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PROXIMITY OF INFORMATION IN GAMES WITH INCOMPLETE INFORMATION

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Existing notions of proximity of information fail to satisfy desired continuity properties of equilibria in games with incomplete information. We demonstrate this failure and propose a topology on information structures which is the minimal one that satisfies those continuity requirements. This topology is defined in terms of the common belief players have about the proximity of each player's information.

1. Introduction. Over the last decade several authors have studied notions of similarity of information (e.g. Allen 1983, 1984; Cotter 1986; VanZandt 1988; and Stinchcombe 1990). In the economic context this similarity should be defined such that agents behave similarly when information is similar. By similarity of information we mean closeness of partitions, or σ-fields, where prior probabilities remain fixed. Milgrom and Weber (1985) and Kajii and Morris (1994) study closeness of information when σ-fields are held fixed while prior probabilities change. That is, behavior should reveal some continuity with respect to the topology defined on information. This paper is motivated by the observation that the existing notions concerning proximity of information, while guaranteeing similarity of behavior for similar information when a single agent is involved, fail to imply an analogous sameness of behavior in models in which agents interact. In the latter models, similarity of information for each single agent does not imply similarity of behavior for the group. We show that similarity of group information, for games with incomplete information, should be defined in terms of the common belief players have concerning the similarity of individuals' information. We define such a metrizable topology on group information, prove its adequacy for certain continuity properties of equilibria and show that it is the coarsest one that satisfies these properties. Continuity refers here to some sort of lower hemi-continuity. Roughly speaking, for a given information structure, a game with this structure, and an equilibrium of the game, close information structures have ϵ-equilibria which are close to the given equilibrium. This type of continuity is considered also in Kajii and Morris (1994), while Milgrom and Weber (1985) consider upper hemi continuity. The basic concept and properties of common belief are taken from Monderer-Samet (1989) (see also Fudenberg-Tirole 1990 for new results) but various variants of common belief are studied here to enable explicit definition of the metric. The notion of common belief, which is so central to our definition of proximity of information structure, does not appear in the works of Milgrom and Weber (1985) or Kajii and Morris (1994), in which prior probabilities vary. The relation between their notions of proximity and ours awaits further research.

In Allen's work, and ensuing studies by others, information is given as a σ-field over a state space. Using a metric, defined by Boylan (1971), on the space of all sub-σ-fields of a given σ-field, Allen proves the continuity of demand and other

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economic quantities. In this paper we restrict ourselves to purely atomic σ-fields or, equivalently, to countable partitions. This is done to drive home our point without using heavy tools of mathematical analysis and more importantly, since we are dealing with equilibria of games with incomplete information, to ensure a setup in which such equilibria always exist.

The continuity properties of equilibria that our newly defined metric exhibits are of the same nature and form as those of single-agent optimal plans with respect to the Boylan metric. We start then with the latter case.

The following statement describes continuity of optimal plans with respect to the Boylan metric and claims that the Boylan topology induced by this metric is the minimal one satisfying this continuity.

> *Two partitions are close iff, for any decision problem over the state space, a plan which is optimal under one of the partitions can, by changing it a little, be made an almost optimal plan under the other partition.*

By an "almost optimal plan" we mean one that pays the agent in expectation, given his information, almost the same as an optimal one (see §6).

An example will clarify this statement. The price of a stock is known to be uniformly distributed between \$0 and \$10. John knows whether it is below \$5 or above it. Suppose the payoffs are such that his optimal plan is to buy the stock if it is below \$5 and sell it otherwise. Consider now a different information structure. John knows whether the price is below \$5.02 or above it. The previous plan is not feasible with this information but a small change makes it feasible: Let John sell the stock when it is below \$5.02 and buy it otherwise. This plan may not be optimal but clearly it is almost optimal.

Consider now a game with incomplete information. We call a list of partitions, one for each player, an *information structure*. We want to define a metrizable topology on information structures such that the continuity of equilibria with respect to it will be analogous to the aforementioned continuity of single-agent optimal plans with respect to the Boylan topology. We simply substitute "information structures" for "partitions," "game" for "decision problem" and "equilibrium" for "optimal plan." Thus our objective is to define proximity of information structures by which:

> *Two information structures are close iff, for any game over the state space, an equilibrium of the game under one information structure can, by changing it a little, be made an almost equilibrium under the other information structure.*

An "almost equilibrium" is defined along the same lines as the almost optimal plan of a single agent. It is a strategy profile such that each player's strategy, given the player's information, pays him almost the same as his best response. Clearly our continuity requirement itself can serve as a definition of a topology on information structures. The point is that we want to express the topology in terms of the information structures themselves and not in terms of all possible games.

In §2 we examine the most natural candidate for the required topology, that is, the product topology on information structures, using the Boylan topology on each coordinate. As we show in Example 2 of this section the product topology fails to achieve the desired continuity: We present a sequence of information structures which converges in the product topology and a game for which continuity of equilibria fails. (A weaker continuity still holds for this topology, one in which almost equilibrium is defined in terms of ex-ante payoffs rather than ex-post payoffs which we use in our definition of continuity.)

Section 3 provides the standard definitions and notations for games with incomplete information. We introduce the notions of belief and common belief as well as some results concerning these concepts.

In §4 we take a closer look at common belief. We introduce new variants of this notion as it appears in the previous sections. General results about belief operators are presented and used to show that all the variants share some common features. These variants are used in the next section to define explicitly the metric that induces our topology.

In §5 we define, for a given pair of information structures, the event that each player considers his partitions in the two information structures to be close. The event that the previous event is common belief is used to formulate several conditions which describe convergence of information structures. The basic idea is that two information structures are close if there is a high probability that players share a common strong belief that all of them consider their partitions to be close. We then state formally a continuity property for this topology and prove that even a weaker one requires a topology as strong as the one we define.

In §6 we look at the case of a single agent, namely, one partition. The previous results apply to this as a special case and we show that the topology thus defined on partitions coincides with the Boylan topology. Moreover, the characterization of the general topology by means of continuity of equilibria gives rise, as a special case, to a characterization of the Boylan topology in terms of continuity of optimal plans.

2. Examples. Our first example shows a case in which information structures converge in the product topology (i.e., each player's partitions converge) and continuity of equilibrium does hold.

EXAMPLE 1. Consider the state space $\Omega = [0,1]$ equipped with the Lebesgue measure. Let $\Pi_1 = \{[0,\frac{1}{2}),[\frac{1}{2},1]\}$ and $\Pi_1^n = \{[0,\frac{1}{2}+\frac{1}{n}),[\frac{1}{2}-\frac{1}{n},1]\}$ be partitions of Ω for agent 1. For big n we consider the partitions Π_1 and Π_1^n to be close and indeed it is hard to think of any notion of proximity for which this is not the case. In particular under the Boylan metric $\Pi_1^n \to \Pi_1$. There are several ways to express the continuity property of optimal plans in a single-agent decision problem with respect to this convergence (most of them are equivalent). We follow the one that was given in the introduction and which can be easily generalized to games. Suppose that agent 1 faces a decision problem over Ω with payoffs bounded to the interval $[-1,1]$. That is, he has to choose an action where his utility depends on this action and the state ω. If δ is an optimal plan when the information is given by Π_1 then for each $n \geq 4$ there exists a plan δ_n which coincides with δ except for a set of measure $\frac{1}{n}$ and is $\frac{8}{n}$-optimal (i.e., given the information agent 1 has, he cannot improve upon δ_n by more than $\frac{8}{n}$). Indeed, suppose δ assigns action a in the interval $[0,\frac{1}{2})$ and action b in the interval $[\frac{1}{2},1]$. Consider the plan δ_n according to which a is selected over $[0,\frac{1}{2}+\frac{1}{n})$ and b over $[\frac{1}{2}+\frac{1}{n},1]$. Clearly δ_n coincides with δ every where except for the interval $[\frac{1}{2},\frac{1}{2}+\frac{1}{n})$ and δ_n can be improved upon, in each of the sets $[0,\frac{1}{2}+\frac{1}{n})$ and $[\frac{1}{2}-\frac{1}{n},1]$, by no more than $\frac{8}{n}$.

Let us now add another agent with information given by $\Pi_2 = \{[0,\frac{1}{3}),[\frac{1}{3},1]\}$ and consider the information structures $\Pi = (\Pi_1,\Pi_2)$, and $\Pi^n = (\Pi_1^n,\Pi_2)$. Here again it seems that any two games which differ only in their information structures, one having Π the other Π^n, are close for big n. The continuity of equilibria with respect to this closeness is very similar to the one we have for optimal plans in the case of a single-agent decision problem. Indeed suppose that Γ is a game with payoffs bounded to the interval $[-1,1]$, and let σ be an equilibrium of $\Gamma(\Pi)$. It is very easy to see that for each n there is a strategy σ^n which coincides with σ except maybe for a set of measure $\frac{1}{n}$ and is $\frac{8}{n}$-ex-post equilibrium in $\Gamma(\Pi^n)$ (i.e. that given his information no player can improve upon σ^n by more than $\frac{8}{n}$). The construction of σ^n is done precisely the same way δ_n was constructed for the decision problem.

The question we pose is what conditions on information structures Π and Π^n guarantee that the continuity property we have described above holds. This example may be misleading as to the simplicity of such conditions. The following example demonstrates the difficulties and hints to the solution. In particular it shows that it is not enough that the partitions of each agent are close in order that the information structures are.

EXAMPLE 2. Consider a two player game where the state space Ω is the set of integers $\{\dots,-2,-1,0,1,2,\dots\}$. Nature chooses $n \in \Omega$ with probability $\mu(n) = (\frac{1}{9})(\frac{4}{5})^{|n|}$. Both players have the same set of strategies {*Even*, *Odd*, *Cooperate*}. The payoffs depend on the actions and the state as follows. If a player chooses *Odd* than he is paid 6 if the state is odd and 0 if it is even, independently of the action of the other player. Similarly if he chooses *Even* he is paid 6 if the state is even and 0 otherwise, independently of the action of the other player. If both players choose *Cooperate* they are paid 4 each, while if only one of them play *Cooperate* he is paid 0.

Consider now the information structure $\Pi = (\Pi_1, \Pi_2)$ where

$$\Pi_1 = \{(2k, 2k+1) | k \text{ is a whole number}\} \quad \text{and}$$

$$\Pi_2 = \{(2k+1, 2k) | k \text{ is a whole number}\}.$$

The resulting game is denoted by $\Gamma(\Pi)$. The strategy σ according to which each player is choosing *Cooperate* regardless of the information he receives is an equilibrium of $\Gamma(\Pi)$. Under this strategy each player receives 4 in each state. to see that deviation to either *Odd* or *Even* is costly note that for each player the posterior probability that the state is odd or even is approximately $\frac{1}{2}$ (more precisely, $(\frac{4}{9},\frac{5}{9})$ or $(\frac{5}{9},\frac{4}{9})$ depending on whether the state is positive or negative). Thus by choosing, in a given information set, *Odd* or *Even* a player can expect approximately $\frac{1}{2}6 + \frac{1}{2}0 = 3$ which is less than the 4 he receives by playing σ.

We change now the information structure of the game and consider games $\Gamma(\Pi^n)$ where the information structure $\Pi^n = (\Pi_1^n, \Pi_2^n)$ is given by

$$\Pi_1^n = \Pi_1 \setminus \{(2n, 2n+1)\} \cup \{(2n)\} \cup \{(2n+1)\} \quad \text{and} \quad \Pi_2^n = \Pi_2.$$

That is, 2's information has not changed and 1's information has changed by splitting his information set $(2n, 2n+1)$ into two.

First we note that for big n, Π_1 and Π_1^n are close in the sense explained above. For any decision problem of agent 1, any optimal plan when information is given by Π_1 can be made by small changes an almost optimal plan under Π_1^n. (In this example the second "almost" is not required. By changing the optimal plan over $(2n)$ and $(2n+1)$, we get an optimal plan for Π_1^n).

Examine now the game $\Gamma(\Pi^n)$. This game has a unique equilibrium. In state $2n$ player 1 can guarantee 6 by playing *Even* and this is the only strategy that pays him so well so he must play *Even* in $2n$. When player 2 is informed $(2n-1, 2n)$ he assigns a probability of approximately $\frac{1}{2}$ that player 1 is playing *Even* and therefore if he plays *Cooperate* he can expect $\frac{1}{2}4 + \frac{1}{2}0 = 2$ (if the other play *Cooperate*) or $\frac{1}{2}0 + \frac{1}{2}0 = 0$ (if the other did not play *Cooperate*). By playing *Even* he can guarantee approximately 3 and therefore this is what he will play. This argument, extended by induction, shows that in the unique equilibrium of this game none of the player plays *Cooperate* in any state. Moreover a simple but tedious computation shows that even if players only ϵ optimize in each state, where $\epsilon \leq 0.1$ than in any ϵ-equilibrium of $\Gamma(\Pi^n)$ the probability of playing *Cooperate* does never accede 0.04 no matter how big n is. Thus, unlike the previous example, an equilibrium of $\Gamma(\Pi)$ cannot be approached by ϵ-equilibria of the games $\Gamma(\Pi^n)$.

We are therefore led to the conclusion that Π and Π^n should not be considered close even for big ns. This is despite the fact the partitions of player 1 are the same in both information structures and those of player 2 are close. The phenomenon observed here is that the small, local, difference between the personal information of individuals in Π_n and Π, propagates in the whole space by the interlacing of partitions of different agents. We use here the notion of common belief to express and measure this propagation effect. It was used in Monderer and Samet (1990) to give a full account of how lack of cooperation propagates in the electronic mail game described by Rubinstein (1989). Such propagation effects are studied in a different way by Sorin (1994). We discuss this example again, at the end of §4, after introducing the formal definition of metric on information structures and stating the continuity properties of equilibrium with respect to this metric.

3. Preliminaries. *States and partitions:* We fix throughout the discussion a measure space (Ω, Σ, μ), where Ω is the **state space**, Σ is a σ field, and μ a probability measure. Denote by Σ^* the subset of Σ which consists of all non-null sets. Let $\mathscr{P}$ be the space of all partitions of Ω, the elements of which are in Σ^*. For each $\Pi \in \mathscr{P}$ and $\omega \in \Omega$, we denote by $\Pi(\omega)$ the element of Π which contains ω. The σ field generated by Π is denoted by Σ_Π.

Games and information structures: A finite set N is the set of **players**. An element of the form $\Pi = (\pi_i)_{i \in N}$ in $\mathscr{P}^N$ is called an **information structure**. A **game** Γ consists of the following elements. For each player $i \in N$ there is a finite set A_i of **actions**, where we denote $A = \Pi_{i \in N} A_i$, a partition $\Pi_i \in \mathscr{P}$ and a **payoff function** u_i: $A \times \Omega \to R$ which is bounded and for each $a \in A$, $u_i(a, \cdot)$ is measurable. By $\Gamma(\Pi')$ we denote the game which is the same as Γ except that the information structure is Π'. The set of **mixed actions** of player i (i.e. probability distribution over A_i) is denoted by Δ_i and we write Δ for the set of mixed actions $\Pi_{i \in N} \Delta_i$. The payoff functions u_i are extended naturally to $\Delta \times \Omega$. We write u for $(u_i)_{i \in N}$. For $x \in R^N$, $\|x\|$ is the max norm of x. We say that Γ is bounded by M if $\|u\| \le M$.

Strategies and equilibria: An **individual strategy** of player i in the game $\Gamma = \Gamma(\Pi)$ is a function σ_i: $\Omega \to \Delta_i$ which is Π_i-measurable. $\sigma = (\sigma_i)_{i \in N}$ is called a **strategy**. For a strategy $\sigma = (\sigma_i)_{i \in N}$ and an individual strategy σ_i', we write $(\sigma | \sigma_i')$ for the strategy which results from replacing σ_i by σ_i'. We extend the payoff functions to strategies by defining it as the expected payoff, i.e., $u_i(\sigma) = E(u_i(\sigma(\cdot), \cdot))$. A strategy σ is an **equilibrium** if for each $i \in N$ and individual strategy σ_i' of i, $u_i(\sigma) \ge u_i(\sigma | \sigma_i')$. It is an **$\epsilon$-ex-post equilibrium** if σ is ϵ-optimal in each information set, that is if for each i and σ_i',

$$E\big(u_i(\sigma(\cdot), \cdot) | \Sigma_{\Pi_i}\big) \ge E\big(u_i((\sigma | \sigma_i')(\cdot), \cdot) | \Sigma_{\Pi_i}\big) - \epsilon.$$

This equilibrium should be contrasted with ϵ-ex-ante equilibrium in which for each i and σ_i', $u_i(\sigma) \ge u_i(\sigma | \sigma_i') - \epsilon$. Observe that there is no distinction between ex-post and ex-ante for equilibria, since $u_i(\sigma) \ge u_i(\sigma | \sigma_i')$ iff

$$E\big(u_i(\sigma(\cdot), \cdot) | \Sigma_{\Pi_i}\big) \ge E\big(u_i((\sigma | \sigma_i')(\cdot), \cdot) | \Sigma_{\Pi_i}\big).$$

Beliefs and common beliefs: We say that player i p-**believes** event E at state ω, if i assigns a posterior probability of at least p to E in state ω, i.e., if $\mu(E | \Pi_i(\omega)) \ge p$.

We denote the event that player i, p-believes E by $P^p_{\Pi_i}(E)$ (or $B^p_i(E)$ when no confusion may result) i.e.,

$$B^p_{\Pi_i}(E) = \{\omega | \mu(E|\Pi_i(\omega)) \geq p\}.$$

The event that all players p-believe E at ω is denoted by $B^p_\Pi(E)$ (or simply B^p when Π is clear from the context), that is

$$B^p_\Pi(E) = \bigcap_{i \in N} B^p_{\Pi_i}(E).$$

We can apply B^p iteratively and thus for example $B^p(B^p(E)) = (B^p)^2(E)$ is the event that all p-believe that all p-believe that E. The event that E is **common p-belief** is

$$C^p_\Pi(E) = \bigcap_{n \geq 1} (B^p_\Pi)^n(E).$$

A state ω is in $C^p_\Pi(E)$ iff all p-believe that all p-believe ... that all p-believe that E, for any number of iterations of "all p-believe that."

4. **Beliefs and common beliefs.** In order to define a metric on $\mathscr{P}^N$ we introduce two variants of common belief—the common repeated belief and the joint common repeated belief. We also define acceptance operators which are useful in handling lack of common belief.

We start with some general remarks concerning operators on Σ. Consider two properties of an operator B: $\Sigma \to \Sigma$.

(a) **Continuous monotonicity**: If $E_n \downarrow E$ (i.e., E_n is a decreasing sequence and $\bigcap E_n = E$) then $B(E_n) \downarrow B(E)$.

(b) **Subpotency**: $(B)^2(E) \subseteq B(E)$.

Any continuously monotonic, subpotent operator B: $\Sigma \to \Sigma$ is called a **belief operator**. It is easy to verify that $B^p_{\Pi_i}$, B^p_Π and C^p_Π are all belief operators. Note also that these three operators satisfy also a third condition:

(c) **Continuous monotonicity in** p: If $p_n \uparrow p$ then $B^{p_n}(E) \downarrow B^p(E)$.

Continuous monotonicity implies simple monotonicity. That is, if $E \subseteq F$ than $B(E) \subseteq B(F)$. The interpretation of this is straight forward: If event E implies event F then if E is believed so must be F. Note that continuous monotonicity holds only for decreasing sequences and not for increasing ones. Subpotency means that beliefs concerning beliefs are always correct. That is, if one believes that he believes E then indeed he does believe E.

The following proposition is proved in Monderer-Samet (1989).

PROPOSITION 4.1. *Let B be a belief operator and E an event. Then $\omega \in \bigcap_{n \geq 1} B^n(E)$ iff there exists an event S such that $\omega \in S$ and*

$$S \subseteq B(S) \cap B(E). \tag{4.1}$$

Moreover, (4.1) *is satisfied with equality by $\cap_{n \geq 1} B^n(E)$ as S.*

For a fixed $E \in \Sigma$ consider the operator $B^p_i(E \cap \cdot)$ which is defined by $B^p_i(E \cap \cdot)(F) = B^p_i(E \cap F)$. Denote also by $B^p_\Pi(E \cap \cdot)$ the intersection $\bigcap_{i \in N} B^p_i(E \cap \cdot)$. The event that E is **common repeated** p**-belief** is:

$$\hat{C}^p_\Pi(E) = \bigcap_{n \geq 1} (B^p_\Pi(E \cap \cdot))^n(E).$$

Thus $\omega \in \hat{C}^p_\Pi(E)$ iff at ω, all p-believe that E and that all p-believe that E and that... all p-believe that E, for any number of iterations of "all p-believe that E and that." The "repeated" in this definition refers to the repetition of the belief in E in each iteration.

PROPOSITION 4.2. *$\omega \in \hat{C}^p_\Pi(E)$ iff there exists an event S such that $\omega \in S$ and*

$$S \subseteq B^p_\Pi(S \cap E). \tag{4.2}$$

Moreover, (4.2) *is satisfied with equality by $\hat{C}^p_\Pi(E)$ as S.*

PROOF. It is easy to see that $B^p_i(E \cap \cdot)$ is a belief operator and therefore by Proposition 4.1 for any F, $\omega \in \bigcap_{n\geq 1}[B^p_\Pi(E \cap \cdot)]^n(F)$ iff there exists an event S such that $\omega \in S$ and

$$S \subseteq B^p_\Pi(E \cap S) \cap B^p_\Pi(E \cap F). \tag{4.3}$$

Moreover, (4.3) is satisfied with equality by $\bigcap_{n\geq 1}[B^p_\Pi(E \cap \cdot)]^n(F)$ as S. Substituting E for F in (4.3) and noting that $B^p_\Pi(E \cap \cdot)$ is monotonic we see that (4.3) is equivalent to (4.2). □

We want now to express the fact that an event E is common p-believed simultaneously in two information structures Π^0 and Π^1. This is done by looking at the operator $B^p_{\Pi^0,\Pi^1}$ which is defined for each E by:

$$B^p_{\Pi^0,\Pi^1}(E) = B^p_{\Pi^0}(E) \cap B^p_{\Pi^1}(E).$$

Using $B^p_{\Pi^0,\Pi^1}$ we define for a given E, $B^p_{\Pi^0,\pi^1}(E \cap \cdot)$ by:

$$B^p_{\Pi^0,\Pi^1}(E \cap \cdot) = B^p_{\Pi^0}(E \cap \cdot) \cap B^p_{\Pi^1}(E \cap \cdot).$$

Thus $B^p_{\Pi^0,\Pi^1}(E \cap \cdot)(F)$ is the event that all p-believe E and F under either information structure. The event that E is **joint common repeated p-belief** is:

$$\hat{C}^p_{\Pi^0,\Pi^1}(E) = \bigcap_{n\geq 1} (B^p_{\Pi^0,\Pi^1}(E \cap \cdot))^n(E).$$

Clearly $\omega \in \hat{C}^p_{\Pi^0,\Pi^1}(E)$ whenever at ω all p-believe under either information structure that E and that all p-believe under either information structure that E and that... all p-believe under either information structure that E, for any number of iterations of "all p-believe under either information structure that E and that."

PROPOSITION 4.3. *$\omega \in \hat{C}^p_{\Pi^0,\Pi^1}(E)$ iff there exists an event S such that $\omega \in S$ and*

$$S \subseteq B^p_{\Pi_0,\Pi^1}(S \cap E) = B^p_{\Pi^0}(S \in E) \cap B^p_{\Pi^1}(S \in E). \tag{4.4}$$

Moreover (4.4), *with equality, is satisfied by $\hat{C}^p_{\Pi^0,\Pi^1}(E)$ as S.*

PROOF. Consider the information structure Π which consists of the $2|N|$ partitions in Π^0 and Π^1. Then $B^p_{\Pi^0,\Pi^1} = B^p_\Pi$ and the claim follows from Proposition 4.3. □

We omit the easy proof of the following proposition.

PROPOSITION 4.4. *$\hat{C}^p_\Pi$ and $\hat{C}^p_{\Pi^0,\Pi^1}$ are belief operators and also continuously monotonic in p.*

In order to describe the event that some other event is not common p-belief we introduce a new operator. We say that player i **p-accepts** E at ω if $(E|\Pi_i(\omega)) > p$. Denote by $A^p_{\Pi_i}(E)$ the event that i p-accepts A, i.e.,

$$A^p_{\Pi_i}(E) = \{\omega | \mu(E|\Pi_i(\omega)) > p\}.$$

The operator $A^p_{\Pi_i}$ is very similar to $B^p_{\Pi_i}$ except that unlike the latter it is defined with a strong inequality. But we are using acceptance here in its statistical meaning and prefer to think of p as being small (as opposed to the p in the belief operator). Thus if we think of E as being a null hypothesis then it is enough that its probability is small, say 0.05, in order that we accept it or rather not reject it.

Clearly, $\overline{B^p_{\Pi_i}(E)} = A^{\bar{p}}_{\Pi_i}(\bar{E})$ where we denote by $\bar{X}$ the complement of an event X and $\bar{p} = 1 - p$. The event that someone p-accepts E is obviously $\bigcup_{i \in N} A^p_{\Pi_i}(E)$, which we denote by $A^p_\Pi(E)$. Here again $\overline{B^p_\Pi(E)} = A^{\bar{p}}_\Pi(\bar{E})$. It easily follows now that:

$$\overline{C^p_\Pi(E)} = \bigcup_{n \geq 1} (A^{\bar{p}}_\Pi)^n(\bar{E}).$$

Thus E is not common p-belief at ω if someone $\bar{p}$-accepts that someone $\bar{p}$-accepts that... that someone $\bar{p}$-accepts $\bar{E}$, for some finite repetition of "someone $\bar{p}$-accepts." The next proposition follows easily from the definition of $A^p_{\Pi_i}$.

PROPOSITION 4.5. *The operators $A^p_{\Pi_i}$ and A^p_Π are continuously monotonic w.r.t. increasing sequences of events. That is, if $E_n \uparrow E$ then $A^p_{\Pi_i}(E) \uparrow A^p_{\Pi_i}(E)$ and $A^p_\Pi(E_n) \uparrow A^p_\Pi(E)$.*

5. The main results. As was demonstrated in Example 2 of §2, two information structures Π and Π' may differ very much game theoretically even when for each i the partitions Π_i and Π'_i are close. We want to define proximity of information structures such that if Π and Π' are close then for any game $\Gamma(\Pi)$, each equilibrium of this game can be approximated by some almost-equilibrium of $\Gamma(\Pi')$. We formalize in this section the following notion of proximity and show its adequacy to our purpose.

> Π *and* Π' *are close if, with high probability, there is a common strong belief that the information each i receives under the partition* Π_i *is almost the same as the one he receives under the partition* Π'_i.

Moreover, we will show that this notion of proximity is also necessary to guarantee the continuity of equilibria that we require.

Since common p-beliefs are applied to events we have to define first the event that the information each i receives under the partition Π_i is almost the same as the one he receives under the partition Π'_i. Clearly ω is in this event if for each i, the sets $\Pi_i(\omega)$ and $\Pi'_i(\omega)$ are very "close." In order to grasp this latter notion of closeness we define a pseudo metric on events which expresses sameness of information. We cannot use the standard pseudo metric on Σ which for given two sets A and B is the measure of the symmetric difference of the sets $\mu((A \setminus B) \cup (B \setminus A))$. Any two small sets are considered close by this metric. But at a given ω in $A \cap B$ the information given by A may be viewed very differently from that given by B, despite the fact that the sets are small. We are looking thus for a metric that measures the differences between the sets, ex-post, i.e. relative to the size of the sets. Therefore we take into account $\mu(A \setminus B|A)$ and $\mu(B \setminus A|B)$. When these terms are small it means that most of A is in B, and most of B is in A.

Formally, we define a pseudo metric d_0 on Σ^* such that for any $A, B \in \Sigma^*$

$$d_0(A, B) = \min\{\max\{\mu(A \setminus B|A), \mu(B \setminus A|B)\}, \tfrac{1}{2}\}.$$

The half in this formula is there to guarantee that d_0 is bounded by $\frac{1}{2}$. This bound is required for the triangle inequality. Since we are interested only in convergence in this metric it does not reflect anywhere in the sequel.

PROPOSITION 5.1. *d_0 is a pseudo metric on Σ^*.*

PROOF. We need only to prove the triangle inequality $d(A,C) \leq d(A,B) + d(B,C)$ for any measurable sets A, B and C. Denote $\mu(A \setminus B|A) = a_1$, $\mu(B \setminus A|B) = a_2$, $\mu(C \setminus B|C) = c_1$, $\mu(B \setminus C|B) = c_2$. We may assume that $a_1, a_2, c_1, c_2 < \frac{1}{2}$, otherwise the inequality holds trivially. Note that

$$(1 - a_1)\mu(A) = \mu(A \cap B) \quad \text{and} \quad (1 - a_2)\mu(B) = \mu(A \cap B).$$

Similar equations hold for B and C. Let

$$a = (1 - a_1)/(1 - a_2) \quad \text{and} \quad c = (1 - c_1)/(1 - c_2).$$

Then,

$$\mu(B) = a\mu(A), \tag{5.1}$$

$$\mu(B) = c\mu(C). \tag{5.2}$$

Now, $\mu(A \setminus C) \leq \mu(A \setminus B) + \mu(B \setminus C) = a_1\,\mu(A) + c_2\,\mu(B)$. Substituting $\mu(B)$ from (5.1) we have $\mu(A \setminus C) \leq \mu(A)(a_1 + c_2 a)$ and therefore,

$$\mu(A \setminus C|A) \leq a_1 + c_2 a. \tag{5.3}$$

Similarly,

$$\mu(C \setminus A|C) \leq c_1 + a_2 c. \tag{5.4}$$

It is enough now to prove that,

$$\max\{a_1 + c_2 a, c_1 + a_2 c\} \leq \max\{a_1, a_2\} + \max\{c_1, c_2\}.$$

Because of the symmetry it suffices to show that the first element on the left hand side is less than the right hand side. We examine two cases. If $a_1 \geq a_2$ then $a \leq 1$ and hence,

$$a_1 + c_2 a \leq a_1 + c_2 \leq \max\{a_1, a_2\} + \max\{c_1, c_2\}.$$

Suppose now that $a_1 < a_2$. It is enough to show that, $a_1 + c_2 a \leq a_2 + c_2$ or equivalently that, $c_2(a - 1) \leq a_2 - a_1$. This is reduced to $c_2(a_2 - a_1)/(1 - a_2) \leq a_2 - a_1$, or $c_2 \leq 1 - a_2$, which follows from the assumption that all as and bs are less than $\frac{1}{2}$. □

Now let Π and Π' be information structures and ω a state. Using this pseudo metric d_0 we can easily measure the indifference of player i between Π_i and Π'_i at ω. This will simply be $d_0(\Pi_i(\omega), \Pi'_i(\omega))$. Thus we say that Π_i and Π_i are **ϵ-close** at ω, if $d_0(\Pi_i(\omega), \Pi'_i(\omega)) \leq \epsilon$. The set of states at which all players' partitions are

ϵ-close is denoted by $I_{\Pi,\Pi'}(\epsilon)$, i.e.,

$$I_{\Pi,\Pi'}(\epsilon) = \bigcap_{i \in N} \{\omega | d_0(\Pi_i(\omega), \Pi'_i(\omega)) \le \epsilon\}.$$

Note that $I_{\Pi,\Pi'}(\epsilon)$ is continuously monotonic in ϵ w.r.t. decreasing sequences, i.e., if $\epsilon_n \downarrow \epsilon$ then $I_{\Pi,\Pi'}(\epsilon_n) \downarrow I_{\Pi,\Pi'}(\epsilon)$.

After identifying the event that the individual partitions are almost the same with the set $I_{\Pi,\Pi'}(\epsilon)$, we go back to the informal definition of proximity that we gave in the beginning of this section. Three parameters determine the proximity of information structures Π and Π'. The measure of sameness of private information—ϵ, the strength of the common belief in this sameness—p, and the probability that such common belief holds—$\mu(C^p_\Pi(I_{\Pi,\Pi'}(\epsilon)))$. The two information structures are closer, the closer is the first parameter, ϵ, to 0 and the other two to 1. The following theorem presents several ways in which a topology on $\mathscr{P}^N$ can be described using these parameters, and states that this topology is metrizable.

THEOREM 5.2. *There exists a pseudo metric d on $\mathscr{P}^N$ such that the following conditions are equivalent*:

(a) $d(\Pi, \Pi^n) \to 0$.

(b) *There exists sequences $p_n \to 1$ and $\epsilon_n \to 0$ such that* $\mu(C^{p_n}_\Pi(I_{\Pi,\Pi^n}(\epsilon_n))) \to 1$.

(c) *For all sequences $p_n \to 1$ and $\epsilon_n \to 0$* $\mu(C^{p_n}_\Pi(I_{\Pi,\Pi^n}(\epsilon_n))) \to 1$.

(d) *There exist sequences $p_n \to 1$ and $\epsilon_n \to 0$ such that* $\mu(C^{p_n}_{\Pi^n}(I_{\Pi,\Pi^n}(\epsilon_n))) \to 1$.

(e) *For all sequences $p_n \to 1$ and $\epsilon_n \to 0$* $\mu(C^{p_n}_{\Pi^n}(I_{\Pi,\Pi^n}(\epsilon_n))) \to 1$.

PROOF. For the purpose of constructing the pseudo metric d we need the stronger notion of common belief $\hat{C}^p_{\Pi^0,\Pi^1}$. Consider the set $\hat{C}^{1-\epsilon}_{\Pi^0,\Pi^1}(I_{\Pi^0,\Pi^1}(\epsilon))$—the event that $I_{\Pi^0,\Pi^1}(\epsilon)$ is joint common repeated $(1-\epsilon)$-belief. For small ϵ there is a strong joint common repeated belief over this set that the players are very indifferent between the two information structures. Consider further the intersection

$$D_{\Pi^0,\Pi^1}(\epsilon) = \hat{C}^{1-\epsilon}_{\Pi^0,\Pi^1}(I_{\Pi^0,\Pi^1}(\epsilon)) \cap I_{\Pi^0,\Pi^1}(\epsilon).$$

Note that both $I_{\Pi^0,\Pi^1}(\epsilon)$ and $\hat{C}^{1-\epsilon}_{\Pi^0,\Pi^1}$ are continuously monotonic in ϵ w.r.t. decreasing sequences and $\hat{C}^{1-\epsilon}_{\Pi^0,\Pi^1}$ is continuously monotonic w.r.t. its arguments and therefore it follows that $D_{\Pi^0,\Pi^1}(\epsilon)$ is also continuously monotonic in ϵ w.r.t. decreasing sequences.

We define now the pseudo metric d on $\mathscr{P}^n$ such that for each $\Pi^0, \Pi^1 \in \mathscr{P}^n$

$$d(\Pi^0, \Pi^1) = \min\{\epsilon | \mu(D_{\Pi^0,\Pi^1}(\epsilon)) \ge 1 - \epsilon\}.$$

Observe that the continuous monotonicity of D_{Π^0,Π^1} and μ guarantee that the minimum in the definition of d is attained.

PROPOSITION 5.2.1. *d is a pseudo metric on $\mathscr{P}^n$.*

In the proof of this proposition we use the following two lemmata.

LEMMA 5.2.2. *Suppose $\epsilon < \frac{1}{2}$. If $\omega \in I_{\Pi^0,\Pi^1}(\epsilon)$ then for each $i \in N$,*

$$I_{\Pi^0,\Pi^1}(\epsilon) \cap \Pi^0_i(\omega) = I_{\Pi^0,\Pi^1}(\epsilon) \cap \Pi^1_i(\omega).$$

PROOF. For such an ω, $d_0(\Pi_i^0(\omega), \Pi_i^1(\omega)) \le \epsilon$ for each $i \in N$. Let $\omega' \in \Pi_i^0(\omega) \setminus \Pi_i^1(\omega)$. Then

$$d_0(\Pi_i^0(\omega'), \Pi_i^1(\omega')) = d_0(\Pi_i^0(\omega), \Pi_i^1(\omega')) \ge \mu(\Pi_i^0(\omega) \setminus \Pi_i^1(\omega') | \Pi_i^0(\omega))$$
$$\ge \mu(\Pi_i^0(\omega) \cap \Pi_i^1(\omega) | \Pi_i^0(\omega)) \ge 1 - \epsilon \ge \epsilon.$$

Thus $\omega' \notin I_{\Pi^0, \Pi^1}(\epsilon)$. □

LEMMA 5.2.3. *If* $\mu(A_1 | B) \ge 1 - \epsilon_1$ *and* $\mu(A_2 | B) \ge 1 - \epsilon_2$, *then* $\mu(A_2 | A_1 \cap B) \ge (1 - \epsilon_1 - \epsilon_2)/(1 - \epsilon_1)$.

PROOF. Denote

$$\alpha = \mu(\bar{A}_1 \cap \bar{A}_2 | B), \qquad \beta = \mu(A_1 \cap \bar{A}_2 | B),$$
$$\gamma = \mu(\bar{A}_1 \cap A_2 | B), \qquad \delta = \mu(A_1 \cap A_2 | B).$$

Clearly these numbers sum to 1, $\alpha + \beta \le \epsilon_2$ and $\alpha + \gamma \le \epsilon_1$. We want to show that $\delta/(\beta + \delta) \ge (1 - \epsilon_1 - \epsilon_2)/(1 - \epsilon_1)$, which is equivalent to $\beta(1 - \epsilon_1 - \epsilon_2) \le \delta\epsilon_2$. Using the above inequalities we conclude

$$\beta(1 - \epsilon_1 - \epsilon_2) \le \beta(1 - (\alpha + \beta) - (\alpha + \gamma))$$
$$= \beta(\delta - \alpha) \le \beta\delta \le \delta(\alpha + \beta) \le \delta\epsilon_2 \quad \square$$

PROOF OF PROPOSITION 5.2.1. We need to prove the triangle inequality. Suppose that $d(\Pi^0, \Pi^j) = \epsilon_j$ for $j = 1, 2$. Since d is bounded by $\frac{1}{2}$ it is enough to consider the case that $\epsilon = \epsilon_1 + \epsilon_2 < \frac{1}{2}$ for $j = 1, 2$. Denote for $j = 1, 2$:

$$I_j = I_{\Pi^0, \Pi^j}(\epsilon), \qquad C_j = \hat{C}^{1-\epsilon_j}_{\Pi^0, \Pi^j}(I_j), \qquad D_j = C_j \cap I_j,$$

and also $D = D_1 \cap D_2$ and $I = I_{\Pi^1, \Pi^2}(\epsilon)$. By the given distances it follows that $\mu(D_j) \ge 1 - \epsilon_j$ and therefore $\mu(D) \ge 1 - \epsilon$. We will show that:

$$D \subseteq D_{\Pi^1, \Pi^2}(\epsilon) \tag{5.5}$$

and thus $\mu(D_{\Pi^1, \Pi^2}(\epsilon)) \ge 1 - \epsilon$ which proves that $d(\Pi^1, \Pi^2) \le \epsilon$. We observe first that if $\omega \in I_1 \cap I_2$, then for each $i \in N$, and $j = 1, 2$, $d(\Pi_i^0(\omega), \Pi_i^j(\omega)) \le \epsilon_j$ and by the triangle inequality $d(\Pi_i^1(\omega), \Pi_i^2(\omega)) \le \epsilon$. Hence $I_1 \cap I_2 \subseteq I$ and therefore $D \subseteq I$. Since $D_{\Pi^1, \Pi^2}(\epsilon) = \hat{C}^{1-\epsilon}_{\Pi^1, \Pi^2}(I) \cap I$ we need only to prove $D \subseteq \hat{C}^{1-\epsilon}_{\Pi^1, \Pi^2}(I)$ in order to show (5.5). By Proposition 4.3 it suffices to show that

$$D \subseteq B^{1-\epsilon}_{\Pi^1}(D \cap I) \cap B^{1-\epsilon}_{\Pi^2}(D \cap I).$$

By symmetry it is enough to prove that $D \subseteq B^{1-\epsilon}_{\Pi^1}(D \cap I)$. To see this we have to show that for each $\omega \in D$ and $i \in N$, $\mu(D \cap I | \Pi_j^1(\omega)) \ge 1 - \epsilon$. But $D \subseteq I$ and therefore the last inequality is reduced to:

$$\mu(D | \Pi_i^1(\omega)) \ge 1 - \epsilon. \tag{5.5}$$

Since $\omega \in D \subseteq C_j$ it follows from Proposition 4.3 that for $j = 1, 2$

$$\mu(D_j | \Pi_i^0(\omega)) \ge 1 - \epsilon_j. \tag{5.7}$$

By Lemma 5.2.3,

$$\mu(D_2|D_1 \cap \Pi_i^0(\omega)) \geq (1-\epsilon)/(1-\epsilon_1). \tag{5.8}$$

Now,

$$\mu(D|\Pi_i^1(\omega)) = \mu(D_1 \cap D_2|\Pi_i^1(\omega)) = \mu(D_2|D_1 \cap \Pi_i^1(\omega))\mu(D_1|\Pi_i^1(\omega)). \tag{5.9}$$

But $\omega \in I_1$ and thus by Lemma 5.2.2, $I_1 \cap \Pi_i^0(\omega) = I_1 \cap \Pi_i^1(\omega)$, and therefore $C_1 \cap I_1 \cap \Pi_i^0(\omega) = C_1 \cap I_1 \cap \Pi_i^1(\omega)$, i.e., $D_1 \cap \Pi_i^0(\omega) = D_1 \cap \Pi_i^1(\omega)$. Substituting $D_1 \cap \Pi_i^0(\omega)$ for $D_1 \cap \Pi_i^1(\omega)$ in (5.9) gives:

$$\mu(D|\Pi_i^1(\omega)) = \mu(D_2|D_1 \cap \Pi_i^0(\omega))\mu(D_1|\Pi_i^1(\omega)).$$

By (5.7) and (5.8) the last equality guarantees

$$\mu(D|\Pi_i^1(\omega)) \geq (1-\epsilon_1)(1-\epsilon)/(1-\epsilon_1) = 1-\epsilon.$$

This is the inequality (5.6) which was needed to complete the proof. □

After constructing the pseudo metric d we turn now to prove the equivalences in Theorem 5.2. We use the following lemma which gives a bound to the distance between two information structures Π and Π' in terms of p, ϵ and $\mu(C_\Pi^p(I_{\Pi,\Pi''}(\epsilon)))$.

LEMMA 5.2.4. *Let* $\epsilon < \frac{1}{2}$ *and denote* $\delta = (2p-1)(1-\epsilon)$ *and* $\eta = (2p-1)\mu(C_\Pi^p(I_{\Pi,\Pi'}(\epsilon)))$. *Then* $d(\Pi, \Pi') \leq \max\{\epsilon, 1-\delta, 1-\eta\}$.

PROOF. Denote $I = I_{\Pi,\Pi'}(\epsilon)$, $C = C_\Pi^p(I_{\Pi,\Pi'}(\epsilon))$ and $D = C \cap I$. We first find a lower bound for the measure of D. By Propositions 3.1 and 3.2,

$$C = B_\Pi^p(C) \cap B_\Pi^p(I) \subseteq B_\Pi^{2p-1}(C \cap I) = B_\Pi^{2p-1}(D). \tag{5.10}$$

By Proposition 3.3, for each i, $\mu(D) \geq (2p-1)\mu(B_{\Pi_i}^{2p-1}(D))$, and since $B_\Pi^{2p-1}(D) \subseteq B_{\Pi_i}^{2p-1}(D)$ it follows using (5.10) that

$$\mu(D) \geq (2p-1)\mu(C) = \eta. \tag{5.11}$$

Now we evaluate the posterior of D at each $\omega \in C$ with respect to Π'. Since $\omega \in I$, $d_0((\Pi_i(\omega), \Pi_i'(\omega)) \leq \epsilon$ for each i and therefore,

$$\mu(\Pi_i(\omega)) \geq \mu(\Pi_i(\omega) \cap \Pi_i'(\omega)) \geq (1-\epsilon)\mu(\Pi_i'(\omega)). \tag{5.12}$$

By Lemma 5.2.2, $D \cap \Pi_i(\omega) = D \cap \Pi_i'(\omega)$. Using this equality, (5.12) and (5.10) we conclude:

$$\begin{aligned}
\mu(D|\Pi_i'(\omega)) &= \mu(D \cap \Pi_i'(\omega))/\mu(\Pi_i'(\omega)) \\
&= \mu(D \cap \Pi_i(\omega))/\mu(\Pi_i'(\omega)) \\
&\geq (1-\epsilon)\mu(D \cap \Pi_i(\omega))/\mu(\Pi_i(\omega)) \\
&= \mu(D|\Pi_i(\omega))(1-\epsilon) \\
&\geq (2p-1)(1-\epsilon) \\
&= \delta,
\end{aligned}$$

i.e., $\omega \in B^{\delta}_{\Pi'}(D)$. This is true for all $\omega \in C$ and hence:

$$C \subseteq B^{\delta}_{\Pi'}(D). \tag{5.13}$$

By (5.13), (5.10), the monotonicity of C^{p}_{Π} in p and since $\delta \leq 2p - 1$: $D \subseteq C \subseteq B^{2p-1}_{\Pi}(D) \cap B^{\delta}_{\Pi'}(D) \subseteq B^{\delta}_{\Pi,\Pi'}(D)$. Therefore by Proposition 4.3:

$$D \subseteq \hat{C}^{\delta}_{\Pi,\Pi'}(D). \tag{5.14}$$

Since $D \subseteq I$ we have from (5.14) and the monotonicity of $\hat{C}$ and $D_{\Pi,\Pi'}$,

$$D \subseteq \hat{C}^{\delta}_{\Pi,\Pi'}(I) \cap I \subseteq D_{\Pi,\Pi'}(\max\{\epsilon, 1 - \delta\}). \tag{5.15}$$

By (5.15) and (5.11): $\mu(D_{\Pi,\Pi'}(\max\{\epsilon, 1 - \delta\}) \geq \eta$, which by the monotonicity of $D_{\Pi,\Pi'}$ shows that $d(\Pi, \Pi') \leq \max(\epsilon, 1 - \delta, 1 - \eta)$. □

We complete the proof of Theorem 5.2. Suppose $d(\Pi, \Pi^n) \to 0$, then there exist a sequence $\epsilon_n \to 0$ such that $\mu(D_{\Pi,\Pi^n}(\epsilon_n)) \geq 1 - \epsilon_n$. By Proposition 4.3, $D_{\Pi,\Pi^n}(\epsilon_n) \subseteq B^{1-\epsilon}_{\Pi}(I_{\Pi,\Pi^n}(\epsilon_n))$, and therefore (b) follows for $p_n = 1 - \epsilon_n$. The same argument applies for (d). (c) and (e) follow by the continuous monotonicity of all the functions involved. Clearly (c) implies (b) and (e) implies (d). Each of (c) and (e) imply (a) by Lemma 5.2.4. □

Note that common belief in the similarity of private information can be measured in either the fixed information structure Π (in (a) and (b)) or from the information structures Π^n which approach Π (in (c) and (d)).

With this topology we now can prove the following continuity properties.

THEOREM 5.3. *For each $\epsilon > 0$ there exists $\delta > 0$ such that for any information structures Π and Π', every game Γ and any equilibrium σ of $\Gamma(\Pi)$, if $d(\Pi, \Pi') \leq \delta$ then $\Gamma(\Pi')$ has an ϵM-equilibrium σ' such that $\mu(\{\omega | \sigma(\omega) \neq \sigma'(\omega)\}) \leq \epsilon$, and therefore also $\mu(\{\omega | u(\sigma(\omega), \omega) \neq u(\sigma'(\omega), \omega)\}) \leq \epsilon$, where M is a bound on the payoffs in Γ.*

PROOF. Let $\delta = \frac{\epsilon}{4}$ and suppose $d(\Pi, \Pi') \leq \delta$. Without loss of generality $\delta < \frac{1}{4}$. Denote $D = D_{\Pi,\Pi'}(\delta)$ and $I = I_{\Pi,\Pi'}(\delta)$. Let Γ be a game with payoffs bounded by M and suppose σ is an equilibrium of $\Gamma(\Pi)$.

We construct a strategy σ' for the game $\Gamma(\Pi')$. Define first σ_i' for each player i over those elements in Π_i' which intersect D. Let $\omega \in D$. Then for each i, $d(\Pi_i(\omega), \Pi_i'(\omega)) \leq \delta$ and moreover since $\delta < \frac{1}{2}$, $\Pi_i(\omega)$ is the only element A in Π_i for which $d(A, \Pi_i'(\omega)) \leq \delta$. Thus we can define unambiguously σ_i' over $\Pi_i'(\omega)$ to be $\sigma_i(\omega)$. Let us denote by Ω_i the set of points over which σ_i' has not been yet defined, i.e., the set of all ωs such that $\Pi_i'(\omega)$ does not intersect D. Consider now the game Γ_0 which is defined as follows. The set of players is $N_0 = \{i | \Omega_i \neq \varnothing\}$. Strategies for player $i \in N_0$ are Π_i' measurable functions τ_i: $\Omega_i \to \Delta_i$. For a strategy $\tau = (\tau_i)_{i \in N_0}$ the payoff for i is given by $E(u_i(\bar{\tau}(\omega), \omega))$ where the expectation is taken over all of Ω and $\bar{\tau}_i$ for each $i \in N_0$ is defined by $\bar{\tau}_i(\omega) = \tau_i(\omega)$ if $\omega \in \Omega_i$ and $\bar{\tau}_i(\omega) = \sigma_i(\omega)$ otherwise. The strategy spaces of the players in Γ_0 are compact metric spaces and the payoff functions are multilinear and thus Γ_0 has an equilibrium. Denote such an equilibrium by τ and for each $i \in N_0$ let $\sigma_i'(\omega) = \tau_i(\omega)$ for each $\omega \in \Omega_i$. This completes the definition of σ'.

Clearly σ' can differ from σ only in $\overline{D}$ which is of measure $\delta = \frac{\epsilon}{4}$ at most.

It remains to show that σ' is an ϵM-equilibrium of $\Gamma(\Pi')$. By the construction of σ', σ_i' is best response over Ω_i. We will show now that over $\overline{\Omega}_i$, σ_i' can be improved by at most ϵM. Let $\hat{\sigma}_i'$ be a strategy in $\Gamma(\Pi')$ and $\hat{\sigma}_i$ any strategy in $\Gamma(\Pi)$ which coincides with $\hat{\sigma}_i'$ on D. Consider the two functions:

$$f(\omega) = u_i(\sigma(\omega), \omega)) - u_i((\sigma|\hat{\sigma}_i)(\omega), \omega),$$

$$g(\omega) = u_i(\sigma'(\omega), \omega)) - u_i((\sigma'|\hat{\sigma}_i)(\omega), \omega).$$

Then $f(\omega) = g(\omega)$ for all $\omega \in D$. We compare now the conditional expectation of g and f over $\Pi_i'(\omega)$ and $\Pi_i(\omega)$ respectively for $\omega \in D$. By the definition of D and Proposition 4.3,

$$D \subseteq \hat{C}^{1-\delta}_{\Pi,\Pi'}(I) = B^{1-\delta}_{\Pi,\Pi'}\big(\hat{C}^{1-\delta}_{\Pi,\Pi'}(I) \cap I\big) = B^{1-\delta}_{\Pi,\Pi'}(D).$$

Therefore for each $\omega \in D$:

$$\mu(D|\Pi_i(\omega)) \geq 1 - \delta, \qquad \mu(D|\Pi_i'(\omega)) \geq 1 - \delta. \tag{5.16}$$

But by Lemma 5.2.2, $D \cap \Pi_i(\omega) = D \cap \Pi_i'(\omega)$. Denote this intersection by E. By (5.16):

$$\mu(E|\Pi_i(\omega)) \geq 1 - \delta, \qquad \mu(E|\Pi_i'(\omega)) \geq 1 - \delta. \tag{5.17}$$

Denote $A = \Pi_i(\omega)$ and $B = \Pi_i'(\omega)$. Now:

$$\begin{aligned}
\frac{1}{\mu(B)}\int_B g - \frac{1}{\mu(A)}\int_A f &\geq -\frac{\mu(B\setminus E)}{\mu(A)}M + \frac{1}{\mu(B)}\int_E g \\
&\quad - \frac{\mu(A\setminus E)}{\mu(A)}M - \frac{1}{\mu(A)}\int_E f \\
&\geq -2\delta M + \left(\frac{1}{\mu(B)} - \frac{1}{\mu(A)}\right)\int_E f \\
&\geq -2\delta M - \frac{2\delta}{\mu(A)}M\mu(E) \\
&\geq -4\delta M \\
&= -\epsilon M.
\end{aligned} \tag{5.18}$$

Since σ is an equilibrium in $\Gamma(\Pi)$, $1/\mu(A)\int_A f \geq 0$ and hence by (5.18), $1/\mu(B)\int_B g \geq -\epsilon M$ which shows that σ' is an ϵM-equilibrium. □

The following theorem states that the topology induced by d is the smallest one that satisfies a continuity property which is even weaker than the one in Theorem 5.3. Note also that the continuity with respect to d, in Theorem 5.3, holds uniformly over information structures while in the following theorem, which is stated in topological terms, such uniformity is not claimed.

THEOREM 5.4. *The topology induced by d on $\mathscr{P}^N$ is the smallest one that has the following uniform (over games) continuity property. For each $\epsilon > 0$, information structure Π and bound M there exists a neighborhood of Π such that for each game Γ with bound M and an equilibrium σ in $\Gamma(\Pi)$, if Π' is in that neighborhood then the game $\Gamma(\Pi')$ has an ϵ-equilibrium σ' such that $E(\|u(\sigma(\cdot),\cdot) - u(\sigma'(\cdot),\cdot)\|) \leq \epsilon M$.*

PROOF. The continuity property of this theorem follows immediately from Theorem 5.1. In order to prove that this is the smallest topology which satisfies this continuity we note that by Theorem 5.2 a neighborhood of Π contains a set of the form $\{\Pi' | \mu(C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\delta))) < 1-\epsilon\}$. Thus we can equivalently prove the following proposition.

PROPOSITION 5.4.1. *Fix M and $\delta < \frac{1}{2}$. For each Π and Π' such that $\mu(C_{\Pi'}^{1-\delta}(I_{\Pi,\Pi'}(\delta))) \leq 1-\delta$, there exist some ϵ, a game Γ bounded by M, and equilibrium σ of $\Gamma(\Pi)$ such that all ϵ-equilibrium σ' of $\Gamma(\Pi')$ satisfy $E(\|u(\sigma(\cdot),\cdot) - u(\sigma'(\cdot),\cdot)\|) > \epsilon$.*

PROOF. A key element in the construction of Γ and σ is the set $\overline{C_{\Pi'}^{1-\delta}(I_{\Pi,\Pi'}(\delta))}$ at which there is no $(1-\delta)$-belief that $I_{\Pi,\Pi'}(\delta)$. As we have seen in §4, this set has an alternative description as $\bigcup_{n\geq 1}(A_{\Pi'}^{\delta})^n(\overline{I_{\Pi,\Pi'}(\delta)})$. We show that a significant part of $\bigcup_{n\geq 1}(A_{\Pi'}^{\delta})^n(\overline{I_{\Pi,\pi'}(\delta)})$ can be expressed in terms of finite number of elements from each Π_i' and Π_i. This finiteness enables us to construct game Γ with finite number of actions. Assume that for each i the information structures Π_i' and Π_i are ordered an denote by $\Pi_i[k]$ and $\Pi_i'[k]$ the sets that contain the first k elements in Π_i and Π_i' respectively. Let $S_{i,k}$ be the union of the elements of $\Pi_i[k]$ and $S_k = \bigcap_{i\in N} S_{i,k}$. The sets $S_{i,k}'$ and S_k' are similarly defined for the partition Π'. We define now an operator $A_{\Pi',k}^p$ as follows. For each $F \in \Sigma$: $A_{\Pi',k}^p(F) = S_k' \cap A_{\Pi'}^p(F)$. Let us also define $E = \overline{I_{\Pi,\Pi'}(\delta)}$ and $E_k = E \cap S_k \cap S_k'$.

LEMMA 5.4.2. *There exists $k > 0$ and $m > 0$ such that $\mu(\bigcup_{j=1}^m A_{\Pi',k}^{\delta}(E_k)) > \delta$.*

PROOF. Since $\mu(\bigcup_{j=1}^{\infty}(A_{\Pi'}^{\delta})^j(E)) > \delta$ there exists some m such that $\mu(\bigcup_{j=1}^m(A_{\Pi'}^{\delta})^j(E)) > \delta$. It is enough now to show that for each $j = 1,\ldots,m$, $(A_{\Pi',k}^{\delta})^j(E_k) \uparrow_k (A_{\Pi'}^{\delta})^j(E)$. This can be easily shown by induction on j since $S_k \uparrow_k \Omega$ and $S_k' \uparrow_k \Omega$, and both $A_{\Pi'}^p$ and $A_{\Pi',k}^p$ satisfy the continuous monotonicity of Proposition 4.5. □

Fix now k and m as in Lemma 5.4.2. We are ready now to define the game Γ starting with the sets of actions A_i. These actions will be interpreted as announcements that i makes concerning the information he has.

Announcement about Π: For each $T \in \Pi_i[k]$ there is an action a_T in A_i which is interpreted as the announcement: "My information is T." There is also an action a in A_i which we interpret as saying: "I am informed according to Π_i."

Announcement about Π': For each $S \in \Pi_i'[k]$ such that for some $T \in \Pi_i[k]$, $\mu(S|T) < 1-\delta$ but $\mu(T|S) \geq 1-\delta$ there is an action b_S which is tantamount to saying: "My information is S." Note that for such S, $d(S,T) > \delta$, but most of S is within T while a "lot" of T can be outside S. In a sense then S is more informative than T.

The generic announcement: The action g in A_i is interpreted as avoiding a specific announcement about the information the player has.

At each ω there is a unique action in A_i which we call the **appropriate** action for i at ω. If ω is in $S_{i,k}$ then $a_{\Pi_i}(\omega)$ is the appropriate action for i at ω. If $\omega \notin S_{i,k}$ then a is that action.

We turn now to the payoff functions. We construct them under the assumption that $M \geq 2$. The rescaling of payoffs for other cases is simple.

Let $\alpha \in A$, then for each i if:

—$\alpha_i = g$ then $u_i(\alpha,\omega) = 1 - \frac{\delta}{2}$ for all ω,

—$\alpha_i = \mathrm{b}_S$ then

$$u_i(\alpha,\omega) = \begin{cases} 1+3\delta & \omega \in T \cap S, \\ -2 & \omega \in T \setminus S, \\ 0 & \omega \notin T, \end{cases}$$

where T is the unique element in Π_i for which $\mu(T|S) \geq 1-\delta$.

—$\alpha_i = a$ or $\alpha_i = a_T$ then $u_i(\alpha, \omega) = 1$ if for each player j, $\alpha_j(\omega)$ is appropriate for j at ω, and $u_i(\alpha, \omega) = 0$ otherwise.

Claim 1. The strategy σ in which the players chose in each ω their appropriate actions is an equilibrium of $\Gamma(\Pi)$.

PROOF. Player i receives 1 at each ω and therefore will not deviate to g which yields only $1 - \frac{\delta}{2}$. A deviation to a or any a_T which are inappropriate to i in ω yields 0. A deviation to b_S can benefit him only if he is informed that he is in $T \cap S$ for the unique T which satisfies $\mu(T|S) \geq 1 - \delta$ because outside this $T \cap S$, b_S plays him naught or -2. But for this T, $\mu(S|T) < 1 - \delta$ and therefore i's expected payoff, using b_S is at most $(1 - \delta)(1 + 3\delta) + \delta(-2)$ which is less than 1. □

Claim 2. Let σ' be an $\frac{\delta}{16}$-equilibrium of $\Gamma(\Pi')$. Then for each $\omega \in \bigcup_{j=1}^{m}(A^{\delta}_{\Pi',k})^j(E_k)$ there exists a player i who assigns a probability of no more than $\frac{1}{4}$ to the action that is appropriate to him at ω.

PROOF. We show first that the claim holds for each ω in E_k. Indeed in each such ω there is a player i for whom $d(\Pi_i(\omega), \Pi'_i(\omega)) > \delta$. There are two possible cases:

1. $\mu(\Pi_i(\omega)|\Pi'_i(\omega)) < 1 - \delta$. By the definition of E_k, $\Pi_i(\omega) \in \Pi_i[k]$ and therefore the appropriate action for i at ω is $a_{\Pi_i(\omega)}$. But this action is inappropriate for i outside $\Pi_i(\omega)$ and thus the expected payoff for i using this action is at most $1 - \delta$, while he can receive $1 - \frac{\delta}{2}$ playing g.

2. $\mu(\Pi_i(\omega)|\Pi'_i(\omega)) \geq 1 - \delta$, but $\mu(\Pi'_i(\omega)|\Pi_i(\omega)) < 1 - \delta$. Again $\Pi'_i(\omega) \in \Pi'_i[k]$ and therefore the action $b_{\Pi_i(\omega)}$ is well defined and yields i at least $(1 - \delta)(1 - 3\delta)$, which for $\delta < \frac{1}{2}$ is more than $1 + \frac{\delta}{2}$.

In either case the appropriate action for i at ω yields at least $\frac{\delta}{2}$ less than the best response at $\Pi_i(\omega)$. We use now and later the following simple fact, which we bring without a proof.

LEMMA 5.4.3. *An action that yields less than ϵ_1 than the best response, can be played in an ϵ_2-equilibrium with probability of at most ϵ_2/ϵ_1.*

It follows now that for each $\omega \in E_k$ there is a player that in any $\frac{\delta}{16}$-equilibrium will use his appropriate action with probability which will not exceed $(\delta/16)/(\delta/2) = \frac{1}{8} < \frac{1}{4}$.

We prove now by induction on j that the claim holds for all $\omega \in (A^{\delta}_{\Pi',k})^j(E_k)$. This was already proved for $j = 0$ where $(A^{\delta}_{\Pi',k})^0(E_k) = E_k$. Suppose now that we proved it for $j - 1$ for some $j \geq 1$. There exists some player i such that

$$(5.19) \qquad \mu\Big(\big(A^{\delta}_{\Pi',k}\big)^{j-1}(E_k)|\Pi'_i(\omega)\Big) > \delta.$$

By the induction hypothesis at each $\omega' \in (A^{\delta}_{\Pi',k})^{j-1}(E_k)$ there is a player i' who employs his appropriate action at ω' with probability of $\frac{1}{4}$ at most. If $i = i'$ then we are finished. Otherwise, by (5.19) the probability that i assigns that not all players their appropriate action is at least $\frac{3}{4}\delta$. Thus by playing in $\Pi_i(\omega)$ his appropriate action at ω, $a_{\Pi_i(\omega)}$, player i can receive at most $1 - \frac{3}{4}\delta$, while he can guarantee $\frac{\delta}{4}$ more by playing g. Thus by Lemma 5.4.7 in any $\frac{\delta}{16}$-equilibrium, i will use at ω his appropriate action with probability of at most $(\delta/16)/(\delta/4) = \frac{1}{4}$. This completes the proof of claim 2. □

We have shown that at each ω in the set $\bigcup_{j=1}^{m}(A^{\delta}_{\Pi',k})^j(E_k)$—which is of measure δ at least—in any $\frac{\delta}{16}$-equilibrium σ' at least one of the players plays with probability $\frac{3}{4}$ at least an action which is inappropriate for him in ω. But for any α which includes an inappropriate action $\|u(\alpha, \omega) - 1\| > \frac{\delta}{2}$. Thus

$$E(\|u(\sigma(\cdot), \cdot) - u(\sigma'(\cdot), \cdot)\|) > \delta\frac{\delta}{2} = \frac{\delta^2}{2}.$$

Any ϵ which does not accede $\min\{\frac{\delta}{16}, \frac{\delta^2}{2}\}$ is satisfactory for Proposition 5.4.1. This completes the proof of Theorem 5.4. □

Reexamining Example 2 we can see now why the continuity of Theorem 5.3 does not hold there. In this example each player has exactly the same information under Π and Π^n in all states other than $2n$ and $2n+1$. Indeed for any $\epsilon < \frac{4}{5}$, $I_{\Pi,\Pi^n}(\epsilon) = \Omega \setminus \{2n, 2n+1\}$. But for any $p > \frac{4}{5}$,

$$B_{\Pi}^{p}(I_{\Pi,\Pi^n}(\epsilon)) = \Omega \setminus \{2n-1, 2n, 2n+1, 2n+2\},$$

since for example at state $2n-1$ player 2 assigns a probability of at least $\frac{5}{9}$ that the state is $2n$ and therefore that the information 1 receives under Π_1 is very different from the one he receives under Π_1^n. Applying B^p again and again gives $C^p(I_{\Pi,\Pi^n}(\epsilon)) = \emptyset$. Thus by Theorem 5.4, the sequence Π^n does not converge to Π and stays outside a neighborhood of Π.

6. The case of a single agent.

When the set of players is a singleton, d becomes a pseudo metric on $\mathscr{P}$. In this case d has a simple form as is stated in the following proposition.

PROPOSITION 6.1. *For Π and Π' in $\mathscr{P}$,*

$$d(\Pi, \Pi') = \min\{\epsilon | \mu(I_{\Pi,\Pi'}(\epsilon)) \geq 1 - \epsilon\}.$$

PROOF. We prove the proposition by showing that for any Π, Π' in $\mathscr{P}$ and ϵ, $D_{\Pi,\Pi'}(\epsilon) = I_{\Pi,\Pi'}(\epsilon)$. Denote $I = I_{\Pi,\Pi'}(\epsilon)$. It suffices to show that $I \subseteq \hat{C}_{\Pi,\Pi'}^{1-\epsilon}(I)$ which will follow easily if we show that for each $n \geq 1$, $I \subseteq (B_{\Pi,\Pi'}^{1-\epsilon}(I \cap \cdot))^n(I)$. For this we need only to prove that $I \subseteq B_{\Pi,\Pi'}^{1-\epsilon}(I)$. This in turn can be proved by showing that $I \subseteq B_{\Pi}^{1-\epsilon}(I)$ and $I \subseteq C_{\Pi'}^{1-\epsilon}(I)$, and by the symmetry it is enough to show only the first of these. Let $\omega \in I$ then $d(\Pi(\omega), \Pi'(\omega)) \leq \epsilon$. This inequality holds for all $\omega' \in \Pi(\omega) \cap \Pi'(\omega)$ and therefore $\Pi(\omega) \cap \Pi'(\omega) \subseteq I$. Hence $\mu(I|\Pi(\omega)) \geq \mu(\Pi(\omega) \cap \Pi'(\omega)|\Pi(\omega)) \geq 1 - \epsilon$. This shows that $\omega \in C_{\Pi}^{1-\epsilon}(I)$ and completes the proof. □

The Boylan pseudo metric δ is defined on the set of all sub σ field of Σ. In particular we can define it on $\mathscr{P}$ by identifying each partition Π with Σ_{Π}—the sub σ field of Σ generated by Π. The Boylan distance of a pair of information structures Π and Π' is:

$$\delta(\Pi, \Pi') = \max_{E \in \Sigma_{\Pi}} \min_{F \in \Sigma_{\Pi'}} \mu(E \Delta F) + \max_{E \in \Sigma_{\Pi'}} \min_{F \in \Sigma_{\Pi}} \mu(E \Delta F),$$

where $E \Delta F = (E \setminus F) \cup (F \setminus E)$.

THEOREM 6.2. *The pseudo metrics d and δ define the same topology on $\mathscr{P}$.*

PROOF. We show first that any δ-ball contains a d-ball by showing that if $d(\Pi, \Pi') < \frac{\epsilon}{4}$ than $\delta(\Pi, \Pi') < \epsilon$. Suppose than that $d(\Pi, \Pi') < \frac{\epsilon}{4}$. Denote $I = I_{\Pi,\Pi'}(\frac{\epsilon}{4})$. Then $\mu(I) > 1 - \frac{\epsilon}{4}$. Now let $A \in \Sigma_{\Pi}$. Write $A = \bigcup_{k \geq 0} A_k$ where for each k, $A_k \in \Pi$, for each $k \geq 1 A_k \cap I \neq \emptyset$ and $A_0 \cap I = \emptyset$. For each $k \geq 1$, there exists a set $B_k \in \Pi'$ such that $\mu(A_k \setminus B_k) \le \frac{\epsilon}{4}\mu(A_k)$. Denote $B = \bigcup_{k \ge 1} B_k$. Then

$$\mu(A \setminus B) \le \mu\Big(\bigcup_{k\ge 1} A_k \setminus B\Big) + \mu(A_0) \le \frac{\epsilon}{4}\mu\Big(\bigcup_{k\ge 1} A_k\Big) + \mu(\bar{A}_0) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

Because of the symmetry of Π and Π' this proves that $\delta(\Pi, \Pi') < \epsilon$.

Conversely, we show now that each δ-ball with radius ϵ centered in Π contains a d-ball. We assume w.o.l.g. that $\epsilon < \frac{1}{2}$. Choose $\eta > 0$ small enough such that $\bar{\Pi} = \{A | A \in \Pi, \mu(A) > \eta\}$ satisfies $\mu(\bigcup \bar{\Pi}) > 1 - \frac{\epsilon}{2}$. Let $\theta = \frac{\eta\epsilon}{2}$. We show that if $\delta(\Pi, \Pi') < \theta$ then $d(\Pi, \Pi') < \epsilon$.

Consider $A \in \Pi$. There exists $B \in \Sigma_{\Pi'}$ such that

$$\mu(A \setminus B) < \theta \quad \text{and} \tag{6.1}$$

$$\mu(B \setminus A) < \theta. \tag{6.2}$$

We can choose B to be a minimal set in $\Sigma_{\Pi'}$ (w.r.t. inclusion) which satisfies (6.2). Observe that $\mu(A) > \eta = \frac{2\theta}{\epsilon} > 4\theta$ and thus by (6.2),

$$\mu(A \cap B) > 3\theta. \tag{6.3}$$

We prove now that B must be in Π'. Suppose to the contrary that $B \notin \Pi'$ then there must be some $C \in \Sigma_{\Pi'}$ which is a proper subset of B. by (6.3) we can choose C such that it satisfies

$$\mu(A \cap C) > \theta, \tag{6.4}$$

(if C does not satisfies it $B \setminus C$ does). By the minimality of B also,

$$\mu(A \setminus C) > \theta. \tag{6.5}$$

Now for C there must be some $F \in \Sigma_{\Pi}$ such that $\Delta(C, F) < \theta$. But either $F \cap A = \varnothing$ in which case by (6.4) $\mu(D \setminus F) \ge \mu(A \cap D) > \theta$, or $A \subseteq F$ in which case by (6.5) $\mu(F \setminus D) \ge \mu(A \setminus F) > \theta$. This contradicts our assumption and proves that $B \in \Pi'$.

Now by (6.2),

$$\mu(A \setminus B | A) < \frac{\theta}{\eta} = \frac{\epsilon}{2}. \tag{6.6}$$

Also $\mu(C) \ge \mu(C \cap A) > \mu(A) - \theta > \eta - \theta$ and thus,

$$\mu(B \setminus A | B) < \frac{\theta}{\eta - \theta} = \frac{\theta}{\eta(1 - \epsilon/2)} = \frac{\epsilon}{1 - \epsilon/2} < \epsilon. \tag{6.7}$$

By (6.6) and (6.7) $d(A, B) < \epsilon$ and hence,

$$A \cap B \subseteq I_{\Pi, \Pi'}(\epsilon). \tag{6.8}$$

Since $\mu(A \cap B | A) = 1 - \mu(A \setminus B | A) > (1 - \frac{\epsilon}{2})$ it follows that $\mu(A \cap C) > \mu(A)(1 - \frac{\epsilon}{2})$. By (6.8) we conclude that $\mu(A \cap I_{\Pi, \Pi'}(\epsilon)) > (1 - \frac{\epsilon}{2})\mu(A)$. This is true for each $A \in \overline{\Pi}$ and thus,

$$\begin{aligned}\mu(I_{\Pi,\Pi'}(\epsilon)) &\geq \mu\left(\bigcup \overline{\Pi} I_{\Pi,\Pi'}(\epsilon)\right)\\ &\geq \left(1 - \frac{\epsilon}{2}\right)\mu\left(\bigcup \overline{\Pi}\right)\\ &\geq \left(1 - \frac{\epsilon}{2}\right)^2\\ &> 1 - \epsilon,\end{aligned}$$

which shows that $d(\Pi, \Pi') < \epsilon$. □

A game with incomplete information of a single player is a decision problem and an equilibrium of such a game is an optimal plan. Thus applying Theorem 4.4 to the case of a single agent we find a continuity property of optimal plans which characterizes the Boylan topology.

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References

Allen, B. (1983). Neighboring information and distribution of agents' characteristics under uncertainty. *J. Math. Econom.* **12** 63–101.

____ (1984). Convergence of σ-fields and application to mathematical economics. In *Selected Topics in Operation Research and Mathematical Economics*, G. Hammer and D. Pallaschke (Eds.) Springer-Verlag, Heidelberg.

Boylan, E. (1971). Equiconvergence of martingales. *Ann. Math. Statistics* **42** 552–559.

Cotter, K. (1986). Similarity of information and behavior with a pointwise convergence topology. *J. Math. Econom.* **15** 25–38.

Fudenberg, D. and J. Tirole (1991). *Game Theory*, MIT Press, Cambridge, MA.

Kajii, A. and S. Morris (1994). *Payoff continuity in incomplete information games.* CARRES Working Paper #94-17, University of Pennsylvania, Philadelphia, PA.

Milgrom, P. and R. Weber (1985). Distributional strategies for games with incomplete information, *Math. Oper. Res.* **10** 619–632.

Monderer, D. and D. Samet (1989). Approximating common knowledge with common beliefs. *Games and Economic Behavior* **1** 170–190.

Rubinstein, A. (1989). The electronic mail game: strategic behavior under "almost common knowledge." *American Economic Review* **79** 385–391.

Sorin, S. (1994). On the impact of an event mimeo.

Stinchcombe, M. (1990). Bayesian information topologies. *J. Math. Economics* **19** 233–253.

VanZandt, T. (1988). Information, measurability and continuous behavior. CARRES Working Paper #88-18, University of Pennsylvania, Philadelphia, PA.

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