . From the previous proof, we have

$$\|\phi_n\|_\varphi = \text{dist}_\varphi(g_n, L^\varphi(A, Y)).$$

It is easy to see that  $\text{dist}_\varphi(g_n, L^\varphi(A, Y)) \rightarrow \text{dist}_\varphi(f, L^\varphi(A, Y))$  as  $n \rightarrow \infty$ . Since  $\phi_n(s) \leq \|g_n(s)\| \leq \|f(s)\|$   $\mu$ -almost everywhere, and  $\phi_n(s) \rightarrow \phi(s)$   $\mu$ -almost everywhere as  $n$  tends to  $\infty$ , by Lemma 2.3.16(c) in [6], we conclude that  $\phi_n \rightarrow \phi$  in  $L^\varphi(A)$ . Hence, letting  $n \rightarrow \infty$ , we see that

$$\|\phi\|_\varphi = \text{dist}_\varphi(f, L^\varphi(A, Y)),$$

which completes the proof of Theorem 1. □

**Corollary 1** *Let  $Y$  be a closed subspace of a Banach space  $X$ . Suppose  $\varphi \in \Phi(A, \mu)$ . An element  $g$  of  $L^\varphi(A, Y)$  is a best approximation to an element  $f$  in  $E^\varphi(A, X)$  if and only if  $g(s)$  is a best approximation in  $Y$  to  $f(s)$  for almost every  $s \in A$ . Furthermore, if  $\varphi$  satisfies the  $\Delta_2$ -condition, then an element  $g$  of  $L^\varphi(A, Y)$  is a best approximation to an element  $f$  in  $L^\varphi(A, X)$  if and only if  $g(s)$  is a best approximation in  $Y$  to  $f(s)$  for almost every  $s \in A$ .*

**Corollary 2** *Let  $Y$  be a Chebyshev subspace of a Banach space  $X$ . Suppose  $\varphi \in \Phi(A, \mu)$  satisfies the  $\Delta_2$ -condition. If  $L^\varphi(A, Y)$  is proximal in  $L^\varphi(A, X)$ , then it is a Chebyshev subspace of  $L^\varphi(A, X)$ .*

**Remark 1** Our results Theorem 1, Corollaries 1 and 2 cover the results for vector Orlicz spaces in [36]. Indeed, in [36] the authors only considered vector Orlicz spaces on finite

measures. Analogous results for best simultaneous approximation were obtained in [37, 38] for vector Orlicz spaces on finite measures.

Next, we transfer the proximality of  $Y$  in  $X$  to  $L^\varphi(A, Y)$  in  $L^\varphi(A, X)$ . To do so, we need some preliminaries.

**Lemma 2** *Let  $(A, \mathcal{A}, \mu)$  be a  $\sigma$ -finite complete measure space. Suppose  $\varphi \in \Phi(A, \mu)$ . Let  $Y$  be a proximinal subspace of a Banach space  $X$ . Suppose  $f \in L^\varphi(A, X)$  and  $g$  is a strongly  $\mu$ -measurable function such that  $g(s)$  is a best approximation to  $f(s)$  from  $Y$  for almost everywhere  $s \in A$ . Then  $g$  is a best approximation to  $f$  from  $L^\varphi(A, Y)$ .*

*Proof* Since  $0 \in Y$ , it follows that  $\|g(s)\| \leq 2\|f(s)\|$   $\mu$ -almost everywhere. Thus,  $g \in L^\varphi(A, Y)$ . For each  $h \in L^\varphi(A, Y)$ , by assumption we know that  $\|f(s) - g(s)\| \leq \|f(s) - h(s)\|$   $\mu$ -almost everywhere. So  $\rho_\varphi(\lambda(f - g)) \leq \rho_\varphi(\lambda(f - h))$  for any  $\lambda > 0$ . Thus,  $\|f - g\|_\varphi \leq \|f - h\|_\varphi$ . This ends the proof.  $\square$

**Definition 5** Let  $(T, \tau)$  be a Polish space (i.e. a topological space which is separable and completely metrizable). A set  $Q \subset T$  is analytic if it is empty or if there exists a continuous mapping  $f : \mathbb{N}^{\mathbb{N}} \rightarrow T$  satisfying  $f(\mathbb{N}^{\mathbb{N}}) = Q$ , where  $\mathbb{N}^{\mathbb{N}}$  denotes the space of all infinite sequences of natural numbers endowed with the Tychonoff topology.

**Definition 6** Let  $(T, \tau)$  be a Polish space,  $H$  a topological space and denote by  $\sigma(\mathcal{A})$  the smallest  $\sigma$ -algebra containing all analytic subsets of  $T$ . Then a mapping  $f : T \rightarrow H$  is said to be analytic measurable if  $f^{-1}(C) \in \sigma(\mathcal{A})$  for every  $C \in \mathcal{B}(H)$ , where  $\mathcal{B}(H)$  is for the Borel sets of  $H$ .

**Definition 7** Let  $H, T$  be topological spaces. Then a multifunction  $F : H \rightarrow 2^T$  is said to be upper semi-continuous if for every  $x \in H$  and for every open set  $U$  satisfying  $F(x) \subset U$ , there exists an open neighborhood  $V$  of  $x$  such that  $F(V) \subset U$ .

**Definition 8** A subset  $C$  of a topological space  $T$  is  $\mathcal{K}$ -analytic if it can be written as  $C = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} F(\sigma)$  for some upper semi-continuous mapping  $F : \mathbb{N}^{\mathbb{N}} \rightarrow 2^T$  with compact values. In the case when  $T$  is a Banach space endowed with its weak topology,  $C$  is said to be weakly  $\mathcal{K}$ -analytic.

For the theory of  $\mathcal{K}$ -analytic sets, we recommend [40]. Specially, all reflexive and all separable Banach spaces are weakly  $\mathcal{K}$ -analytic. The following lemma is just Theorem 3.3 in [31].

**Lemma 3** *Let  $(X, \|\cdot\|)$  be a real Banach space and let  $Y$  be a proximinal, weakly  $\mathcal{K}$ -analytic convex subset of  $X$ . Then, for each closed and separable set  $M \subset X$ , there exists an analytic measurable mapping  $h : M \rightarrow Y$  such that  $h(M)$  is separable in  $Y$  and  $h(x)$  is a best approximation of  $x$  in  $Y$  for any  $x \in M$ .*

Thus, following the argument of the proof of (i)  $\rightarrow$  (ii) in [39, p.185], we have the following conclusion, the details being omitted.

**Theorem 2** *Let  $(A, \mathcal{A}, \mu)$  be a  $\sigma$ -finite complete measure space. Suppose  $\varphi \in \Phi(A, \mu)$ , and  $Y$  is a weakly  $\mathcal{K}$ -analytic linear subspace of a real Banach space  $X$ . If  $Y$  is proximal in  $X$ , then  $L^\varphi(A, Y)$  is proximal in  $L^\varphi(A, X)$ .*

**Theorem 3** *Let  $(A, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space. Suppose  $\varphi \in \Phi(A, \mu)$  such that  $E^\varphi(A) = L^\varphi(A)$ . Let  $Y$  be a linear subspace of a real Banach space  $X$ . If  $L^\varphi(A, Y)$  is proximal in  $L^\varphi(A, X)$ , then  $Y$  is proximal in  $X$ .*

*Proof* Since the measure  $\mu$  is  $\sigma$ -finite, let us choose positive measure set  $Q$  such that  $\chi_Q \in L^\varphi(A)$ . For any  $x \in X$ , let  $f(t) := \chi_Q(t) \cdot x$ ,  $t \in A$ . Then  $f \in L^\varphi(A, X)$ . By the assumption, we know that there is a  $g$  in  $L^\varphi(A, Y)$  which is a best approximation element of  $f$ . Consequently,  $g(s)$  is a best approximation to  $f(s)$  in  $Y$  for almost every  $s \in A$  by Corollary 1. Therefore, there is a best approximation element of  $x$  in  $Y$ . Thus,  $Y$  is proximal in  $X$ .  $\square$

From Theorems 2 and 3, we deduce the following corollary.

**Corollary 3** *Let  $(A, \mathcal{A}, \mu)$  be a  $\sigma$ -finite complete measure space. Suppose  $\varphi \in \Phi(A, \mu)$  such that  $E^\varphi(A) = L^\varphi(A)$ . Let  $Y$  be a weakly  $\mathcal{K}$ -analytic linear subspace of a real Banach space  $X$ . Then the following conditions are equivalent:*

- (i)  $Y$  is proximal in  $X$ ;
- (ii)  $L^\varphi(A, Y)$  is proximal in  $L^\varphi(A, X)$ .

**Remark 2** An analog to Corollary 3 in terms of a modular was obtained in [39].

Finally, we give a characterization of proximity of  $L^\varphi(A, Y)$  in  $L^\varphi(A, X)$  via the proximity of  $L^1(A, Y)$  in  $L^1(A, X)$ . When  $L^\varphi(A, X)$  is a Bochner-Lebesgue space, which was obtained in [29] and [32, 33] on finite measure spaces and  $\sigma$  finite measure spaces, respectively.  $L^\varphi(A, X)$  is a Bochner-Orlicz space, which was discussed in [35].

**Theorem 4** *Let  $(A, \mathcal{A}, \mu)$  be a  $\sigma$ -finite complete measure space. Suppose  $\varphi \in \Phi(A, \mu)$  such that the set of simple functions,  $S(A, \mu)$ , satisfies  $S(A, \mu) \subset L^{\varphi^*}(A, \mu)$ , where  $\varphi^*$  is the conjugate function of  $\varphi$  (see [6]). Let  $Y$  be a closed subspace of a Banach space  $X$ . If  $L^1(A, Y)$  is proximal in  $L^1(A, X)$ , then  $L^\varphi(A, Y)$  is proximal in  $L^\varphi(A, X)$ .*

*Proof* Since  $A$  is  $\sigma$ -finite, we may write  $A = \bigcup_{i=1}^\infty A_i$ , where  $\{A_i\}$  is a sequence of disjoint measurable sets each of finite measure. Let  $f \in L^\varphi(A, X)$ . For any  $n \in \mathbb{N}$ , since  $\mu(A_n) < \infty$ , then  $\chi_{A_n} \in L^{\varphi^*}(A)$ . Thus, by the norm conjugate formula (see Corollary 2.7.5 in [6]), we find that  $f\chi_{A_n} \in L^1(A, X)$ . By assumption, we know that there exists  $g_n \in L^1(A, Y)$  such that

$$\|f\chi_{A_n} - g_n\|_{L^1} \leq \|f\chi_{A_n} - h\|_{L^1}, \quad \forall h \in L^1(A, Y).$$

By Corollary 1, we have, for all  $y \in Y$ ,

$$\|f(t)\chi_{A_n} - g_n(t)\| \leq \|f(t)\chi_{A_n} - y\|$$

$\mu$ -almost everywhere. Therefore  $g_n(t) = 0$   $\mu$ -almost every  $t \in A_n^c$ . Let  $g = \sum_{n=1}^\infty g_n$ . Since  $f = \sum_{n=1}^\infty f\chi_{A_n}$ , it follows that for all  $h \in L^\varphi(A, Y)$ ,

$$\|f(t) - g(t)\| \leq \|f(t) - h(t)\|$$

$\mu$ -almost everywhere. Because  $0 \in Y$ , it follows that  $\|g(t)\| \leq 2\|f(t)\|$ . Thus,  $g \in L^\varphi(A, Y)$  and

$$\|f - g\|_\varphi \leq \|f - h\|_\varphi$$

for all  $h \in L^\varphi(A, Y)$ . This finishes the proof.  $\square$

**Theorem 5** *Let  $(A, \mathcal{A}, \mu)$  be a  $\sigma$ -finite complete measure space. Suppose  $\varphi \in \Phi(A, \mu)$  satisfies  $E^\varphi(A) = L^\varphi(A)$  and, for each  $t \in A$ ,  $\varphi(t, \cdot)$  is strictly increasing. Let  $Y$  be a closed subspace of a Banach space  $X$ . If  $L^\varphi(A, Y)$  is proximal in  $L^\varphi(A, X)$ , then  $L^1(A, Y)$  is proximal in  $L^1(A, X)$ .*

*Proof* We use the idea from [35]. Indeed in [35] the authors only considered Banach space valued Orlicz spaces on the unit interval. Since, for each  $t \in A$ ,  $\varphi(t, \cdot)$  is strictly increasing, let  $\varphi^{-1}(t, \cdot)$  be its inverse function, which means, for each  $s \in [0, \infty)$ ,  $\varphi(t, \varphi^{-1}(t, s)) = s$ . Define the map  $J : L^1(A, X) \rightarrow L^\varphi(A, X)$  by setting

$$J(f)(t) := \begin{cases} \frac{\varphi^{-1}(t, \|f(t)\|)}{\|f(t)\|} f(t), & f(t) \neq 0; \\ 0, & f(t) = 0. \end{cases}$$

Then  $\|J(f)(t)\| = \varphi^{-1}(t, \|f(t)\|)$ . Therefore  $\rho_\varphi(J(f)) = \|f\|_{L^1}$ . So  $J$  is injective. Moreover, if  $g \in L^\varphi(A, X)$ , let

$$f(t) := \begin{cases} \frac{\varphi(t, \|g(t)\|)}{\|g(t)\|} g(t), & g(t) \neq 0; \\ 0, & g(t) = 0. \end{cases}$$

Then  $f(t) \in X$  and  $\|f(t)\| = \varphi(t, \|g(t)\|)$ . Thus,  $f \in L^1(A, X)$ . In addition, for  $g(t) \neq 0$ ,

$$J(f)(t) = \frac{\varphi^{-1}(t, \varphi(t, \|g(t)\|))}{\varphi(t, \|g(t)\|)} f(t) = \frac{\|g(t)\|}{\varphi(t, \|g(t)\|)} f(t) = g(t).$$

If  $g(t) = 0$ , then  $f(t) = 0$  also, thus  $J(f)(t) = 0 = g(t)$ . Hence  $J$  is surjective and  $J(L^1(A, Y)) = L^\varphi(A, Y)$  also.

Now, let  $f \in L^1(A, X)$ . Without loss of generality we may suppose that  $f(t) \neq 0$   $\mu$ -almost everywhere, for otherwise we can restrict our measure to the support of  $f$ . Since  $J(f) \in L^\varphi(A, X)$ , by the assumption, we know that there exists some  $g \in L^1(A, Y)$  such that

$$\|J(f) - J(g)\|_\varphi \leq \|J(f) - J(v)\|_\varphi$$

for all  $v \in L^1(A, Y)$ . By Corollary 1, we see that, for all  $y \in Y$ ,

$$\|J(f)(t) - J(g)(t)\| \leq \|J(f)(t) - y\|$$

$\mu$ -almost everywhere. Multiplying both sides of the last inequality by  $\frac{\|f(t)\|}{\varphi^{-1}(t, \|f(t)\|)}$ , we obtain, for all  $y \in Y$ ,

$$\left\| f(t) - \frac{\|f(t)\|}{\varphi^{-1}(t, \|f(t)\|)} \frac{\varphi^{-1}(t, \|g(t)\|)}{\|g(t)\|} g(t) \right\| \leq \|f(t) - y\|.$$

Let  $h(t) = \frac{\|f(t)\|}{\varphi^{-1}(t, \|f(t)\|)} \frac{\varphi^{-1}(t, \|g(t)\|)}{\|g(t)\|} g(t)$ . Since  $h(t)$  is a best approximation of  $f(t)$  in  $Y$ , and  $0 \in Y$ , it follows that  $\|h(t)\| \leq 2\|f(t)\|$ . Therefore,  $h \in L^1(A, Y)$ . Thus, for all  $w \in L^1(A, Y)$ ,

$$\|f(t) - h(t)\| \leq \|f(t) - w(t)\|$$

$\mu$ -almost everywhere. Thus, by Corollary 1  $h$  is a best approximation of  $f$  in  $L^1(A, Y)$ . This finishes the proof.  $\square$

From Theorems 4 and 5, we deduce the following corollary.

**Corollary 4** *Let  $(A, \mathcal{A}, \mu)$  be a  $\sigma$ -finite complete measure space. Suppose  $\varphi \in \Phi(A, \mu)$  such that  $E^\varphi(A) = L^\varphi(A)$ , for each  $t \in A$ ,  $\varphi(t, \cdot)$  is strictly increasing and the set of simple functions  $S(A, \mu) \subset L^{\varphi^*}(A, \mu)$ . Let  $Y$  be a closed subspace of a Banach space  $X$ . Then the following conditions are equivalent:*

- (i)  $L^1(A, Y)$  is proximal in  $L^1(A, X)$ ;
- (ii)  $L^\varphi(A, Y)$  is proximal in  $L^\varphi(A, X)$ .

**Remark 3** When  $(A, \mu)$  is a finite measure and  $\varphi$  is a Orlicz function that satisfies the  $\Delta_2$ -condition, the result of Corollary 4 was obtained in [36]. While  $(A, \mu)$  is the unit interval and  $\varphi$  is a Young function that satisfies the  $\Delta_2$ -condition, the result of Corollary 4 was obtained in [35].

#### Competing interests

The author declare that he has no competing interests.

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#### References

1. Musielak, J: Orlicz Spaces and Modular Spaces. Springer, Berlin (1983)
2. Kováčik, O, Rákosník, J: On spaces  $L^{p(x)}$  and  $W^{k,p(x)}$ . Czechoslov. Math. J. **41**, 592-618 (1991)
3. Almeida, A, Drihem, D: Maximal, potential and singular type operators on Herz spaces with variable exponents. J. Math. Anal. Appl. **394**, 781-795 (2012)
4. Almeida, A, Hästö, P: Besov spaces with variable smoothness and integrability. J. Funct. Anal. **258**, 1628-1655 (2010)
5. Diening, L, Hästö, P, Roudenko, S: Function spaces of variable smoothness and integrability. J. Funct. Anal. **256**, 1731-1768 (2009)
6. Diening, L, Harjulehto, P, Hästö, P, Růžička, M: Lebesgue and Sobolev Spaces with Variable Exponents. Springer, Berlin (2011)
7. Harjulehto, P, Hästö, P, Le, UV, Nuortio, M: Overview of differential equations with non-standard growth. Nonlinear Anal. **72**, 4551-4574 (2010)
8. Izuki, M: Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization. Anal. Math. **36**, 33-50 (2010)
9. Izuki, M: Boundedness of commutators on Herz spaces with variable exponent. Rend. Circ. Mat. Palermo **59**, 199-213 (2010)
10. Izuki, M: Vector-valued inequalities on Herz spaces and characterizations of Herz-Sobolev spaces with variable exponent. Glas. Mat. **45**, 475-503 (2010)
11. Kempka, H: 2-Microlocal Besov and Triebel-Lizorkin spaces of variable integrability. Rev. Mat. Complut. **22**, 227-251 (2009)
12. Kempka, H: Atomic, molecular and wavelet decomposition of generalized 2-microlocal Besov spaces. J. Funct. Spaces Appl. **8**, 129-165 (2010)
13. Nakai, E, Sawano, Y: Hardy spaces with variable exponents and generalized Campanato spaces. J. Funct. Anal. **262**, 3665-3748 (2012)
14. Samko, S: Variable exponent Herz spaces. Mediterr. J. Math. **10**, 2007-2025 (2013)
15. Xu, J: Variable Besov and Triebel-Lizorkin spaces. Ann. Acad. Sci. Fenn., Math. **33**, 511-522 (2008)



16. Xu, J: The relation between variable Bessel potential spaces and Triebel-Lizorkin spaces. *Integral Transforms Spec. Funct.* **19**, 599-605 (2008)
17. Cheng, C, Xu, J: Geometric properties of Banach space valued Bochner-Lebesgue spaces with variable exponent. *J. Math. Inequal.* **7**, 461-475 (2013)
18. Cao, J, Chang, DC, Yang, D, Yang, S: Weighted local Orlicz-Hardy spaces on domains and their applications in inhomogeneous Dirichlet and Neumann problems. *Trans. Am. Math. Soc.* **365**, 4729-4809 (2013)
19. Liang, Y, Huang, J, Yang, D: New real-variable characterizations of Musielak-Orlicz Hardy spaces. *J. Math. Anal. Appl.* **395**, 413-428 (2012)
20. Yang, D, Yang, S: Orlicz-Hardy spaces associated with divergence operators on unbounded strongly Lipschitz domains of  $\mathbb{R}^n$ . *Indiana Univ. Math. J.* **61**, 81-129 (2012)
21. Yang, D, Yang, S: Local Hardy spaces of Musielak-Orlicz type and their applications. *Sci. China Math.* **55**, 1677-1720 (2012)
22. Yang, D, Yang, S: Musielak-Orlicz-Hardy spaces associated with operators and their applications. *J. Geom. Anal.* **24**, 495-570 (2014)
23. Chen, Y, Levine, S, Rao, R: Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.* **66**, 1383-1406 (2006)
24. Li, F, Li, Z, Pi, L: Variable exponent functionals in image restoration. *Appl. Math. Comput.* **216**, 870-882 (2010)
25. Růžička, M: *Electrorheological Fluids: Modeling and Mathematical Theory*. Springer, Berlin (2000)
26. Abu-Sirhan, E: Simultaneous approximation in function spaces. In: Neamtu, M, Schumaker, L (eds.) *Approximation Theory XIII: San Antonio*, pp. 321-329 (2010)
27. Abu-Sirhan, E: Best simultaneous approximation in function and operator spaces. *Turk. J. Math.* **36**, 101-112 (2012)
28. Khalil, R: Best approximation in  $L_p(I, X)$ . *Math. Proc. Camb. Philos. Soc.* **94**, 277-279 (1983)
29. Khalil, R, Deeb, W: Best approximation in  $L_p(I, X)$ . *J. Approx. Theory* **59**, 296-299 (1989)
30. Khalil, R, Saidi, F: Best approximation in  $L_1(I, X)$ . *Proc. Am. Math. Soc.* **123**, 183-190 (1995)
31. Light, WA: Proximality in  $L_p(S, Y)$ . *Rocky Mt. J. Math.* **19**, 251-259 (1989)
32. Mendoza, J: Proximality in  $L_p(\mu, X)$ . *J. Approx. Theory* **93**, 331-343 (1998)
33. You, ZY, Guo, TX: Pointwise best approximation in the space of strongly measurable functions with applications to best approximation in  $L^p(\mu, X)$ . *J. Approx. Theory* **78**, 314-320 (1994)
34. Xu, J: Proximality in Banach space valued Bochner-Lebesgue spaces with variable exponent (submitted)
35. Khandaqji, M, Khalil, R, Hussein, D: Proximality in Orlicz-Bochner function spaces. *Tamkang J. Math.* **34**, 71-75 (2003)
36. Al-Minawi, H, Ayes, S: Best approximation in Orlicz spaces. *Int. J. Math. Math. Sci.* **14**, 245-252 (1991)
37. Khandaqji, M, Al-Sharif, S: Best simultaneous approximation in Orlicz spaces. *Int. J. Math. Math. Sci.* **2007**, Article ID 68017 (2007)
38. Shen, ZS, Yang, ZY: Best simultaneous approximation in  $L^\Phi(I, X)$ . *J. Math. Res. Expo.* **30**, 863-868 (2010)
39. Micherda, B: On proximal subspaces of vector-valued Orlicz-Musielak spaces. *J. Approx. Theory* **174**, 182-191 (2013)
40. Kakol, J, Kubiś, W, López-Pellicer, M: *Descriptive Topology in Selected Topics of Functional Analysis*. Springer, Berlin (2011)

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