Pseudo-amenability of Brandt semigroup algebras

Maysam Maysami Sadr

Abstract. In this paper it is shown that for a Brandt semigroup S over a group G with an arbitrary index set I, if G is amenable, then the Banach semigroup algebra $\ell^1(S)$ is pseudo-amenable.

Keywords: pseudo-amenability, Brandt semigroup algebra, amenable group

Classification: 43A20, 43A07, 46H20

1. Introduction

The concept of amenability for Banach algebras was introduced by Johnson in 1972 [6]. Several modifications of this notion, such as approximate amenability and pseudo-amenability, were introduced in [2] and [4]. In the current paper we investigate the pseudo-amenability of Brandt semigroup algebras. It was shown in [2] and [4] that for the group algebra $L^1(G)$, amenability, approximate amenability and pseudo-amenability coincide and are equivalent to the amenability of locally compact group G. In the semigroup case we know that, if S is a discrete semigroup, then amenability of $\ell^1(S)$ implies that S is regular and amenable [1]. Ghahramani et al. [3] have shown that, if $\ell^1(S)$ is approximately amenable, then S is regular and amenable. The present author and Pourabbas in [9] have shown that for a Brandt semigroup S over a group G with an index set I, the following are equivalent.

(i) $\ell^1(S)$ is amenable.
(ii) $\ell^1(S)$ is approximately amenable.
(iii) I is finite and G is amenable.

This result corrects [7, Theorem 1.8]. In the present paper we show that for a Brandt semigroup S over a group G with an arbitrary (finite or infinite) index set I, amenability of G implies pseudo-amenability of $\ell^1(S)$.

2. Preliminaries

Throughout $\hat{\otimes}$ denotes the completed projective tensor product. For an element x of a set X, δ_x is its point mass measure in $\ell^1(X)$. Also, we frequently use the identification $\ell^1(X \times Y) = \ell^1(X)\hat{\otimes}\ell^1(Y)$ for the sets X and Y.

A Banach algebra A is called (approximately) amenable, if for any *dual* Banach A-bimodule E, every bounded *derivation* from A to E is (*approximately*) *inner*. It is well known that amenability of A is equivalent to existence of a *bounded approximate diagonal*, that is a bounded net $(m_i) \in A\hat{\otimes}A$ such that for every

$a \in A$, $a \cdot m_i - m_i \cdot a \longrightarrow 0$ and $\pi(m_i)a \longrightarrow a$, where $\pi : A\hat{\otimes}A \longrightarrow A$ is the continuous bimodule homomorphism defined by $\pi(a \otimes b) := ab$ $(a, b \in A)$, and called the *diagonal map*. The famous Johnson Theorem [6], says that, for any locally compact group G, amenability of G and $L^1(G)$ are equivalent. For a modern account on amenability see [8] and for approximate amenability see the original papers [2] and [3].

A Banach algebra A is called pseudo-amenable ([4]) if there is a net $(n_i) \in A\hat{\otimes}A$, called an *approximate diagonal* for A, such that $a \cdot n_i - n_i \cdot a \longrightarrow 0$ and $\pi(n_i)a \longrightarrow a$ for each $a \in A$.

Let I be a nonempty set and let G be a discrete group. Consider the set $T := I \times G \times I$, add a null element $\emptyset$ to T, and define a semigroup multiplication on $S := T \cup \{\emptyset\}$, as follows. For $i, i', j, j' \in I$ and $g, g' \in G$, let

$$(i, g, j)(i', g', j') = \begin{cases} (i, gg', j') & \text{if } j = i', \\ \emptyset & \text{if } j \neq i', \end{cases}$$

also let $\emptyset(i, g, j) = (i, g, j)\emptyset = \emptyset$ and $\emptyset\emptyset = \emptyset$. Then S becomes a semigroup that is called Brandt semigroup over G with index I, and usually denoted by $B(I, G)$. For more details see [5].

The Banach space $\ell^1(T)$, with the convolution product,

$$(ab)(i, g, j) = \sum_{k\in I, h\in G} a(i, gh^{-1}, k)b(k, h, j),$$

for $a, b \in \ell^1(T)$, $i, j \in I$, $g \in G$, becomes a Banach algebra. (Note that if G is the one point group, and I is finite, then $\ell^1(T)$ is an ordinary matrix algebra.) We have a closed relation between the Banach algebra $\ell^1(T)$ and the Banach semigroup algebra $\ell^1(S)$:

Lemma 1. *There exists a homeomorphic isomorphism $\ell^1(S) \cong \ell^1(T) \oplus \mathbb{C}$ of Banach algebras, where the multiplication of $\ell^1(T) \oplus \mathbb{C}$ is coordinatewise.*

PROOF: Consider the following short exact sequence of Banach algebras and continuous algebra homomorphisms:

$$0 \longrightarrow \ell^1(T) \longrightarrow \ell^1(S) \longrightarrow \mathbb{C} \longrightarrow 0,$$

where the second arrow $\Psi : \ell^1(T) \longrightarrow \ell^1(S)$ is defined by $\Psi(b)(t) := b(t)$ and $\Psi(b)(\emptyset) := -\sum_{s\in T} b(s)$, for $b \in \ell^1(T)$ and $t \in T \subset S$, and the third arrow $\Phi : \ell^1(S) \longrightarrow \mathbb{C}$ is the integral functional, $\Phi(a) := \sum_{s\in S} a(s)$ $(a \in \ell^1(S))$. Now, let $\Theta : \ell^1(S) \longrightarrow \ell^1(T)$ be the restriction map, $\Theta(a) := a\,|_T$. Then Θ is a continuous algebra homomorphism and $\Theta\Psi = \mathrm{Id}_{\ell^1(T)}$. Thus the exact sequence splits and we have $\ell^1(S) \cong \ell^1(T) \oplus \mathbb{C}$. □

Lemma 2. *If $\ell^1(T)$ is pseudo-amenable, then so is $\ell^1(S)$.*

PROOF: Suppose that $\ell^1(T)$ is pseudo-amenable. Then by Lemma 1 and [4, Proposition 2.1], $\ell^1(S)$ is pseudo-amenable. □

3. The main result

Let S, T, G and I be as above. We need some other notations and computations:

For $a \in \ell^1(T)$ and every $u, v \in I$, let $a_{(u,v)}$ be an element of $\ell^1(G)$ defined by $a_{(u,v)}(g) := a(u,g,v)$ $(g \in G)$. Note that

$$\|a\|_{\ell^1(T)} = \sum_{u,v \in I} \|a_{(u,v)}\|_{\ell^1(G)}.$$

For $b \in \ell^1(G \times G)$, $c \in \ell^1(G)$ and any $i, j, i', j' \in I$, let $E^b_{(i,j,i',j')}$ and $H^c_{(i,j)}$ be elements of $\ell^1(T \times T)$ and $\ell^1(T)$ respectively, defined by

$$E^b_{(i,j,i',j')}(u,g,v,u',g',v') = \begin{cases} b(g,g') & \text{if } u = i, v = j, u' = i', v' = j', \\ 0 & \text{otherwise,} \end{cases}$$

$$H^c_{(i,j)}(u,g,v) = \begin{cases} c(g) & \text{if } u = i, v = j, \\ 0 & \text{otherwise,} \end{cases}$$

where $u, v, u', v' \in I$ and $g, g' \in G$. Also note that

$$\text{(1)} \qquad \|E^b_{(i,j,i',j')}\|_{\ell^1(T \times T)} = \|b\|_{\ell^1(G \times G)}, \qquad \|H^c_{(i,j)}\|_{\ell^1(T)} = \|c\|_{\ell^1(G)}.$$

For $u, v \in I$ and $g \in G$, the module action of $\ell^1(T)$ on $\ell^1(T \times T)$ becomes

$$\text{(2)} \qquad \delta_{(u,g,v)} \cdot E^b_{(i,j,i',j')} = \begin{cases} E^{\delta_g \cdot b}_{(u,j,i',j')} & \text{if } i = v, \\ 0 & \text{if } i \neq v, \end{cases}$$

$$\text{(3)} \qquad E^b_{(i,j,i',j')} \cdot \delta_{(u,g,v)} = \begin{cases} E^{b \cdot \delta_g}_{(i,j,i',v)} & \text{if } j' = u, \\ 0 & \text{if } j' \neq u. \end{cases}$$

For the multiplication of $\ell^1(T)$ we have

$$\text{(4)} \quad \delta_{(u,g,v)} H^c_{(i,j)} = \begin{cases} H^{\delta_g c}_{(u,j)} & \text{if } i = v, \\ 0 & \text{if } i \neq v, \end{cases} \qquad H^c_{(i,j)} \delta_{(u,g,v)} = \begin{cases} H^{c\delta_g}_{(i,v)} & \text{if } j = u, \\ 0 & \text{if } j \neq u. \end{cases}$$

And finally, the diagonal maps $\pi : \ell^1(T \times T) \longrightarrow \ell^1(T)$ and $\pi : \ell^1(G \times G) \longrightarrow \ell^1(G)$ have the relation

$$\text{(5)} \qquad \pi(E^b_{(i,j,i',j')}) = \begin{cases} H^{\pi(b)}_{(i,j')} & \text{if } j = i', \\ 0 & \text{if } j \neq i'. \end{cases}$$

We are now ready to prove our main result:

Theorem 3. *Suppose that G is amenable. Then $\ell^1(S)$ is pseudo-amenable.*

Proof: Let $(m_\lambda)_{\lambda\in\Lambda}\in\ell^1(G\times G)$ be a bounded approximate diagonal for the amenable Banach algebra $\ell^1(G)$. For any finite nonempty subset F of I and $\lambda\in\Lambda$, let

$$W_{F,\lambda}:=\frac{1}{\#F}\sum_{i,j\in F}E^{m_\lambda}_{(i,j,j,i)},$$

where $\#F$ denotes the cardinal of F. We show that the net $(W_{F,\lambda})\in\ell^1(T\times T)$ over the directed set $\Gamma\times\Lambda$, where Γ is the directed set of finite subsets of I ordered by inclusion, is an approximate diagonal for $\ell^1(T)$.

For any $u,v\in I$ and $g\in G$, by equations (2) and (3), we have,

$$\delta_{(u,g,v)}\cdot W_{F,\lambda}=\begin{cases}\frac{1}{\#F}\sum_{j\in F}E^{\delta_g\cdot m_\lambda}_{(u,j,j,v)} & \text{if } v\in F,\\ 0 & \text{if } v\notin F,\end{cases}$$

$$W_{F,\lambda}\cdot\delta_{(u,g,v)}=\begin{cases}\frac{1}{\#F}\sum_{j\in F}E^{m_\lambda\cdot\delta_g}_{(u,j,j,v)} & \text{if } u\in F,\\ 0 & \text{if } u\notin F,\end{cases}$$

and thus,

$$\delta_{(u,g,v)}\cdot W_{F,\lambda}-W_{F,\lambda}\cdot\delta_{(u,g,v)}=\begin{cases}\frac{1}{\#F}\sum_{j\in F}E^{\delta_g\cdot m_\lambda-m_\lambda\cdot\delta_g}_{(u,j,j,v)} & \text{if } u\in F, v\in F,\\ \frac{1}{\#F}\sum_{j\in F}E^{\delta_g\cdot m_\lambda}_{(u,j,j,v)} & \text{if } v\in F, u\notin F,\\ -\frac{1}{\#F}\sum_{j\in F}E^{m_\lambda\cdot\delta_g}_{(u,j,j,v)} & \text{if } u\in F, v\notin F,\\ 0 & \text{if } v\notin F, u\notin F.\end{cases}$$

Then, for $a=\sum_{u,v\in I,g\in G}a(u,g,v)\delta_{(u,g,v)}$ in $\ell^1(T)$ we have

$$\begin{aligned}a\cdot W_{F,\lambda}-W_{F,\lambda}\cdot a=&\frac{1}{\#F}\sum_{j,u,v\in F}E^{a_{(u,v)}\cdot m_\lambda-m_\lambda\cdot a_{(u,v)}}_{(u,j,j,v)}\\ &+\frac{1}{\#F}\sum_{j,v\in F,u\in I-F}E^{a_{(u,v)}\cdot m_\lambda}_{(u,j,j,v)}\\ &-\frac{1}{\#F}\sum_{j,u\in F,v\in I-F}E^{m_\lambda\cdot a_{(u,v)}}_{(u,j,j,v)},\end{aligned}$$

and thus, by (1),

$$\begin{aligned}\left\|a\cdot W_{F,\lambda}-W_{F,\lambda}\cdot a\right\|\le&\sum_{u,v\in F}\left\|a_{(u,v)}\cdot m_\lambda-m_\lambda\cdot a_{(u,v)}\right\|\\ &+\sum_{v\in F,u\in I-F}\left\|a_{(u,v)}\cdot m_\lambda\right\|\\ &+\sum_{u\in F,v\in I-F}\left\|m_\lambda\cdot a_{(u,v)}\right\|.\end{aligned}\tag{6}$$

Now, suppose that $M > 0$ is a bound for the norms of m_λ's. Let $\epsilon > 0$ be arbitrary, and let F_0 be an element of Γ such that

$$\sum_{(u,v)\in J_0, g\in G} |a(u,g,v)| = \sum_{(u,v)\in J_0} \left\| a_{(u,v)} \right\| < \epsilon,$$

where $J_0 = (I \times (I - F_0)) \cup ((I - F_0) \times I)$. And choose a $\lambda_0 \in \Lambda$ such that for every $\lambda \geq \lambda_0$,

$$\sum_{u,v\in F_0} \left\| a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)} \right\| < \epsilon.$$

Now, if $(F, \lambda) \in \Gamma \times \Lambda$ such that $F_0 \subseteq F$, $\lambda \geq \lambda_0$, then we have,

$$\begin{aligned}\sum_{u,v\in F} \left\| a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)} \right\| &\leq \sum_{u,v\in F_0} \left\| a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)} \right\| \\ &\quad + \sum_{(u,v)\in J_0} \left\| a_{(u,v)} \cdot m_\lambda \right\| \\ &\quad + \sum_{(u,v)\in J_0} \left\| m_\lambda \cdot a_{(u,v)} \right\| \\ &< \epsilon + \epsilon M + \epsilon M,\end{aligned}$$

and analogously,

$$\sum_{v\in F, u\in I-F} \| a_{(u,v)} \cdot m_\lambda \| < \epsilon M$$

and

$$\sum_{u\in F, v\in I-F} \| m_\lambda \cdot a_{(u,v)} \| < \epsilon M.$$

Thus by (6), we have $\| a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a \| < \epsilon + 4\epsilon M$.

Therefore, we proved that $a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a \longrightarrow 0$, for every $a \in \ell^1(T)$.

Now, we prove that $\pi(W_{F,\lambda})a \longrightarrow a$ for any $a \in \ell^1(T)$.

By (5), we have

$$\pi(W_{F,\lambda}) = \frac{1}{\#F} \sum_{i,j\in F} H_{(i,i)}^{\pi(m_\lambda)} = \sum_{i\in F} H_{(i,i)}^{\pi(m_\lambda)}.$$

Thus, (4) implies that

$$\pi(W_{F,\lambda})a = \sum_{i\in F, v\in I} H_{(i,v)}^{\pi(m_\lambda)a_{(i,v)}},$$

since $a=\sum_{u,v\in I} H_{(u,v)}^{a_{(u,v)}}$. Then we have,

$$\left\|\pi(W_{F,\lambda})a-a\right\| \leq \sum_{i\in F, v\in I}\left\|H_{(i,v)}^{\pi(m_\lambda)a_{(i,v)}-a_{(i,v)}}\right\| + \sum_{v\in I, u\in I-F}\left\|H_{(u,v)}^{a_{(u,v)}}\right\|. \tag{7}$$

Let $\epsilon>0$ be arbitrary, and let F_0 and J_0 be as above. Choose a $\lambda_1\in\Lambda$ such that for every $\lambda\geq\lambda_1$,

$$\sum_{i,j\in F_0}\left\|\pi(m_\lambda)a_{(i,j)}-a_{(i,j)}\right\|<\epsilon.$$

Now, if $(F,\lambda)\in\Gamma\times\Lambda$ is such that $F_0\subseteq F$, $\lambda\geq\lambda_1$, then by (1) we have,

$$\begin{aligned}\sum_{i\in F, v\in I}\left\|H_{(i,v)}^{\pi(m_\lambda)a_{(i,v)}-a_{(i,v)}}\right\| &\leq \sum_{i,j\in F_0}\left\|\pi(m_\lambda)a_{(i,j)}-a_{(i,j)}\right\| \\ &\quad+\sum_{(u,v)\in J_0}\left\|\pi(m_\lambda)a_{(u,v)}\right\|+\sum_{(u,v)\in J_0}\left\|a_{(u,v)}\right\| \\ &<\epsilon+\epsilon M+\epsilon,\end{aligned}$$

and

$$\sum_{v\in I, u\in I-F}\|H_{(u,v)}^{a_{(u,v)}}\| = \sum_{v\in I, u\in I-F}\|a_{(u,v)}\|<\epsilon.$$

Thus, by (7) we have

$$\left\|\pi(W_{F,\lambda})a-a\right\|<3\epsilon+\epsilon M.$$

This completes the proof. □

We end with a natural question:

Question 4. *Does pseudo-amenability of $\ell^1(B(I,G))$ imply amenability of G?*

References

[1] Duncan J., Namioka I., *Amenability of inverse semigroups and their semigroup algebras*, Proc. Royal Soc. Edinburgh Sect. A **80** (1978), 309–321.

[2] Ghahramani F., Loy R.J., *Generalized notion of amenability*, J. Funct. Anal. **208** (2004), 229–260.

[3] Ghahramani F., Loy R.J., Zhang Y., *Generalized notions of amenability II*, J. Funct. Anal. **254** (2008), 1776–1810.

[4] Ghahramani F., Zhang Y., *Pseudo-amenable and pseudo-contractible Banach algebras*, Math. Proc. Cambridge Philos. Soc. **142** (2007), 111–123.

[5] Howie J.M., *An Introduction to Semigroup Theory*, Academic Press, London, 1976.

[6] Johnson B.E., *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127** (1972), pp 96.

[7] Lashkarizadeh Bami M., Samea H., *Approximate amenability of certain semigroup algebras*, Semigroup Forum **71** (2005), 312–322.

[8] Runde V., *Lectures on amenability*, Lecture Notes in Mathematics, 1774, Springer, Berlin, 2002.
[9] Sadr M.M., Pourabbas A., *Approximate amenability of Banach category algebras with application to semigroup algebras*, Semigroup Forum, to appear.

Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), P.O. Box 45195-1159, Zanjan, Iran
Email: sadr@iasbs.ac.ir

(*Received* March 25, 2009)