Prym Varieties and Teichmüller Curves

Foliations of Hilbert modular surfaces

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Abstract

The Hilbert modular surface X_D is the moduli space of Abelian varieties A with real multiplication by a quadratic order of discriminant $D > 1$. The locus where A is a product of elliptic curves determines a finite union of algebraic curves $X_D(1) \subset X_D$.

In this paper we show the lamination $X_D(1)$ extends to an essentially unique foliation $\mathcal{F}_D$ of X_D by complex geodesics. The geometry of $\mathcal{F}_D$ is related to Teichmüller theory, holomorphic motions, polygonal billiards and Lattès rational maps. We show every leaf of $\mathcal{F}_D$ is either closed or dense, and compute its holonomy. We also introduce refinements $T_N(\nu)$ of the classical modular curves on X_D, leading to an explicit description of $X_D(1)$.

Contents

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1 Introduction

Let $D > 1$ be an integer congruent to 0 or $1 \bmod 4$, and let $\mathcal{O}_D$ be the real quadratic order of discriminant D. The *Hilbert modular surface*

$$X_D = (\mathbb{H} \times \mathbb{H}) / \operatorname{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$$

is the moduli space for principally polarized Abelian varieties

$$A_\tau = \mathbb{C}^2 / (\mathcal{O}_D \oplus \mathcal{O}_D^\vee \tau)$$

with real multiplication by $\mathcal{O}_D$.

Let $X_D(1) \subset X_D$ denote the locus where A_τ is isomorphic to a polarized product of elliptic curves $E_1 \times E_2$. The set $X_D(1)$ is a finite union of disjoint, irreducible algebraic curves (§4), forming a *lamination* of X_D. Note that $X_D(1)$ is preserved by the twofold symmetry $\iota(\tau_1, \tau_2) = (\tau_2, \tau_1)$ of X_D.

In this paper we will show:

Theorem 1.1 *Up to the action of ι, the lamination $X_D(1)$ extends to a unique foliation $\mathcal{F}_D$ of X_D by complex geodesics.*

(Here a Riemann surface in X_D is a *complex geodesic* if it is isometrically immersed for the Kobayashi metric.)

Holomorphic graphs. The preimage $\widetilde{X}_D(1)$ of $X_D(1)$ in the universal cover of X_D gives a lamination of $\mathbb{H} \times \mathbb{H}$ by the graphs of countably many Möbius transformations. To foliate X_D itself, in §6 we will show:

Theorem 1.2 *For any $(\tau_1, \tau_2) \notin \widetilde{X}_D(1)$, there is a unique holomorphic function*

$$f : \mathbb{H} \to \mathbb{H}$$

such that $f(\tau_1) = \tau_2$ and the graph of f is disjoint from $\widetilde{X}_D(1)$.

The graphs of such functions descend to X_D, and form the leaves of the foliation $\mathcal{F}_D$ (§7). The case $D = 4$ is illustrated in Figure 1.

Modular curves. To describe the lamination $X_D(1)$ explicitly, recall that the Hilbert modular surface X_D is populated by infinitely many *modular curves* F_N [Hir], [vG]. The endomorphism ring of a generic Abelian variety in F_N is a quaternionic order R of discriminant N^2.

In general F_N can be reducible, and R is not determined up to isomorphism by N. In §3 we introduce a refinement $F_N(\nu)$ of the traditional modular curves, such that the isomorphism class of R is constant along

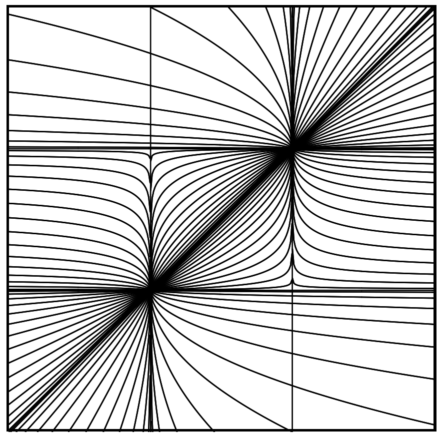

Figure 1. Foliation of the Hilbert modular surface X_D, $D = 4$.

$F_N(\nu)$ and $F_N = \bigcup F_N(\nu)$. The additional finite invariant ν ranges in the ring $\mathcal{O}_D/(\sqrt{D})$ and its norm satisfies $\mathrm{N}(\nu) = -N \bmod D$. The curves $T_N = \bigcup F_{N/\ell^2}$ can be refined similarly, and we obtain:

Theorem 1.3 *The locus $X_D(1) \subset X_D$ is given by*

$$X_D(1) \quad = \quad \bigcup T_N((e+\sqrt{D})/2),$$

where the union is over all integral solutions to $e^2 + 4N = D$, $N > 0$.

Remark. Although $X_D(1) = \bigcup T_{(D-e^2)/4}$ when D is prime, in general (e.g. for $D = 12, 16, 20, 21, \ldots$) the locus $X_D(1)$ cannot be expressed as a union of the traditional modular curves T_N (§3).

Here is a corresponding description of the lamination $\widetilde{X}_D(1)$. Given $N > 0$ such that $D = e^2 + 4N$, let

$$\Lambda_D^N \quad = \quad \left\{ U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} \; : \; \begin{array}{l} a, b \in \mathbb{Z},\, \mu \in \mathcal{O}_D,\, \det(U) = N \\ \text{and } \mu \equiv \pm(e+\sqrt{D})/2 \text{ in } \mathcal{O}_D/(\sqrt{D}) \end{array} \right\}.$$

Let Λ_D be the union of all such Λ_D^N. Choosing a real place $\iota_1 : \mathcal{O}_D \to \mathbb{R}$, we can regard Λ_D as a set of matrices in $\mathrm{GL}_2^+(\mathbb{R})$, acting by Möbius transformations on $\mathbb{H}$.

Theorem 1.4 *The lamination $\widetilde{X}_D(1)$ of $\mathbb{H} \times \mathbb{H}$ is the union of the loci $\tau_2 = U(\tau_1)$ over all $U \in \Lambda_D$.*

We also obtain a description of the locus $X_D(E) \subset X_D$ where A_τ admits an action of both $\mathcal{O}_D$ and $\mathcal{O}_E$ (§3).

Quasiconformal dynamics. Although its leaves are Riemann surfaces, $\mathcal{F}_D$ is not a holomorphic foliation. Its transverse dynamics is given instead by quasiconformal maps, which can be described as follows.

Let $q = q(z)\,dz^2$ be a meromorphic quadratic differential on $\mathbb{H}$. We say a homeomorphism $f : \mathbb{H} \to \mathbb{H}$ is a *Teichmüller mapping* relative to q if it satisfies $\overline{\partial} f/\partial f = \alpha q/|q|$ for some complex number $|\alpha| < 1$; equivalently, if f has the form of an orientation-preserving real-linear mapping

$$f(x+iy) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = D_q(f) \begin{pmatrix} x \\ y \end{pmatrix}$$

in local charts where $q = dz^2 = (dx + i\,dy)^2$.

Fix a transversal $\mathbb{H}_s = \{s\} \times \mathbb{H}$ to $\widetilde{\mathcal{F}}_D$. Any $g \in \mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ acts on $\mathbb{H} \times \mathbb{H}$, permuting the leaves of $\widetilde{\mathcal{F}}_D$. The permutation of leaves is recorded by the *holonomy map*

$$\phi_g : \mathbb{H}_s \to \mathbb{H}_s,$$

characterized by the property that $g(s,z)$ and $(s, \phi_g(z))$ lie on the same leaf of $\widetilde{\mathcal{F}}_D$.

In §8 we will show:

Theorem 1.5 *The holonomy acts by Teichmüller mappings relative to a fixed meromorphic quadratic differential q on $\mathbb{H}_s$. For $s = i$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have*

$$D_q(\phi_g) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R}).$$

On the other hand, for $z \in \partial\mathbb{H}_s$ we have

$$\phi_g(z) = (a'z - b')/(-c'z + d');$$

in particular, the holonomy acts by Möbius transformations on $\partial\mathbb{H}_s$.

Here $(x + y\sqrt{D})' = (x - y\sqrt{D})$. Note that both Galois conjugate actions of g on $\mathbb{R}^2$ appear, as different aspects of the holonomy map ϕ_g.

Quantum Teichmüller curves. For comparison, consider an isometrically immersed *Teichmüller curve*

$$f : V \to \mathcal{M}_g,$$

generated by a holomorphic quadratic differential (Y, q) of genus g. For simplicity assume $\mathrm{Aut}(Y)$ is trivial. Then the pullback of the universal curve $X = f^*(\mathcal{M}_{g,1})$ gives an algebraic surface

$$p : X \to V$$

with $p^{-1}(v) = Y$ for a suitable basepoint $v \in V$. The surface X carries a canonical foliation $\mathcal{F}$, transverse to the fibers of p, whose leaves map to Teichmüller geodesics in $\mathcal{M}_{g,1}$. The holonomy of $\mathcal{F}$ determines a map

$$\pi_1(V, v) \to \mathrm{Aff}^+(Y, q)$$

giving an action of the fundamental group by Teichmüller mappings; and its linear part yields the isomorphism

$$\pi_1(V, v) \cong \mathrm{PSL}(Y, q) \subset \mathrm{PSL}_2(\mathbb{R}),$$

where $\mathrm{PSL}(Y, q)$ is the stabilizer of (Y, q) in the bundle of quadratic differentials $Q\mathcal{M}_g \to \mathcal{M}_g$. (See e.g. [V1], [Mc4, §2].)

The foliated Hilbert modular surface $(X_D, \mathcal{F}_D)$ presents a similar structure, with the fibration $p : X \to V$ replaced by the holomorphic foliation $\mathcal{A}_D$ coming from the level sets of τ_1 on $\widetilde{X}_D = \mathbb{H} \times \mathbb{H}$. This suggests that one should regard $(X_D, \mathcal{A}_D, \mathcal{F}_D)$ as a *quantum* Teichmüller curve, in the same sense that a 3-manifold with a measured foliation can be regarded as a quantum Teichmüller geodesic [Mc3].

Question. Does every fibered surface $p : X \to C$ admit a foliation $\mathcal{F}$ by Riemann surfaces transverse to the fibers of p?

Complements. We conclude in §9 by presenting the following related results.

1. Every leaf of $\mathcal{F}_D$ is either closed or dense.
2. When $D \neq d^2$, there are infinitely many eigenforms for real multiplication by $\mathcal{O}_D$ that are isoperiodic but not isomorphic.
3. The Möbius transformations Λ_D give a maximal top-speed holomorphic motion of a discrete subset of $\mathbb{H}$.
4. The foliation $\mathcal{F}_4$ also arises as the motion of the Julia set in a Lattès family of iterated rational maps.

The link with complex dynamics was used to produce Figure 1.

Notes and references. The foliation $\mathcal{F}_D$ is constructed using the connection between polygonal billiards and Hilbert modular surfaces presented in [Mc4]. For more on the interplay of dynamics, holomorphic motions and quasiconformal mappings, see e.g. [MSS], [BR], [Sl], [Mc2], [Sul], [McS], [EKK] and [Dou]. A survey of the theory of *holomorphic* foliations of surfaces appears in [Br1]; see also [Br2] for the Hilbert modular case.

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2 Quaternion algebras

In this section we consider a real quadratic order $\mathcal{O}_D$ acting on a symplectic lattice L, and classify the quaternionic orders $R \subset \mathrm{End}(L)$ extending $\mathcal{O}_D$.

Quadratic orders. Given an integer $D > 0$, $D \equiv 0$ or $1 \bmod 4$, the *real quadratic order* of discriminant D is given by

$$\mathcal{O}_D = \mathbb{Z}[T]/(T^2 + bT + c), \quad \text{where } D = b^2 - 4c.$$

Let $K_D = \mathcal{O}_D \otimes \mathbb{Q}$. Provided D is not a square, K_D is a real quadratic field. Fixing an embedding $\iota_1 : K_D \to \mathbb{R}$, we obtain a unique basis

$$K_D = \mathbb{Q} \cdot 1 \oplus \mathbb{Q} \cdot \sqrt{D}$$

such that $\iota_1(\sqrt{D}) > 0$. The conjugate real embedding $\iota_2 : K_D \to \mathbb{R}$ is given by $\iota_2(x) = \iota_1(x')$, where $(a + b\sqrt{D})' = (a - b\sqrt{D})$.

Square discriminants. The case $D = d^2$ can be treated similarly, so long as we regard $x = \sqrt{d^2}$ as an element of K_D satisfying $x^2 = d^2$ but $x \notin \mathbb{Q}$. In this case the algebra $K_D \cong \mathbb{Q} \oplus \mathbb{Q}$ is not a field, so we must take care to distinguish between elements of the algebra such as

$$x = d - \sqrt{d^2} \in K_D,$$

and the corresponding real numbers

$$\iota_1(x) = d - d = 0, \quad \text{and} \quad \iota_2(x) = d + d = 2d.$$

Trace, norm and different. For simplicity of notation, we fix D and denote $\mathcal{O}_D$ and K_D by K and $\mathcal{O}$.

The trace and norm on K are the rational numbers $\mathrm{Tr}(x) = x + x'$ and $\mathrm{N}(x) = xx'$. The *inverse different* is the fractional ideal

$$\mathcal{O}^\vee = \{x \in K \,:\, \mathrm{Tr}(xy) \in \mathbb{Z}\ \forall y \in \mathcal{O}\}.$$

It is easy to see that $\mathcal{O}^\vee = D^{-1/2}\,\mathcal{O}$, and thus the *different* $\mathcal{D} = (\mathcal{O}^\vee)^{-1} \subset \mathcal{O}$ is the principal ideal $(\sqrt{D})$. The trace and norm descend to give maps

$$\mathrm{Tr}, \mathrm{N} : \mathcal{O}/\mathcal{D} \to \mathbb{Z}/D,$$

satisfying

$$\mathrm{Tr}(x)^2 = 4\,\mathrm{N}(x) \bmod D. \tag{2.1}$$

When D is odd, $\mathrm{Tr} : \mathcal{O}/\mathcal{D} \to \mathbb{Z}/D$ is an isomorphism, and thus (2.1) determines the norm on $\mathcal{O}/\mathcal{D}$. On the other hand, when $D = 4E$ is even, we have an isomorphism

$$\mathcal{O}/\mathcal{D} \cong \mathbb{Z}/2E \oplus \mathbb{Z}/2$$

given by $a + b\sqrt{E} \mapsto (a, b)$, and the trace and norm on $\mathcal{O}/\mathcal{D}$ are given by

$$\mathrm{Tr}(a, b) = 2a \bmod D, \quad \mathrm{N}(a, b) = a^2 - Eb^2 \bmod D.$$

Symplectic lattices. Now let $L \cong (\mathbb{Z}^{2g}, \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right))$ be a unimodular symplectic lattice of genus g. (This lattice is isomorphic to the first homology group $H_1(\Sigma_g, \mathbb{Z})$ of an oriented surface of genus g with the symplectic form given by the intersection pairing.)

Let $\mathrm{End}(L) \cong \mathrm{M}_{2g}(\mathbb{Z})$ denote the endomorphism ring of L as a $\mathbb{Z}$-module. The *Rosati involution* $T \mapsto T^*$ on $\mathrm{End}(L)$ is defined by the condition $\langle Tx, y\rangle = \langle x, T^*y\rangle$; it satisfies $(ST)^* = T^*S^*$, and we say T is *self-adjoint* if $T = T^*$.

Specializing to the case $g = 2$, let L denote the lattice

$$L = \mathcal{O} \oplus \mathcal{O}^\vee$$

with the unimodular symplectic form

$$\langle x, y\rangle = \mathrm{Tr}(x \wedge y) = \mathrm{Tr}_{\mathbb{Q}}^K(x_1y_2 - x_2y_1).$$

A standard symplectic basis for L (satisfying $\langle a_i \cdot b_j\rangle = \delta_{ij}$) is given by

$$(a_1, a_2, b_1, b_2) = ((1, 0), (\gamma, 0), (0, -\gamma'/\sqrt{D}), (0, 1/\sqrt{D})), \tag{2.2}$$

where $\gamma = (D + \sqrt{D})/2$.

The lattice L comes equipped with a proper, self-adjoint action of $\mathcal{O}$, given by

$$k \cdot (x_1, x_2) = (kx_1, kx_2). \tag{2.3}$$

Conversely, any proper, self-adjoint action of $\mathcal{O}$ on a symplectic lattice of genus two is isomorphic to this model (see e.g. [Ru], [Mc7, Thm 4.1]). (Here an action of R on L is *proper* if it is indivisible: if whenever $T \in \mathrm{End}(L)$ and $mT \in R$ for some integer $m \neq 0$, then $T \in R$.)

Matrices. The natural embedding of $L = \mathcal{O} \oplus \mathcal{O}^\vee$ into $K \oplus K$ determines an embedding of matrices

$$\mathrm{M}_2(K) \to \mathrm{End}(L \otimes \mathbb{Q}),$$

and hence a diagonal inclusion

$$K \to \mathrm{End}(L \otimes \mathbb{Q})$$

extending the natural action (2.3) of $\mathcal{O}$ on L. Every $T \in \mathrm{End}(L \otimes \mathbb{Q})$ can be uniquely expressed in the form

$$T(x) = Ax + Bx', \quad A, B \in \mathrm{M}_2(K),$$

where $(x_1, x_2)' = (x_1', x_2')$; and we have

$$T^*(x) = A^\dagger x + (B^\dagger)' x', \tag{2.4}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\dagger = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

The automorphisms of L as a symplectic $\mathcal{O}$-module are given, as a subgroup of $M_2(K)$, by

$$\mathrm{SL}(\mathcal{O} \oplus \mathcal{O}^\vee) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O} & \mathcal{D} \\ \mathcal{O}^\vee & \mathcal{O} \end{pmatrix} \;:\; ad - bc = 1 \right\}.$$

Compare [vG, p.12].

Integrality. An endomorphism $T \in \mathrm{End}(L \otimes \mathbb{Q})$ is *integral* if it satisfies $T(L) \subset L$.

Lemma 2.1 *The endomorphism $\phi(x) = ax + bx'$ of K satisfies $\phi(\mathcal{O}) \subset \mathcal{O}$ iff $a, b \in \mathcal{O}^\vee$ and $a + b \in \mathcal{O}$.*

Proof. Since $x - x' \in \sqrt{D}\mathbb{Z}$ for all $x \in \mathcal{O}$, the conditions on a, b imply $\phi(x) = a(x - x') + (a + b)x' \in \mathcal{O}$ for all $x \in \mathcal{O}$. Conversely, if ϕ is integral, then $\phi(1) = a + b \in \mathcal{O}$, and thus $a(x - x') \in \mathcal{O}$ for all $x \in \mathcal{O}$, which implies $a \in D^{-1/2}\,\mathcal{O} = \mathcal{O}^\vee$. ■

Corollary 2.2 *The endomorphism* $T(x) = kx + \left(\begin{smallmatrix} a & bD \\ c & d \end{smallmatrix}\right) x'$ *is integral iff we have*

$$a, b, c, d, k \in \mathcal{O}^\vee \quad \textit{and} \quad k + a, k - d \in \mathcal{O}\,.$$

Proof. This follows from the preceding Lemma, using the fact that $kx + dx'$ maps $\mathcal{O}^\vee$ to $\mathcal{O}^\vee$ iff $kx - dx'$ maps $\mathcal{O}$ to $\mathcal{O}$. ∎

Quaternion algebras. A rational *quaternion algebra* is a central simple algebra of dimension 4 over $\mathbb{Q}$. Every such algebra has the form

$$Q \;\cong\; \mathbb{Q}[i,j]/(i^2 = a, j^2 = b, ij = -ji) \;\;=\;\; \left(\frac{a,b}{\mathbb{Q}}\right)$$

for suitable $a, b \in \mathbb{Q}^*$. Any $q \in Q$ satisfies a quadratic equation

$$q^2 - \mathrm{Tr}(q)q + \mathrm{N}(q) = 0,$$

where $\mathrm{Tr}, \mathrm{N} : Q \to \mathbb{Q}$ are the *reduced trace and norm.*

An *order* $R \subset Q$ is a subring such that, as an additive group, we have $R \cong \mathbb{Z}^4$ and $\mathbb{Q} \cdot R = Q$. Its *discriminant* is the square integer

$$N^2 = |\det(\mathrm{Tr}(q_i q_j))| > 0,$$

where $(q_i)_1^4$ is an integral basis for R. The discriminants of a pair of orders $R_1 \subset R_2$ are related by $N_1/N_2 = |R_2/R_1|^2$.

Generators. We say $V \in \mathrm{End}(L)$ is a *quaternionic generator* if:

1. $V^* = -V$,
2. $V^2 = -N \in \mathbb{Z}$, $N \neq 0$,
3. $Vk = k'V$ for all $k \in K$, and
4. $k + D^{-1/2}V \in \mathrm{End}(L)$ for some $k \in K$.

These conditions imply that $Q = K \oplus KV$ is a quaternion algebra isomorphic to $\left(\frac{D,-N}{\mathbb{Q}}\right)$. Conversely, we have:

Theorem 2.3 *Any Rosati-invariant quaternion algebra* Q *with*

$$K \subset Q \subset \mathrm{End}(L \otimes \mathbb{Q})$$

contains a unique pair of primitive quaternionic generators $\pm V$.

(A generator is *primitive* unless $(1/m)V, m > 1$ is also a generator.)

Proof. By a standard application of the Skolem-Noether theorem, we can write $Q = K \oplus KW$ with $0 \neq W^2 \in \mathbb{Q}$ and $Wk = k'W$ for all $k \in K$. Then KW coincides with the subalgebra of Q anticommuting with the self-adjoint element $\sqrt{D}$, so it is Rosati-invariant. The eigenspaces of $*|KW$ are exchanged by multiplication by $\sqrt{D}$, so up to a rational multiple there is a unique nonzero $V \in KW$ with $V^* = -V$. A suitable integral multiple of V is then a generator, and a rational multiple is primitive. ■

Corollary 2.4 *Quaternionic extensions $K \subset Q \subset \mathrm{End}(L)$ correspond bijectively to pairs of primitive generators $\pm V \in \mathrm{End}(L)$.*

Generator matrices. We say $U \in M_2(K)$ is a *quaternionic generator matrix* if it has the form

$$U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} \tag{2.5}$$

with $a, b \in \mathbb{Z}$, $\mu \in \mathcal{O}$ and $N = \det(U) \neq 0$.

Theorem 2.5 *The endomorphism $V(x) = Ux'$ is a quaternionic generator iff U is a quaternionic generator matrix.*

Proof. By (2.4) the condition $V = -V^*$ is equivalent to $U^\dagger = -U'$, and thus U can be written in the form (2.5) with $a, b \in \mathbb{Q}$ and $\mu \in K$. Assuming $U^\dagger = -U'$, we have

$$N = \det(U) = UU^\dagger = -UU' = -V^2,$$

so $V^2 \neq 0 \iff \det(U) \neq 0$. The condition that $D^{-1/2}(k+V)$ is integral for some k implies, by Corollary 2.2, that the coefficients of U satisfy $a, b \in \mathbb{Z}$ and $\mu \in \mathcal{O}$; and given such coefficients for U, the endomorphism $D^{-1/2}(k+V)$ is integral when $k = -\mu$. ■

The invariant $\nu(U)$. Given generator matrix $U = \left(\begin{smallmatrix} \mu & bD \\ -a & -\mu' \end{smallmatrix}\right)$, let $\nu(U)$ denote the image of μ in the finite ring $\mathcal{O}/\mathcal{D}$. It is easy to check that

$$\nu(U) = \pm\nu(g'Ug^{-1})$$

for all $g \in \mathrm{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)$, and that its norm satisfies

$$\mathrm{N}(\nu(U)) \equiv -N \bmod D. \tag{2.6}$$

Quaternionic orders. Let $V(x) = Ux'$, and let

$$R_U = (K \oplus KV) \cap \mathrm{End}(L).$$

Then R_U is a Rosati-invariant order in the quaternion algebra generated by V. Clearly $\mathcal{O} \subset R_U$, so we can also regard $(R_U, *)$ as an involutive algebra over $\mathcal{O}$. We will show that $N = \det(U)$ and $\nu(U)$ determine $(R_U, *)$ up to isomorphism.

Models. We begin by constructing a model algebra $(R_N(\nu), *)$ over $\mathcal{O}_D$ for every $\nu \in \mathcal{O}/\mathcal{D}$ with $\mathrm{N}(\nu) = -N \neq 0 \bmod D$.

Let $Q_N = K \oplus KV$ be the abstract quaternion algebra with the relations $V^2 = -N$ and $Vk = k'V$. Define an involution on Q_N by $(k_1 + k_2 V)^* = (k_1 - k_2' V)$, and let $R_N(\nu)$ be the order in Q_N defined by

$$R_N(\nu) = \{\alpha + \beta V \ : \ \alpha, \beta \in \mathcal{O}^\vee, \alpha + \beta\nu \in \mathcal{O}\,.\} \tag{2.7}$$

Note that $\mathcal{O}^\vee \cdot \mathcal{D} \subset \mathcal{O}$, so the definition of $R_N(\nu)$ depends only on the class of ν in $\mathcal{O}/\mathcal{D}$. To check that $R_N(\nu)$ is an order, note that

$$(\alpha + \beta V)(\gamma + \delta V) = (\kappa + \lambda V) = (\alpha\gamma - N\beta\delta') + (\alpha\delta + \beta\gamma')V;$$

since $-N \equiv \mathrm{N}(\nu) = \nu\nu' \bmod D$, we have

$$\begin{aligned} \kappa + \nu\lambda &\equiv (\alpha\gamma + \nu\nu'\beta\delta') + \nu(\alpha\delta + \beta\gamma') \\ &= (\alpha + \beta\nu)(\gamma' + \delta'\nu') + \alpha(\gamma - \gamma' + \nu\delta - \nu'\delta') \\ &\equiv 0 + 0 \mod \mathcal{O}, \end{aligned}$$

and thus R_U is closed under multiplication.

Theorem 2.6 *The quaternionic order $R_N(\nu)$ has discriminant N^2.*

Proof. Note that the inclusions

$$\mathcal{O} \oplus \mathcal{O}\, V \subset R_N(\nu) \subset \mathcal{O}^\vee \oplus \mathcal{O}^\vee\, V$$

each have index D. The quaternionic order $\mathcal{O} \oplus \mathcal{O}\, V$ has discriminant $D^2 N^2$, since $V^2 = -N$ and $\mathrm{Tr}\,|\,\mathcal{O}\, V = 0$, and thus $R_N(\nu)$ has discriminant N^2. ∎

Theorem 2.7 *We have* $(R_N(\nu), *) \cong (R_M(\mu), *)$ *iff* $N = M$ *and* $\nu = \pm\mu$.

Proof. The element $V \in R_N(\nu)$ is, up to sign, the order's unique primitive generator, in the sense that $V^* = -V$, $Vk = k'V$ for all $k \in \mathcal{O}_D$, $V^2 \neq 0$, $k + D^{-1/2}V \in R_N(\nu)$ for some $k \in K$, and V is not a proper multiple of another element in $R_N(\nu)$ with the same properties. Thus the structure of $(R_N(\nu), *)$ as an $\mathcal{O}_D$-algebra determines $V \in R_N(\nu)$ up to sign, and V determines $N = -V^2$ and the constant $\nu \in \mathcal{O}/\mathcal{D}$ in the relation $\alpha + \beta\nu \in \mathcal{O}$ defining $R_N(\nu) \subset K \oplus KV$. ∎

Theorem 2.8 *If* U *is a primitive generator matrix, then we have*

$$(R_U, *) \cong (R_N(\nu), *)$$

where $N = \det(U)$ *and* $\nu = \nu(U)$.

Proof. Setting $V(x) = Ux'$, we need only verify that $(K \oplus KV) \cap \mathrm{End}(L)$ coincides with the order $R_N(\nu)$ defined by (2.7). To see this, let

$$T(x) = \alpha x + \beta V(x) = \alpha x + \beta \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} x'$$

in $K \oplus KV$. By Corollary 2.2, T is integral iff

(i) $a\beta, b\beta, \mu\beta, \mu'\beta \in \mathcal{O}^\vee$,

(ii) $\alpha \in \mathcal{O}^\vee$,

(iii) $\alpha + \beta\mu \in \mathcal{O}$ and

(iv) $\alpha + \beta\mu' \in \mathcal{O}$.

Using (iii), condition (iv) can be replaced by

(iv′) $\beta(\mu - \mu')/\sqrt{D} \in \mathcal{O}^\vee$.

Since U is primitive, the ideal $(a, b, \mu, (\mu - \mu')/\sqrt{D})$ is equal to $\mathcal{O}$. Thus (i) and (iv′) together are equivalent to the condition $\beta \in \mathcal{O}^\vee$, and we are left with the definition of $R_N(\nu)$. ∎

Remark. In general, the invariants $\det(U)$ and $\nu(U)$ do not determine the embedding $R_U \subset \mathrm{End}(L)$ up to conjugacy. For example, when D is odd, the generator matrices $U_1 = \begin{pmatrix} 0 & D^2 \\ -D & 0 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 0 & D^3 \\ -1 & 0 \end{pmatrix}$ have the same invariants, but the corresponding endomorphisms are not conjugate in $\mathrm{End}(L)$ because

$$L/V_1(L) \cong (\mathbb{Z}/D \times \mathbb{Z}/D^2)^2$$

while

$$L/V_2(L) \cong \mathbb{Z}/D \times \mathbb{Z}/D^2 \times \mathbb{Z}/D^3.$$

Extra quadratic orders. Finally we determine when the algebra $R_N(\nu)$ contains a second, independent quadratic order $\mathcal{O}_E$.

Theorem 2.9 *The algebra $(R_N(\nu), *)$ contains a self-adjoint element $T \notin \mathcal{O}_D$ generating a copy of $\mathcal{O}_E$ iff there exist $e, \ell \in \mathbb{Z}$ such that*

$$ED = e^2 + 4N\ell^2, \quad \ell \neq 0$$

and $(e + E\sqrt{D})/2 + \ell\nu = 0 \bmod \mathcal{D}$.

Proof. Given e, ℓ as above, let

$$T = \alpha + \beta V = D^{-1/2}\left(\frac{e + E\sqrt{D}}{2} + \ell V\right).$$

Then we have $T = T^*$, $T \in R_N(\nu)$ and $T^2 - eT + (E - E^2)/4 = 0$; therefore $\mathbb{Z}[T] \cong \mathcal{O}_E$. A straightforward computation shows that, conversely, any independent copy of $\mathcal{O}_E$ in $R_N(\nu)$ arises as above. ■

For additional background on quaternion algebras, see e.g. [Vi], [MR] and [Mn].

3 Modular curves and surfaces

In this section we describe modular curves on Hilbert modular surfaces from the perspective of the Abelian varieties they determine.

Abelian varieties. A *principally polarized Abelian variety* is a complex torus $A \cong \mathbb{C}^g/L$ equipped with a unimodular symplectic form $\langle x, y\rangle$ on $L \cong \mathbb{Z}^{2g}$, whose extension to $L \otimes \mathbb{R} \cong \mathbb{C}^g$ satisfies

$$\langle x, y\rangle = \langle ix, iy\rangle \quad \text{and} \quad \langle x, ix\rangle \geq 0.$$

The ring $\mathrm{End}(A) = \mathrm{End}(L) \cap \mathrm{End}(\mathbb{C}^g)$ is Rosati invariant, and coincides with the endomorphism ring of A as a complex Lie group. We have $\mathrm{Tr}(TT^*) \geq 0$ for all $T \in \mathrm{End}(A)$.

Every Abelian variety can be presented in the form

$$A = \mathbb{C}^g/(\mathbb{Z}^g \oplus \Pi\mathbb{Z}^g),$$

where Π is an element of the Siegel upper halfplane

$$\mathfrak{H}_g = \{\Pi \in M_g(\mathbb{C}) \, : \, \Pi^t = \Pi \text{ and } \mathrm{Im}(\Pi) \text{ is positive-definite}\}.$$

The symplectic form on $L = \mathbb{Z}^g \oplus \Pi\mathbb{Z}^g$ is given by $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Any two such presentations of A differ by an automorphism of L, so the moduli space of abelian varieties of genus g is given by the quotient space

$$\mathcal{A}_g = \mathfrak{H}_g / \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Real multiplication. As in §2, let $D > 0$ be the discriminant of a real quadratic order $\mathcal{O}_D$, and let $K = \mathcal{O} \otimes \mathbb{Q}$. Fix a real place $\iota_1 : K \to \mathbb{R}$, and set $\iota_2(k) = \iota_1(k')$.

We will regard K as a subfield of the reals, using the fixed embedding $\iota_1 : K \subset \mathbb{R}$. The case $D = d^2$ is treated with the understanding that the real numbers (k, k') implicitly denote $(\iota_1(k), \iota_2(k))$, $k \in K$.

An Abelian variety $A \in \mathcal{A}_2$ admits *real multiplication* by $\mathcal{O}_D$ if there is a self-adjoint endomorphism $T \in \mathrm{End}(A)$ generating a proper action of $\mathbb{Z}[T] \cong \mathcal{O}_D$ on A. Any such variety can be presented in the form

$$A_\tau = \mathbb{C}^2/(\mathcal{O}_D \oplus \mathcal{O}_D^\vee \tau) = \mathbb{C}^2/\phi_\tau(L), \tag{3.1}$$

where $\tau = (\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}$ and where $L = \mathcal{O} \oplus \mathcal{O}^\vee$ is embedded in $\mathbb{C}^2$ by the map

$$\phi_\tau(x_1, x_2) = (x_1 + x_2\tau_1, x_1' + x_2'\tau_2).$$

As in §2, the symplectic form on L is given by $\langle x, y\rangle = \mathrm{Tr}_{\mathbb{Q}}^K(x \wedge y)$, and the action of $\mathcal{O}_D$ on $\mathbb{C}^2 \supset L$ is given simply by $k \cdot (z_1, z_2) = (kz_1, k'z_2)$.

Eigenforms. The Abelian variety A_τ comes equipped with a distinguished pair of normalized *eigenforms* $\eta_1, \eta_2 \in \Omega(A_\tau)$. Using the isomorphism $H_1(A_\tau, \mathbb{Z}) \cong L$, these forms are characterized by the property that

$$\phi_\tau(C) = \left(\int_C \eta_1, \int_C \eta_2\right). \tag{3.2}$$

Modular surfaces. If we change the identification $L \cong H_1(A_\tau, \mathbb{Z})$ by an automorphism g of L, we obtain an isomorphic Abelian variety $A_{g\cdot\tau}$. Thus the moduli space of Abelian varieties with real multiplication by $\mathcal{O}_D$ is given by the Hilbert modular surface

$$X_D = (\mathbb{H} \times \mathbb{H}) / \operatorname{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee).$$

The point $g(\tau)$ is characterized by the property that

$$\phi_{g\cdot\tau} = \chi(g, \tau)\ \phi_\tau \circ g^{-1}$$

for some matrix $\chi(g, \tau) \in \operatorname{GL}_2(\mathbb{C})$; explicitly, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau_1, \tau_2) = \left(\frac{a\tau_1 - b}{-c\tau_1 + d}, \frac{a'\tau_2 - b'}{-c'\tau_2 + d'} \right) \tag{3.3}$$

and

$$\chi(g, \tau) = \begin{pmatrix} (d - c\tau_1)^{-1} & 0 \\ 0 & (d' - c'\tau_2)^{-1} \end{pmatrix}. \tag{3.4}$$

A point $[\tau] \in X_D$ gives an Abelian variety $[A_\tau] \in \mathcal{A}_2$ with a *chosen* embedding $\mathcal{O}_D \to \operatorname{End}(A_\tau)$. Similarly, a point $\tau \in \widetilde{X}_D = \mathbb{H} \times \mathbb{H}$ gives an Abelian variety with a distinguished isomorphism or *marking*, $L \cong H_1(A_\tau, \mathbb{Z})$, sending $\mathcal{O}_D$ into $\operatorname{End}(A_\tau)$.

Modular embedding. The *modular embedding*

$$p_D : X_D \to \mathcal{A}_2$$

is given by $[\tau] \mapsto [A_\tau]$. To write p_D explicitly, note that the embedding $\phi_\tau : L \to \mathbb{C}^2$ can be expressed with respect to the basis (a_1, a_2, b_1, b_2) for L given in (2.2) by the matrix

$$\phi_\tau = \begin{pmatrix} 1 & \gamma & -\tau_1\gamma'/\sqrt{D} & \tau_1/\sqrt{D} \\ 1 & \gamma' & \tau_2\gamma/\sqrt{D} & -\tau_2/\sqrt{D} \end{pmatrix} = (A, B).$$

Consequently we have $A_\tau \cong \mathbb{C}^2/(\mathbb{Z}^2 \oplus \Pi\mathbb{Z}^2)$, where

$$\Pi = \widetilde{p_D}(\tau) = A^{-1}B = \frac{1}{D} \begin{pmatrix} \tau_1(\gamma')^2 + \tau_2\gamma^2 & -\tau_1\gamma' - \tau_2\gamma \\ -\tau_1\gamma' - \tau_2\gamma & \tau_1 + \tau_2 \end{pmatrix}.$$

The map $X_D \to p_D(X_D)$ has degree two.

Modular curves. Given a matrix $U(x) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(K) \cap \mathrm{End}(L)$ such that $U' = -U^*$, let $V(x) = Ux'$ and define

$$\mathbb{H}_U = \{\tau \in \mathbb{H} \times \mathbb{H} \,:\, V \in \mathrm{End}(A_\tau)\}.$$

It is straightforward to check that

$$\mathbb{H}_U = \left\{(\tau_1, \tau_2) \,:\, \tau_2 = \frac{d\tau_1 + b}{c\tau_1 + a}\right\}; \tag{3.5}$$

indeed, when τ_1 and τ_2 are related as above, the map $\phi_\tau : L \to \mathbb{C}^2$ satisfies

$$\phi_\tau(V(x)) = \begin{pmatrix} 0 & a + c\tau_1 \\ a' + c'\tau_2 & 0 \end{pmatrix} \phi_\tau(x),$$

exhibiting the complex-linearity of V. Note that $\mathbb{H}_U = \emptyset$ if $\det(U) < 0$.

We now restrict attention to the case where U is a generator matrix. Then by the results of §2, we have:

Theorem 3.1 *The ring* $\mathrm{End}(A_\tau)$ *contains a quaternionic order extending* $\mathcal{O}_D$ *if and only if* $\tau \in \mathbb{H}_U$ *for some generator matrix* U*.*

Let $F_U \subset X_D$ denote the projection of $\mathbb{H}_U$ to the quotient $(\mathbb{H}\times\mathbb{H})/\,\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$. Following [Hir, §5.3], we define the *modular curve* F_N by

$$F_N = \bigcup \{F_U \,:\, U \text{ is a primitive generator matrix with } \det(U) = N.\}.$$

It can be shown that F_N is an algebraic curve on X_D.

To describe this curve more precisely, let

$$F_N(\nu) = \{F_U \,:\, U \text{ is primitive, } \det(U) = N \text{ and } \nu(U) = \pm\nu\},$$

where $\nu \in \mathcal{O}_D\,/\mathcal{D}_D$. Note that we have

$$F_N(\nu) \neq \emptyset \iff N(\nu) = -N \bmod D$$

by equation (2.6), $F_N(\nu) = F_N(-\nu)$, and $F_N = \bigcup F_N(\nu)$.

The results of §2 give the structure of the quaternion ring generated by $V(x) = Ux'$.

Theorem 3.2 *The curve* $F_N(\nu) \subset X_D$ *coincides with the locus of Abelian varieties such that*

$$\mathcal{O}_D \subset R \subset \mathrm{End}(A_\tau),$$

for some properly embedded quaternionic order $(R, *)$ *isomorphic to* $(R_N(\nu), *)$*.*

Corollary 3.3 *The curve F_N is the locus where $\mathcal{O}_D \subset \mathrm{End}(A_\tau)$ extends to a properly embedded, Rosati-invariant quaternionic order of discriminant N^2.*

Two quadratic orders. We can now describe the locus $X_D(E)$ of Abelian varieties with an independent, self-adjoint action of $\mathcal{O}_E$. (We do not require the action of $\mathcal{O}_E$ to be proper.)

To state this description, it is useful to define:

$$T_N = \bigcup \{F_U \ : \ \det(U) = N\} = \bigcup F_{N/\ell^2},$$

and

$$T_N(\nu) = \bigcup \{F_U \ : \ \det(U) = N, \nu(U) = \pm\nu\}.$$

Then Theorem 2.9 implies:

Theorem 3.4 *The locus $X_D(E)$ is given by*

$$X_D(E) = \bigcup T_N((e + E\sqrt{D})/2),$$

where the union is over all $N > 0$ and $e \in \mathbb{Z}$ such that $ED = e^2 + 4N$.

Corollary 3.5 *We have $X_D(1) = \bigcup \{T_N((e + \sqrt{D})/2) \ : \ e^2 + 4N = D\}$.*

Refined modular curves. To conclude we show that in general the expression $F_N = \bigcup F_N(\nu)$ gives a proper refinement of F_N. First note:

Theorem 3.6 *We have $F_N(\nu) = F_N$ iff $\pm\nu$ are the only solutions to*

$$N(\xi) = -N \bmod D, \quad \xi \in \mathcal{O}_D / \mathcal{D}_D.$$

Corollary 3.7 *If $D = p$ is prime, then $F_N = F_N(\nu)$ whenever $F_N(\nu) \neq \emptyset$.*

Proof. In this case, according to (2.1), the norm map

$$N : \mathcal{O}_D / \mathcal{D}_D \overset{\mathrm{Tr}}{\cong} \mathbb{Z}/p \to \mathbb{Z}/p$$

is given by $N(\xi) = \xi^2/4$. Since $F_N(\nu) \neq \emptyset$, we have $N(\nu) = -N$; and since $\mathbb{Z}/p$ is a field, $\pm\nu$ are the only solutions to this equation. ∎

Corollary 3.8 *When D is prime, we have $X_D(E) = \bigcup T_{(ED-e^2)/4}$.*

Now consider the case $D = 21$, the first odd discriminant which is not a prime. Then the norm map is still given by $N(\xi) = \xi^2/4$ on $\mathcal{O}_D / \mathcal{D}_D \cong \mathbb{Z}/D$, but now $\mathbb{Z}/D$ is not a field. For example, the equation $\xi^2 = 1 \bmod D$ has four solutions, namely $\xi = 1, 8, 13$ or 20. These give four solutions to the equation $N(\xi) = -5$, and hence contribute two distinct terms to the expression

$$F_5 = \bigcup F_5(\nu) = F_5((1+\sqrt{21})/2) \cup F_5((8+\sqrt{21})/2).$$

Only one of these terms appears in the expression for $X_D(1)$. In fact, since $21 = 1^2 + 4 \cdot 5 = 3^2 + 4 \cdot 3$, by Corollary 3.5 we have

$$\begin{aligned} X_{21}(1) &= F_3 \cup F_5((1+\sqrt{21})/2) \\ &\neq F_3 \cup F_5. \end{aligned}$$

(The full curve F_3 appears because the only solutions to $N(\xi) = \xi^2/4 = -3 \bmod 21$ are $\xi = \pm 3$.)

Using Theorem 3.6, it is similarly straightforward to check other small discriminants; for example:

Theorem 3.9 *For $D \leq 30$ we have $X_D(1) = \bigcup_{e^2+4N=D} T_N$ when $D = 4, 5, 8, 9, 13, 17, 25$ and 29, but not when $D = 12, 16, 20, 21, 24$ or 28.*

Notes. For more background on modular curves and surfaces, see [Hir], [HZ2], [HZ1], [BL], [Mc7, §4] and [vG]. Our $U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix}$ corresponds to the skew-Hermitian matrix $B = \sqrt{D}\begin{pmatrix} a & \mu \\ \mu' & bD \end{pmatrix}$ in [vG, Ch. V]. Note that (3.3) agrees with the standard action $(a\tau + b)/(c\tau + d)$ up to the automorphism $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ of $\mathrm{SL}_2(K)$. We remark that X_D can also be presented as the quotient $(\mathbb{H} \times -\mathbb{H}) / \mathrm{SL}_2(\mathcal{O}_D)$, using the fact that $\sqrt{D}' = -\sqrt{D}$; on the other hand, the surfaces $(\mathbb{H} \times \mathbb{H}) / \mathrm{SL}_2(\mathcal{O}_D)$ and X_D are generally not isomorphic (see e.g. [HH].)

It is known that the intersection numbers $\langle T_N, T_M \rangle$ form the coefficients of a modular form [HZ1], [vG, Ch. VI]. The results of [GKZ] suggest that the intersection numbers of the refined modular curves $T_N(\nu)$ may similarly yield a Jacobi form.

4 Laminations

In this section we show algebraically that $\widetilde{X}_D(1)$ gives a lamination of $\mathbb{H} \times \mathbb{H}$ by countably many disjoint hyperbolic planes. We also describe these laminations explicitly for small values of D. Another proof of laminarity appears in §7.

Jacobian varieties. Let $\Omega(X)$ denote the space of holomorphic 1-forms on a compact Riemann surface X. The *Jacobian* of X is the Abelian variety $\mathrm{Jac}(X) = \Omega(X)^*/H_1(X,\mathbb{Z})$, polarized by the intersection pairing on 1-cycles.

In the case of genus two, any principally polarized Abelian variety A is either a Jacobian or a product of polarized elliptic curves. The latter case occurs iff A admits real multiplication by $\mathcal{O}_1$, generated by projection to one of the factors of $A \cong B_1 \times B_2$. In particular, we have:

Theorem 4.1 *For any $D \geq 4$, the locus of Jacobian varieties in X_D is given by $X_D - X_D(1)$.*

Laminations. To describe $X_D(1)$ in more detail, given $N > 0$ such that $D = e^2 + 4N$ let

$$\begin{aligned}\Lambda_D^N &= \{U \in M_2(K) \,:\, U \text{ is a generator matrix, } \det(U) = N \text{ and} \\ &\quad\ \nu(U) \equiv \pm(e+\sqrt{D})/2 \bmod \mathcal{D}_D\},\end{aligned}$$

and let Λ_D be the union of all such Λ_D^N. Note that if U is in Λ_D, then $-U, U'$ and U^* are also in Λ_D.

By Corollary 3.5, the preimage of $X_D(1)$ in $\widetilde{X}_D = \mathbb{H} \times \mathbb{H}$ is given by:

$$\widetilde{X}_D(1) \;=\; \bigcup \{\mathbb{H}_U \,:\, U \in \Lambda_D\}.$$

Note that each $\mathbb{H}_U$ is the graph of a Möbius transformation.

Theorem 4.2 *The locus $\widetilde{X}_D(1)$ gives a lamination of $\mathbb{H} \times \mathbb{H}$ by countably many hyperbolic planes.*

(This means any two planes in $\widetilde{X}_D(1)$ are either identical or disjoint.)

For the proof, it suffices to show that the difference $g \circ h^{-1}$ of two Möbius transformations in Λ_D is never elliptic. Since Λ_D is invariant under $U \mapsto U^* = (\det U)U^{-1}$, this in turn follows from:

Theorem 4.3 *For any $U_1, U_2 \in \Lambda_D$, we have* $\mathrm{Tr}(U_1U_2)^2 \geq 4\det(U_1U_2)$.

Proof. By the definition of Λ_D, we can write $D = e_i^2 + 4\det(U_i) = e_i^2 + 4N_i$, where $e_i \geq 0$. We can also assume that

$$U_i = \begin{pmatrix} \mu_i & b_i D \\ -a_i & -\mu_i' \end{pmatrix}$$

satisfies

$$\mu_i \equiv (x_i + y_i\sqrt{D})/2 \equiv (e_i + \sqrt{D})/2 \bmod \mathcal{D}_D$$

(replacing U_i with $-U_i$ if necessary). It follows that y_i is odd and $x_i = e_i \bmod D$, which implies

$$\operatorname{Tr}(U_1U_2) \equiv \operatorname{Tr}(\mu_1\mu_2) = (x_1x_2 + Dy_1y_2)/2 \equiv (e_1e_2 - D)/2 \bmod D. \quad (4.1)$$

(The factor of 1/2 presents no difficulties, because x_i is even when D is even.)

Now suppose

$$\operatorname{Tr}(U_1U_2)^2 < 4\det(U_1U_2) = 4N_1N_2. \quad (4.2)$$

Then we have $|\operatorname{Tr}(U_1U_2)| < 2\sqrt{N_1N_2} \leq D/2$, and thus (4.1) implies

$$\operatorname{Tr}(U_1U_2) = (e_1e_2 - D)/2.$$

But this implies

$$\begin{aligned} 4\operatorname{Tr}(U_1U_2)^2 &= (D - e_1e_2)^2 \\ &\geq (D - e_1^2)(D - e_2^2) = (4N_1)(4N_2) = 16\det(U_1U_2), \end{aligned}$$

contradicting (4.2). ■

Small discriminants. To conclude we record a few cases where Λ_D admits a particularly economical description.

For concreteness, we will present Λ_D as a set matrices in $\mathrm{GL}_2^+(\mathbb{R})$ using the chosen real place $\iota_1 : K \to \mathbb{R}$. This works even when $D = d^2$, since both μ and μ' appear on the diagonal of $U \in \Lambda_D$ (no information is lost). Under the standard action $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = (az + b)/(cz + d)$ of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{H}$, we can then write

$$\widetilde{X}_D(1) = \bigcup_{\Lambda_D} \{(\tau_1, \tau_2) : \tau_2 = U(\tau_1)\}.$$

This holds despite the twist in the definition (3.5) of $\mathbb{H}_U$, because Λ_D is invariant under $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$.

Theorem 4.4 *For $D = 4, 5, 8, 9$ and 13 respectively, we have:*

$$\begin{aligned} \Lambda_4 &= \{U \in \mathrm{M}_2(\mathbb{Z}) : \det(U) = 1 \text{ and } U \equiv \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \bmod 4\}, \\ \Lambda_5 &= \{U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : \det(U) = 1\}, \\ \Lambda_8 &= \Lambda_8^1 \cup \Lambda_8^2 = \left\{U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : \det(U) = 1 \text{ or } 2\right\}, \\ \Lambda_9 &= \{U \in \mathrm{M}_2(\mathbb{Z}) : \det(U) = 2 \text{ and } U \equiv \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \bmod 9\}, \quad \textit{and} \\ \Lambda_{13} &= \Lambda_{13}^1 \cup \Lambda_{13}^3 = \left\{U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : \det(U) = 1 \text{ or } 3\right\}, \end{aligned}$$

where it is understood that $a, b \in \mathbb{Z}$ and $\mu \in \mathcal{O}_D$.

Proof. Recall from Theorem 3.9 that $X_D(1) = \bigcup_{e^2+4N=D} T_N$ when $D = 4, 5, 8, 9$ and 13. When this equality holds, we can ignore the condition on $\nu(U)$ in the definition of Λ_D. The cases $D = 5, 8$ and 13 then follow directly from the definition of Λ_D^N. For $D = 9$, we note that any integral matrix satisfying $\det \left(\begin{smallmatrix} x & 9b \\ -a & y \end{smallmatrix}\right) = 2$ also satisfies $x + y = 0 \bmod 3$, and thus it can be written in the form $\left(\begin{smallmatrix} \mu & bD \\ -a & -\mu' \end{smallmatrix}\right)$ with

$$\mu = \frac{(x-y) + (x+y)\sqrt{9}/3}{2}.$$

Similar considerations apply when $D = 4$. ∎

5 Foliations of Teichmüller space

In this section we introduce a family of foliations $\mathcal{F}_i$ of Teichmüller space, related to normalized Abelian differentials and their periods $\tau_{ij} = \int_{b_i} \omega_j$. We then show:

Theorem 5.1 *There is a unique holomorphic section of the period map*

$$\tau_{ii} : \mathcal{T}_g \to \mathbb{H}$$

through any $Y \in \mathcal{T}_g$. Its image is the leaf of $\mathcal{F}_i$ containing Y.

The case $g = 2$ will furnish the desired foliations of Hilbert modular surfaces.

Abelian differentials. Let Z_g be a smooth oriented surface of genus g. Let $\mathcal{T}_g$ be the Teichmüller space of Riemann surfaces Y, each equipped with an isotopy class of homeomorphism or *marking* $Z_g \to Y$. The marking determines a natural identification between $H_1(Z_g)$ and $H_1(Y)$ used frequently below.

Let $\Omega\mathcal{T}_g \to \mathcal{T}_g$ denote the bundle of nonzero Abelian differentials (Y, ω), $\omega \in \Omega(Y)$. For each such form we have a *period map*

$$I(\omega) : H_1(Z_g, \mathbb{Z}) \to \mathbb{C}$$

given by $I(\omega) : C \to \int_C \omega$. There is a natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\Omega\mathcal{T}_g$, satisfying

$$I(A \cdot \omega) = A \circ I(\omega) \tag{5.1}$$

under the identification $\mathbb{C} = \mathbb{R}^2$ given by $x + iy = (x, y)$.

Each orbit $\mathrm{GL}_2^+(\mathbb{R}) \cdot (Y, \omega)$ projects to a *complex geodesic*

$$f : \mathbb{H} \to \mathcal{T}_g,$$

which can be normalized so that $f(i) = Y$ and

$$\nu = \left.\frac{df}{dt}\right|_{t=i} = \frac{i}{2}\frac{\overline{\omega}}{\omega}.$$

The subspace of $H^1(Z_g, \mathbb{R})$ spanned by $(\operatorname{Re}\omega, \operatorname{Im}\omega)$ is constant along each orbit (cf. [Mc7, §3]).

Symplectic framings. Now let $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ be a real symplectic basis for $H_1(Z_g, \mathbb{R})$ (with $\langle a_i, b_i\rangle = -\langle b_i, a_i\rangle = 1$ and all other products zero). Then for each $Y \in \mathcal{T}_g$, there exists a unique basis $(\omega_1, \ldots, \omega_g)$ of $\Omega(Y)$ such that $\int_{a_i} \omega_j = \delta_{ij}$. The *period matrix*

$$\tau_{ij}(Y) = \int_{b_i} \omega_j$$

then determines an embedding

$$\tau : \mathcal{T}_g \to \mathfrak{H}_g.$$

This agrees with the usual Torelli embedding, up to composition with an element of $\mathrm{Sp}_{2g}(\mathbb{R})$. Note that $\operatorname{Im}(\tau_{ii}(Y)) > 0$ since $\operatorname{Im}\tau$ is positive definite.

The normalized 1-forms (ω_i) give a splitting

$$\Omega(Y) = \oplus_1^g \mathbb{C}\omega_i = \oplus_1^g F_i(Y),$$

and corresponding subbundles $F_i\mathcal{T}_g \subset \Omega T_g$.

Complex subspaces. Let (a_i^*, b_i^*) denote the dual basis for $H^1(Z_g, \mathbb{R})$, and let S_i be the span of (a_i^*, b_i^*). It easy to check that the following conditions are equivalent:

1. S_i is a complex subspace of $H^1(Y, \mathbb{R}) \cong \Omega(Y)$.
2. S_i is spanned by $(\operatorname{Re}\omega_i, \operatorname{Im}\omega_i)$.
3. The period matrix $\tau(Y)$ satisfies $\tau_{ij} = 0$ for all $j \neq i$.

Let $\mathcal{T}_g(S_i) \subset \mathcal{T}_g$ denote the locus where these condition hold. Note that condition (3) defines a totally geodesic subset

$$H_i \cong \mathbb{H} \times \mathfrak{H}_{g-1} \subset \mathfrak{H}_g$$

such that $\mathcal{T}_g(S_i) = \tau^{-1}(H_i)$.

Foliations. Next we show that the complex geodesics generated by the forms (Y, ω_i) give a foliation of Teichmüller space.

Theorem 5.2 *The sub-bundle $F_i\mathcal{T}_g \subset \Omega\mathcal{T}_g$ is invariant under the action of $\mathrm{GL}_2^+(\mathbb{R})$, as is its restriction to $\mathcal{T}_g(S_i)$.*

Proof. The invariance of $F_i\mathcal{T}_g$ is immediate from (5.1). To handle the restriction to $\mathcal{T}_g(S_i)$, recall that the span W of $(\mathrm{Re}\,\omega_i, \mathrm{Im}\,\omega_i)$ is constant along orbits; thus the condition $W = S_i$ characterizing $\mathcal{T}_g(S_i)$ is preserved by the action of $\mathrm{GL}_2^+(\mathbb{R})$. ■

Corollary 5.3 *The foliation of $F_i\mathcal{T}_g$ by $\mathrm{GL}_2^+(\mathbb{R})$ orbits projects to a foliation $\mathcal{F}_i$ of $\mathcal{T}_g$ by complex geodesics.*

Corollary 5.4 *The locus $\mathcal{T}_g(S_i)$ is also foliated by $\mathcal{F}_i$: any leaf meeting $\mathcal{T}_g(S_i)$ is entirely contained therein.*

Proof of Theorem 5.1. The proof uses Ahlfors' variational formula [Ah] and follows the same lines as the proof of [Mc4, Thm. 4.2]; it is based on the fact that the leaves of $\mathcal{F}_i$ are the geodesics along which the periods of ω_i change most rapidly.

Let $s : \mathbb{H} \to \mathcal{T}_g$ be a holomorphic section of τ_{ii}. Let $v \in \mathrm{T}\mathbb{H}$ be a unit tangent vector with respect to the hyperbolic metric $\rho = |dz|/(2\,\mathrm{Im}\,z)$ of constant curvature -4, mapping to $Ds(v) \in \mathrm{T}_Y\mathcal{T}_g$. By the equality of the Teichmüller and Kobayashi metrics [Gd, Ch. 7], $Ds(v)$ is represented by a Beltrami differential $\nu = \nu(z)d\overline{z}/dz$ on Y with $\|\nu\|_\infty \le 1$. But s is a section, so the composition

$$\tau_{ii} \circ s : \mathbb{H} \to \mathbb{H}$$

is the identity; thus the norm of its derivative, given by Ahlfors' formula as

$$\|D(\tau_{ii} \circ s)(\nu)\| \;=\; \left|\int_Y \omega_i^2 \nu\right| \Big/ \int_Y |\omega_i|^2 ,$$

is one. It follows that $\nu = \overline{\omega}_i/\omega_i$ up to a complex scalar of modulus one, and thus $Ds(v)$ is tangent to the complex geodesic generated by (Y, ω_i). Equivalently, $s(\mathbb{H})$ is everywhere tangent to the foliation $\mathcal{F}_i$; therefore its image is the unique leaf through Y. ■

6 Genus two

We can now obtain results on Hilbert modular surfaces by specializing to the case of genus two. In this section we will show:

Theorem 6.1 *There is a unique holomorphic section of* τ_1 *passing through any given point of* $\mathbb{H} \times \mathbb{H} - \widetilde{X}_D(1)$.

Here $\tau_1 : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is simply projection onto the first factor. This result is a restatement of Theorem 1.2; as in §1, we assume $D \geq 4$.

Framings for real multiplication. Let $g = 2$, and choose a symplectic isomorphism

$$L = H_1(Z_g, \mathbb{Z}) \cong \mathcal{O}_D \oplus \mathcal{O}_D^\vee .$$

We then have an action of $\mathcal{O}_D$ on $H_1(Z_g, \mathbb{Z})$, and the elements $\{a, b\} = \{(1,0),(0,1)\}$ in L give a distinguished basis for

$$H_1(Z_g, \mathbb{Q}) = L \otimes \mathbb{Q} \cong K^2$$

as a vector space over $K = \mathcal{O}_D \otimes \mathbb{Q}$. Using the two Galois conjugate embeddings $K \to \mathbb{R}$, we obtain an orthogonal splitting

$$H_1(Z_g, \mathbb{R}) = L \otimes \mathbb{R} = V_1 \oplus V_2$$

such that $k \cdot (C_1, C_2) = (kC_1, k'C_2)$. The projections (a_i, b_i) of $a, b \in L$ to each summand yield bases for V_i, which taken together give a standard symplectic basis for $H_1(Z_g, \mathbb{R})$. (Note that (a_i, b_i) is generally *not* an integral sympletic basis; indeed, when K is a field, the elements (a_i, b_i) do not even lie in $H_1(Z_g, \mathbb{Q})$.)

Let $S_i^D \subset H^1(Z_g, \mathbb{R})$ be the span of the dual basis a_i^*, b_i^*.

Theorem 6.2 *The ring* $\mathcal{O}_D \subset \mathrm{End}(L)$ *acts by real multiplication on* $\mathrm{Jac}(Y)$ *if and only if* $Y \in \mathcal{T}_g(S_1^D)$.

Proof. Since $g = 2$ we have $S_2^D = (S_1^D)^\perp$, and thus $\mathcal{T}_g(S_1^D) = \mathcal{T}_g(S_2^D)$. But $\mathrm{Jac}(Y)$ has real multiplication iff S_1^D and S_2^D are complex subspaces of $H^1(Y, \mathbb{R}) \cong \Omega(Y)$ so the result follows. (Cf. [Mc4, Lemma 7.4].) ∎

Sections. Let $E_D = X_D - X_D(1)$ denote the space of Jacobians in X_D, and $\widetilde{E}_D = \mathbb{H}\times\mathbb{H} - \widetilde{X}_D(1)$ its preimage in the universal cover. (The notation comes from [Mc7, §4], where we consider the space of eigenforms ΩE_D as a closed, $\mathrm{GL}_2^+(\mathbb{R})$-invariant subset of $\Omega\mathcal{M}_g$.)

By the preceding result, the Jacobian of any $Y \in \mathcal{T}_g(S_1^D)$ is an Abelian variety with real multiplication. Moreover, the marking of Y determines a marking

$$L \cong H_1(Y,\mathbb{Z}) \cong H_1(\mathrm{Jac}(Y),\mathbb{Z})$$

of its Jacobian, and thus a map

$$\mathrm{Jac} : \mathcal{T}_g(S_1^D) \to \widetilde{E}_D = \widetilde{X}_D - \widetilde{X}_D(1).$$

The basis (a_i, b_i) yields a pair of normalized forms $\omega_1, \omega_2 \in \Omega(Y)$. Similarly, we have a pair of normalized eigenforms $\eta_1, \eta_2 \in \Omega(A_\tau)$ for each $\tau \in \widetilde{X}_D$, characterized by (3.2). Under the identification $\Omega(Y) = \Omega(\mathrm{Jac}(Y))$, we find:

Theorem 6.3 *The forms ω_i and η_i are equal for any $Y \in \mathcal{T}_g(S_1^D)$. Thus* $\mathrm{Jac}(Y) = A_{(\tau_1,\tau_2)}$, *where*

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = \tau_{ij}(Y) = \left(\int_{b_i} \omega_j \right). \tag{6.1}$$

Proof. The period map $\phi_\tau : L \to \mathbb{C}^2$ for $A_\tau = \mathrm{Jac}(Y)$ is given by

$$\phi_\tau(C) = \left(\int_C \eta_1, \int_C \eta_2 \right) = (x_1 + x_2\tau_1, x_1' + x_2'\tau_2),$$

where $C = (x_1, x_2) \in \mathcal{O}_D \oplus \mathcal{O}_D^\vee$; in particular, we have

$$\phi_\tau(a) = \phi_\tau(1,0) = (1,1).$$

Since ϕ_τ diagonalizes the action of K, we also have

$$\phi_\tau(C) = \left(\int_{C_1} \eta_1, \int_{C_2} \eta_2 \right)$$

for any $C = C_1 + C_2 \in L \otimes \mathbb{R} = V_1 \oplus V_2$. Setting $C = a$, this implies $\phi_\tau(a_1) = (1,0)$ and $\phi_\tau(a_2) = (0,1)$; thus $\int_{a_i} \eta_j = \delta_{ij}$, and therefore $\eta_i = \omega_i$ for $i = 1,2$. Similarly, we have

$$\phi_\tau(b) = (\tau_1, \tau_2) = (\tau_{11}, \tau_{22}),$$

which implies Y and A_τ are related by (6.1). ∎

Corollary 6.4 *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{T}_g(S_1^D) & \xrightarrow{\text{Jac}} & \widetilde{E}_D \\ & \searrow^{\tau_{11}} & \downarrow \tau_1 \\ & & \mathbb{H}. \end{array}$$

Proof of Theorem 6.1. Using the Torelli theorem, it follows easily that $\mathrm{Jac} : \mathcal{T}_g(S_1^D) \to \widetilde{E}_D$ is a holomorphic covering map. Since $\mathbb{H}$ is simply-connected, any section s of τ_1 lifts to a section $\mathrm{Jac}^{-1} \circ s$ of τ_{11}. Thus Theorem 5.1 immediately implies Theorem 6.1. ∎

7 Holomorphic motions

In this section we use the theory of holomorphic motions to define and characterize the foliation $\mathcal{F}_D$.

Holomorphic motions. Given a set $E \subset \widehat{\mathbb{C}}$ and a basepoint $s \in \mathbb{H}$, a *holomorphic motion* of E over $(\mathbb{H}, s)$ is a family of injective maps

$$F_t : E \to \widehat{\mathbb{C}}, \quad t \in \mathbb{H},$$

such that $F_s(z) = z$ and $F_t(z)$ is a holomorphic function of t.

A holomorphic motion of E has a unique extension to a holomorphic motion of its closure $\overline{E}$; and each map $F_t : E \to \widehat{\mathbb{C}}$ extends to a quasiconformal homeomorphism of the sphere. In particular, $F_t|\operatorname{int}(E)$ is quasiconformal (see e.g. [Dou]).

These properties imply:

Theorem 7.1 *Let P be a partition of $\mathbb{H} \times \mathbb{H}$ into disjoint graphs of holomorphic functions. Then:*

1. *P is the set of leaves of a transversally quasiconformal foliation $\mathcal{F}$ of $\mathbb{H} \times \mathbb{H}$; and*

2. *If we adjoin the graphs of the constant functions $f : \mathbb{H} \to \partial\mathbb{H}$ to P, we obtain a continuous foliation of $\mathbb{H} \times \overline{\mathbb{H}}$.*

The foliation $\mathcal{F}_D$. Recall that every component of $\widetilde{X}_D(1) \subset \mathbb{H} \times \mathbb{H}$ is the graph of a Möbius transformation. By Theorem 6.1, there is a unique partition of $\mathbb{H} \times \mathbb{H} - \widetilde{X}_D(1)$ into the graphs of holomorphic maps as well.

Taken together, these graphs form the leaves of a foliation $\widetilde{\mathcal{F}}_D$ of $\mathbb{H} \times \mathbb{H}$ by the preceding result. Since $\widetilde{X}_D(1)$ is invariant under $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$, the foliation $\widetilde{\mathcal{F}}_D$ descends to a foliation $\mathcal{F}_D$ of X_D.

To characterize $\mathcal{F}_D$, recall that the surface X_D admits a holomorphic involution $\iota(\tau_1, \tau_2) = (\tau_2, \tau_1)$ which preserves $X_D(1)$.

Theorem 7.2 *The only leaves shared by $\mathcal{F}_D$ and $\iota(\mathcal{F}_D)$ are the curves in $X_D(1)$.*

Proof. Let $f : \mathbb{H} \to \mathbb{H}$ be a holomorphic function whose graph F is both a leaf of $\widetilde{\mathcal{F}}_D$ and $\iota(\widetilde{\mathcal{F}}_D)$. Then $\iota(F)$ is also a graph, so f is an isometry. But if $F \cap \widetilde{X}_D(1) = \emptyset$, then F lifts to a leaf of the foliation $\mathcal{F}_1$ of Teichmüller space, and hence f is a contraction by [Mc4, Thm. 4.2]. ■

Corollary 7.3 *The only leaves of $\widetilde{\mathcal{F}}_D$ that are graphs of Möbius transformations are those belonging to $\widetilde{X}_D(1)$.*

Complex geodesics. Let us say $\mathcal{F}$ is a foliation by *complex geodesics* if each leaf is a hyperbolic Riemann surface, isometrically immersed for the Kobayashi metric. We can then characterize $\mathcal{F}_D$ as follows.

Theorem 7.4 *Up to the action of ι, $\mathcal{F}_D$ is the unique extension of the lamination $X_D(1)$ to a foliation of X_D by complex geodesics.*

Proof. Let $\mathcal{F}$ be a foliation by complex geodesics extending $X_D(1)$. Then every leaf of its lift $\widetilde{\mathcal{F}}$ to $\widetilde{X}_D$ is a Kobayashi geodesic for $\mathbb{H} \times \mathbb{H}$. But a complex geodesic in $\mathbb{H} \times \mathbb{H}$ is either the graph of a holomorphic function or its inverse, so every leaf belongs to either $\widetilde{\mathcal{F}}_D$ or $\iota(\widetilde{\mathcal{F}}_D)$. Consequently every leaf of $\mathcal{F}$ is a leaf of $\mathcal{F}_D$ or $\iota(\mathcal{F}_D)$. Since these foliations have no leaves in common on the open set $U = X_D - X_D(1)$, $\mathcal{F}$ coincides with one or the other. ■

Stable curves. The Abelian varieties $E \times F$ in $X_D(1)$ are the Jacobians of certain *stable curves* with real multiplication, namely the nodal curves $Y = E \vee F$ obtained by gluing E to F at a single point. If we adjoin these stable curves to $\mathcal{M}_2$, we obtain a partial compactification $\mathcal{M}_2^*$ which maps isomorphically to $\mathcal{A}_2$. The locus $X_D(1)$ can then be regarded as the projection to X_D of a finite set of $\mathrm{GL}_2^+(\mathbb{R})$ orbits in $\Omega\mathcal{M}_2^*$, giving another proof that it is a lamination.

8 Quasiconformal dynamics

In this section we use the relative period map $\rho = \int_{y_1}^{y_2} \eta_1$ to define a meromorphic quadratic differential $q = (d\rho)^2$ transverse to $\mathcal{F}_D$. We then show the transverse dynamics of $\mathcal{F}_D$ is given by Teichmüller mappings relative to q.

Absolute periods. The level sets of τ_1 form the leaves of a holomorphic foliation $\widetilde{\mathcal{A}}_D$ on $\mathbb{H} \times \mathbb{H}$ which covers foliation $\mathcal{A}_D$ of X_D. By (3.2), every $\tau = (\tau_1, \tau_2)$ determines a pair of eigenforms $\eta_1, \eta_2 \in \Omega(A_\tau)$ such that the *absolute periods*

$$\int_C \eta_1, \quad C \in H_1(A_\tau, \mathbb{Z})$$

are constant along the leaves of $\widetilde{\mathcal{A}}_D$. Since every leaf of $\widetilde{\mathcal{F}}_D$ is the graph of a function $f : \mathbb{H} \to \mathbb{H}$, we have:

Theorem 8.1 *The foliation $\mathcal{A}_D$ is transverse to $\mathcal{F}_D$.*

The Weierstrass curve. Recall that $E_D \subset X_D$ denotes the locus of Jacobians with real multiplication by $\mathcal{O}_D$. For $[A_\tau] = \operatorname{Jac}(Y) \in E_D$ we can regard the eigenforms η_1, η_2 as holomorphic 1-forms in $\Omega(Y) \cong \Omega(A_\tau)$.

Let $W_D \subset E_D$ denote the locus where η_1 has a double zero on Y. By [Mc5] we have:

Theorem 8.2 *The locus W_D is an algebraic curve with one or two irreducible components, each of which is a leaf of $\mathcal{F}_D$.*

We refer to W_D as the *Weierstrass curve*, since η_1 vanishes at a Weierstrass point of Y.

Relative periods. Let $E_D(1,1) = X_D - (W_D \cup X_D(1))$ denote the Zariski open set where η_1 has a pair of simple zeros, and let $\widetilde{E}_D(1,1)$ be its preimage in the universal cover $\widetilde{X}_D$. Let

$$\mathbb{H}_s = \{s\} \times \mathbb{H} \subset \mathbb{H} \times \mathbb{H},$$

and let $\mathbb{H}_s^* = \mathbb{H}_s \cap \widetilde{E}_D(1,1)$.

For each $\tau \in \mathbb{H}_s^*$, let y_1, y_2 denote the zeros of the associated form $\eta_1 \in \Omega(Y)$. We can then define the (multivalued) *relative period map* $\rho_s : \mathbb{H}_s^* \to \mathbb{C}$ by

$$\rho_s(\tau) = \int_{y_1}^{y_2} \eta_1.$$

To make $\rho_s(\tau)$ single-valued, we must (locally) choose (i) an ordering of the zeros y_1 and y_2, and (ii) a path on Y connecting them.

Quadratic differentials. Let z be a local coordinate on $\mathbb{H}_s$, and recall that the absolute periods of η_1 are constant along $\mathbb{H}_s$. Thus if we change the choice of path from y_1 to y_2, the derivative $d\rho/dz$ remains the same; and if we interchange y_1 and y_2, it changes only by sign. Thus the *quadratic differential*

$$q = (d\rho/dz)^2 \, dz^2$$

is globally well-defined on $\mathbb{H}_s^*$.

Theorem 8.3 *The form q extends to a meromorphic quadratic differential on $\mathbb{H}_s$, with simple zeros where $\mathbb{H}_s$ meets $\widetilde{W}_D$, and simple poles where it meets $\widetilde{X}_D(1)$.*

Proof. It is a general result that the period map provides holomorphic local coordinates on any stratum of $\Omega\mathcal{M}_g$ (see [V2], [MS, Lemma 1.1], [KZ]). Thus $\rho_s|\mathbb{H}_s^*$ is holomorphic with $d\rho_s \neq 0$, and hence $q|\mathbb{H}_s^*$ is a nowhere vanishing holomorphic quadratic differential.

To see q acquires a simple zero when η_1 acquires a double zero, note that the relative period map

$$\rho(t) = \int_{-\sqrt{t}}^{\sqrt{t}} (z^2 - t)\, dz = (-4/3)t^{3/2}$$

of the local model $\eta_t = (z^2 - t)\, dz$ satisfies $(d\rho/dt)^2 = 4t$. Similarly, a point of $\mathbb{H}_s \cap \widetilde{X}_D(1)$ is locally modeled by the family of connected sums

$$(Y_t, \eta_t) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2),$$

with $I = [0, \rho(t)] = [0, \pm\sqrt{t}]$. Since $(d\rho/dt)^2 = 1/(4t)$, at these points q has simple poles. ∎

See [Mc7, §6] for more on connected sums.

Teichmüller maps. Now let $f : \mathbb{H}_s \to \mathbb{H}_t$ be a quasiconformal map. We say f is a *Teichmüller map*, relative to a holomorphic quadratic differential q, if its complex dilatation satisfies

$$\mu(f) = \left(\frac{\partial f / \partial \overline{z}}{\partial f / \partial z} \right) \frac{d\overline{z}}{dz} = \alpha \frac{\overline{q}}{|q|}$$

for some $\alpha \in \mathbb{C}^*$. This is equivalent to the condition that $w = f(z)$ is real-linear in local coordinates where $q = dz^2$ and dw^2 respectively. In such charts we can write

$$w = w_0 + D_q(f) \cdot z,$$

with $D_q(f) \in \mathrm{SL}_2(\mathbb{R})$. We refer to $D_q(f)$ as the *linear part* of f; it is only well-defined up to sign, since $z \mapsto -z$ preserves dz^2.

Theorem 8.4 *Given $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ and $s \in \mathbb{H}$, let $\mathbb{H}_t = g(\mathbb{H}_s)$. Then the linear part of $g : \mathbb{H}_s \to \mathbb{H}_t$ is given by $D_q(g) \cdot z = (d - cs)^{-1} z$.*

Proof. Since the Riemann surfaces Y at corresponding points of $\mathbb{H}_s$ and $\mathbb{H}_t$ differ only by marking, the relative period maps ρ_s and ρ_t differ only by the normalization of η_1. This discrepancy is accounted for by equation (3.4), which gives $\rho_t/\rho_s = \chi(g, s) = (d - cs)^{-1}$. Since the coordinates ρ_s and ρ_t linearize q, the map $D_q(g)$ is given by multiplication by $(d - cs)^{-1}$. ■

Now let $C_{st} : \mathbb{H}_s \to \mathbb{H}_t$ be the unique map such that z and $C_{st}(z)$ lie on the same leaf of $\widetilde{\mathcal{F}}_D$.

Theorem 8.5 *The linear part of C_{st} is given by $D_q(C_{st}) = A_t A_s^{-1}$, where $A_u = \left(\begin{smallmatrix} 1 & \mathrm{Re}(u) \\ 0 & \mathrm{Im}(u) \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$.*

Proof. By the definition of $\mathcal{F}_D$, the forms η_1 at corresponding points of $\mathbb{H}_s^*$ and $\mathbb{H}_t^*$ are related by some element $B \in \mathrm{GL}_2^+(\mathbb{R})$ acting on $\Omega\mathcal{T}_g$. Thus $\rho_t = B \circ \rho_s$ and therefore $D_q(C_{st}) = B$. Since the action of B on the absolute periods of η_1 satisfies

$$B(\mathcal{O}_D \oplus \mathcal{O}_D^\vee\, s) = \mathcal{O}_D \oplus \mathcal{O}_D^\vee\, t$$

(in the sense of equation (3.1)), we have $B(1) = 1$ and $B(s) = t$, and thus $B = A_t A_s^{-1}$ as above. ■

Dynamics. Every leaf of $\widetilde{\mathcal{F}}_D$ meets the transversal $\mathbb{H}_s$ in a single point. Thus the action of $g \in \mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ on the space of leaves determines a *holonomy map*

$$\phi_g : \mathbb{H}_s \to \mathbb{H}_s,$$

characterized by the property that $(s, \phi_g(z))$ lies on the same leaf as $g(s, z)$.

Theorem 8.6 *The group $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ acts on $\mathbb{H}_s$ by Teichmüller mappings, satisfying $D_q(\phi_g) = g$ in the case $s = i$.*

(As usual we regard g as a real matrix using $\iota_1 : K \to \mathbb{R}$.)

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and $t = (as - b)/(-cs + d)$; then $\mathbb{H}_t = g(\mathbb{H}_s)$.

Since $\phi_g(z)$ is obtained from $g(s, z)$ by combing it along the leaves of $\widetilde{\mathcal{F}}_D$ back into $\mathbb{H}_s$, we have $\phi_g(s, z) = C_{ts}(g(s, z))$. Thus the chain rule implies

$$D_q(\phi_g) \cdot z = B \cdot z = A_s \circ A_t^{-1}(z/(-cs + d)).$$

Now assume $s = i$. Then we have $B(ai - b) = A_t^{-1}(t) = i$ and $B(-ci + d) = A_t^{-1}(1) = 1$; therefore $B^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ and thus $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = g$. ■

Corollary 8.7 *The foliation $\mathcal{F}_D$ carries a natural transverse invariant measure.*

Proof. Since $\det D_q(\phi_g) = 1$ for all g, the form $|q|$ gives a holonomy-invariant measure on the transversal $\mathbb{H}_s$. ■

Finally we show that, although $\phi_g|\mathbb{H}_s$ is quasiconformal, its continuous extension to $\partial\mathbb{H}_s$ is a Möbius transformation.

Theorem 8.8 *For any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ and $z \in \partial\mathbb{H}_s$, we have*

$$\phi_g(z) = (a'z - b')/(-c'z + d').$$

Proof. By Theorem 7.1, the combing maps C_{st} extend to the identity on $\partial\mathbb{H}_s$. Thus $(t, \phi_g(z)) = g(s, z)$, and the result follows from equation (3.3). ■

Note: if we use the transversal $\mathbb{H}_t$ instead of $\mathbb{H}_s$, the holonomy simply changes by conjugation by C_{st}.

9 Further results

In this section we summarize related results on the density of leaves, isoperiodic forms, holomorphic motions and iterated rational maps.

I. Density of leaves. By [Mc7], the closure of the complex geodesic $f : \mathbb{H} \to \mathcal{M}_2$ generated by a holomorphic 1-form is either an algebraic curve, a Hilbert modular surface or the whole moduli space. Since the leaves of $\mathcal{F}_D$ are examples of such complex geodesics, we obtain:

Theorem 9.1 *Every leaf of $\mathcal{F}_D$ is either a closed algebraic curve, or a dense subset of X_D.*

It is easy to see that the union of the closed leaves is dense when $D = d^2$. On the other hand, the classification of Teichmüller curves in [Mc5] and [Mc6] implies:

Theorem 9.2 *If D is not a square, then $\mathcal{F}_D$ has only finitely many closed leaves. These consist of the components of $W_D \cup X_D(1)$ and, when $D = 5$, the Teichmüller curve generated by the regular decagon.*

II. Isoperiodic forms. Next we discuss interactions between the foliations $\mathcal{F}_D$ and $\mathcal{A}_D$. When $D = d^2$ is a square, the surface X_D is finitely covered by a product, and hence every leaf of $\mathcal{A}_D$ is closed.

Theorem 9.3 *If D is not a square, then every leaf L of $\mathcal{A}_D$ is dense in X_D, and $L \cap F$ is dense in F for every leaf F of $\mathcal{F}_D$.*

Proof. The first result follows from the fact that $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ is a dense subgroup of $\mathrm{SL}_2(\mathbb{R})$, and the second follows from the first by transversality of $\mathcal{A}_D$ and $\mathcal{F}_D$. ■

Let us say a pair of 1-forms $(Y_i, \omega_i) \in \Omega\mathcal{M}_g$ are *isoperiodic* if there is a symplectic isomorphism

$$\phi : H_1(Y_1, \mathbb{Z}) \to H_1(Y_2, \mathbb{Z})$$

such that the period maps

$$I(\omega_i) : H_1(Y_i, \mathbb{Z}) \to \mathbb{C}$$

satisfy $I(\omega_1) = I(\omega_2) \circ \phi$. Since the absolute periods of η_1 are constant along the leaves of $\mathcal{A}_D$, from the preceding result we obtain:

Corollary 9.4 *The $\mathrm{SL}_2(\mathbb{R})$-orbit of any eigenform for real multiplication by $\mathcal{O}_D$, $D \neq d^2$, contains infinitely many isoperiodic forms.*

For a concrete example, let $Q \subset \mathbb{C}$ be a regular octagon containing $[0, 1]$ as an edge. Identifying opposite sides of Q, we obtain the *octagonal form*

$$(Y, \omega) = (Q, dz)/\sim$$

of genus two.

Let $\mathbb{Z}[\zeta] \subset \mathbb{C}$ denote the ring generated by $\zeta = (1+i)/\sqrt{2} = \exp(2\pi i/8)$, equipped with the symplectic form

$$\langle z_1, z_2 \rangle = \mathrm{Tr}_{\mathbb{Q}}^{\mathbb{Q}(\zeta)}((\zeta + \zeta^2 + \zeta^3) z_1 \overline{z}_2 / 4).$$

Then it is easy to check that:

1. The octagonal form ω has a single zero of order 2, and
2. Its period map $I(\omega)$ sends $H_1(Y, \mathbb{Z})$ to $\mathbb{Z}[\zeta]$ by a symplectic isomorphism.

However, these two properties do *not* determine (Y, ω) uniquely. Indeed, ω is an eigenform for real multiplication by $\mathcal{O}_8$, so the preceding Corollary ensures there are infinitely many isoperiodic forms (Y_i, ω_i) in its $\mathrm{SL}_2(\mathbb{R})$ orbit. In other words we have:

Corollary 9.5 *There are infinite many fake octagonal forms in $\Omega\mathcal{M}_2$.*

Note that the forms (Y_i, ω_i) cannot be distinguished by their relative periods either, since they all have double zeros.

A similar statement can be formulated for the pentagonal form on the curve $y^2 = x^5 - 1$.

III. Top-speed motions. Let $F_t : E \to \mathbb{H}$ be a holomorphic motion of $E \subset \mathbb{H}$ over $(\mathbb{H}, s)$. By the Schwarz lemma, we have $\|dF_t(z)/dt\| \le 1$ with respect to the hyperbolic metric on $\mathbb{H}$. Let us say F_t is a *top-speed* holomorphic motion if equality holds everywhere; equivalently, if $t \mapsto F_t(z)$ is an isometry of $\mathbb{H}$ for every $z \in E$.

A top-speed holomorphic motion is *maximal* if it cannot be extended to a top-speed motion of a larger set $E' \supset E$.

Theorem 9.6 *For any discriminant $D \ge 4$, the map*

$$F_t(U(s)) = U(t), \quad U \in \Lambda_D$$

gives a maximal top-speed holomorphic motion of $E = \Lambda_D \cdot s$ over $(\mathbb{H}, s)$.

Proof. Let $t \mapsto f(t) = F_t(z)$ be an extension of the motion to a point $z \notin E$. Then the graph of f is a leaf of $\widetilde{\mathcal{F}}_D$, since it is disjoint from $\widetilde{X}_D(1)$. But the only leaves that are graphs of Möbius transformations are those in $\widetilde{X}_D(1)$, by Corollary 7.3. ∎

Corollary 9.7 *The group* $\Gamma(2) = \{A \in \mathrm{SL}_2(\mathbb{Z}) \ : \ A \equiv I \bmod 2\}$ *gives a maximal top-speed holomorphic motion of* $E = \Gamma(2) \cdot s$ *over* $(\mathbb{H}, s)$.

Proof. We have $\Gamma(2) = g\Lambda_4 g^{-1}$, where $g = \left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ (Theorem 4.4). ■

IV. Iterated rational maps. Finally we explain how the foliation $\mathcal{F}_4$ of X_4 arises in complex dynamics.

First recall that the moduli space of elliptic curves can be described as the quotient orbifold $\mathcal{M}_1 = \widetilde{\mathcal{M}}_1/S_3$, where

$$\widetilde{\mathcal{M}}_1 = \mathbb{H}/\Gamma(2) \cong \mathbb{C} - \{0, 1\}.$$

The deck group S_3 also acts diagonally on $\widetilde{\mathcal{M}}_1 \times \widetilde{\mathcal{M}}_1$, preserving the diagonal Δ.

Theorem 9.8 *For* $D = 4$, *we have* $(X_D, X_D(1)) \cong (\widetilde{\mathcal{M}}_1 \times \widetilde{\mathcal{M}}_1, \Delta)/S_3$.

Proof. Since $\mathcal{O}_4^\vee = (1/2)\,\mathcal{O}_4$, the surface X_4 is isomorphic to $(\mathbb{H} \times \mathbb{H})/\operatorname{SL}_2(\mathcal{O}_4)$. In these coordinates we have $\Lambda_4 = \Gamma(2)$. Since

$$\mathrm{SL}_2(\mathcal{O}_4) \cong \{(A_1, A_2) \in \mathrm{SL}_2(\mathbb{Z}) \ : \ A_1 \equiv A_2 \bmod 2\}$$

contains $\Gamma(2) \times \Gamma(2)$ as a subgroup of index 6, the result follows. ■

Now consider, for each $t \in \widetilde{\mathcal{M}}_1$, the elliptic curve E_t defined by $y^2 = x(x-1)(x-t)$. There is a unique rational map $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$x(2P) = f_t(x(P))$$

with respect to the usual group law on E_t. Indeed, using the fact that $-2P$ lies on the tangent line to E_t at P, we find

$$f_t(z) = \frac{(z^2 - t)^2}{4z(z-1)(z-t)}.$$

Note that the *postcritical set*

$$P(f_t) = \bigcup \{f_t^n(z) \ : \ n > 0, f_t'(z) = 0\}$$

coincides with the branch locus $\{0, 1, t, \infty\}$ of the map $x : E_t \to \mathbb{P}^1$.

The rational maps $f_t(z)$ form a *stable family of Lattès examples.* It is well-known that the Julia set of any Lattès example is the whole Riemann sphere; and that in any stable family, the Julia set varies by a holomorphic motion respecting the dynamics (see e.g. [MSS], [Mc1, Ch. 4], [Mil].)

Theorem 9.9 *As t varies in $\widetilde{\mathcal{M}}_1$, the holomorphic motion of $J(f_t)$ sweeps out the lift of the foliation $\mathcal{F}_4$ to the covering space $\widetilde{\mathcal{M}}_1 \times \widetilde{\mathcal{M}}_1$ of X_4.*

Proof. Let $\mathcal{G}$ be the foliation of $\widetilde{\mathcal{M}}_1 \times \mathbb{P}^1$ swept out by $J(f_t)$. Since the holomorphic motion respects the dynamics, it preserves the post-critical set, and thus the leaves of $\mathcal{G}$ include the loci $z = 0, 1, \infty$ as well as the diagonal $t = z$. In particular, $\mathcal{G}$ restricts to a foliation of the finite cover $\widetilde{\mathcal{M}}_1 \times \widetilde{\mathcal{M}}_1 - \Delta$ of $X_4 - X_4(1)$. Since each leaf of $\mathcal{G}$ lifts to the graph of a holomorphic function in the universal cover $\mathbb{H} \times \mathbb{H}$, it lies over a leaf of $\mathcal{F}_D$ by the uniqueness part of Theorem 1.2. ∎

Algebraic curves. The loci $f_t^n(z) = \infty$ form a dense set of algebraic leaves of $\mathcal{G}$ that can easily be computed inductively. The real points of these curves are graphed in Figure 1; thus the figure depicts the lift of $\mathcal{F}_4$ to the finite cover $\widetilde{\mathcal{M}}_1 \times \widetilde{\mathcal{M}}_1$ of X_4.

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