

# Pacific Journal of Mathematics

**PSEUDO-ANALYTIC VECTORS ON PSEUDO-KÄHLERIAN  
MANIFOLD**

SHIGEO SASAKI AND KENTARO YANO

# PSEUDO-ANALYTIC VECTORS ON PSEUDO-KÄHLERIAN MANIFOLDS

SHIGEO SASAKI AND KENTARO YANO

**1. Introduction.** A pseudo-Kählerian manifold is by definition a Riemannian manifold  $M^{2n}$  of class  $C^r$  ( $r \geq 2$ ) which has a skew-symmetric tensor field  $I_{AB}^1$  of class  $C^{r-1}$  with non-vanishing determinant satisfying following two conditions:

$$(1) \quad I^A_B I^B_C = -\delta^A_C, \quad (I^{AB} I_{BC} = -\delta^A_C)$$

$$(2) \quad I_{AB, C} = 0,$$

where

$$(3) \quad I^A_B = g^{AE} I_{EB}, \quad I^{AB} = g^{AE} g^{BF} I_{EF},$$

and a comma denotes the covariant differentiation with respect to  $g_{AB}$ . It is known that the real representation of a Kählerian manifold of complex dimension  $n$  is a pseudo-Kählerian manifold of dimension  $2n$  and of class  $C^\omega$  and the converse is also true. However, the problem whether a pseudo-Kählerian manifold  $M^{2n}$  of class  $C^r$  ( $r \neq \omega$ ) can be regarded, by introducing suitable complex coordinate systems on  $M^{2n}$ , as a real representation of a (complex) Kählerian manifold or not is, as far as we know, still an open problem. In this paper we shall generalize some theorems which concern analytic vectors on Kählerian manifolds to pseudo-Kählerian manifolds.

## 2. Definitions of pseudo-analyticity.

**DEFINITION 1.** A set of *functions*  $(\phi, \psi)$  defined over a pseudo-Kählerian manifold  $M^{2n}$  is said to be *pseudo-analytic* if

$$(4) \quad I^A_B \phi_{,A} = \psi_{,B}.$$

If  $(\phi, \psi)$  is pseudo-analytic, then  $(-\psi, \phi)$  is pseudo-analytic too.

**DEFINITION 2.** A *contravariant vector field*  $u^A$  defined over  $M^{2n}$  is said to be *pseudo-analytic* if

$$(5) \quad I^A_B u^B_{,C} = u^A_{,B} I^B_C.$$

Received June 14, 1954.

<sup>1</sup> We assume that the indices run as follows:

$$\begin{aligned} \alpha, \beta, \gamma, \dots &= 1, 2, \dots, n, \\ A, B, C, \dots &= 1, 2, \dots, n, n+1, \dots, 2n. \end{aligned}$$

DEFINITION 3. A covariant vector field  $u_A$  defined over  $M^{2n}$  is said to be *pseudo-analytic* if

$$(6) \quad I^A_B u_{A,C} = I^A_C u_{B,A}.$$

If we denote a complex coordinate system of a Kählerian manifold  $K^n$  by  $z^\alpha (=x^\alpha + iy^\alpha)$  and take  $(x^\alpha, y^\alpha)$  as coordinates of the real representation of  $K^n$ , then (cf. [3])

$$(7) \quad I^\alpha_\beta = I^{n+\alpha}_{n+\beta} = 0, \quad I^\alpha_{n+\beta} = -I^{n+\alpha}_\beta = \delta^\alpha_\beta.$$

In this case, (4), (5) and (6) are nothing but Cauchy-Riemann equations for a complex analytic function  $\phi + i\psi$ , for a self-adjoint complex analytic contravariant vector  $u^\alpha + iu^{n+\alpha}$  and for a self-adjoint complex analytic covariant vector  $u_\alpha + iu_{n+\alpha}$ . (We must take account of the fact that the real representation of a contravariant vector  $u^\alpha + iu^{n+\alpha}$  is  $(u^\alpha, u^{n+\alpha})$  and that of a covariant vector  $u_\alpha + iu_{n+\alpha}$  is  $(2u_\alpha, -2u_{n+\alpha})$ ). Hence, the Definitions 1, 2 and 3 are appropriate.

When  $I^A_B$  takes the value (7), (5) means that  $u^A_{,B}$  is a matrix which is the real representation of a unitary  $(n \times n)$  matrix. Hence we may say that  $u^A_{,B}$  is pseudo-unitary.

THEOREM 1. *If a set of functions  $(\phi, \psi)$  is pseudo-analytic, then  $\phi$  and  $\psi$  are both harmonic functions on our pseudo-Kählerian manifold.*

*Proof.* By hypothesis

$$\psi_{,B} = I^A_B \phi_{,A},$$

hence we get

$$\Delta\psi = \psi_{,BC} g^{BC} = I^{AC} \phi_{,AC} = 0.$$

As (4) can be written also in the form

$$\phi_{,B} = -I^A_B \psi_{,A},$$

we get in the same way  $\Delta\phi = 0$ .

THEOREM 2. *If a contravariant vector field  $u^A$  and its associated covariant vector  $u_A$  are both pseudo-analytic, then  $u^A$  is a parallel vector field.*

*Proof.* If we use covariant components of  $u^A$ , then (5) can be written as

$$-I^A_B u_{A,C} = I^A_C u_{B,A}.$$

Comparing the last equation with (6), we can immediately see that our assertion is true.

**THEOREM 3.** *If  $u^A$  is a pseudo-analytic contravariant vector field, then  $I^A{}_B u^B$  is also pseudo-analytic.*

*Proof.* 
$$\begin{aligned} (I^A{}_B u^B)_{,c} I^c{}_D &= I^A{}_B u^B_{,c} I^c{}_D \\ &= I^A{}_B I^B{}_c u^c_{,D} = I^A{}_B (I^B{}_c u^c)_{,D}. \end{aligned}$$

We shall remark that equation (6) can be written also as

(8) 
$$XI^A{}_c = 0$$

where  $X = u^A(\partial/\partial x^A)$  and  $XI^A{}_c$  is the Lie derivative of  $I^A{}_c$  (cf. [5]). We put further  $Y = v^A(\partial/\partial x^A)$  and

$$[uv]^A = u^B(v^A{}_{,B}) - v^B(u^A{}_{,B})$$

and similarly define  $[Iu, v]^A$ ,  $[u, Iv]^A$ ,  $[Iu, Iv]^A$ . Then we get the following

**THEOREM 4.** *Let  $u^A$  and  $v^A$  be two pseudo-analytic contravariant vector fields, then  $[uv]^A$ ,  $[Iu, v]^A$ ,  $[u, Iv]^A$  and  $[Iu, Iv]^A$  are pseudo-analytic too.*

*Proof.* It is sufficient to prove the pseudo-analyticity of  $[uv]^A$ . As

$$(XY - YX)f = [uv]^A \frac{\partial f}{\partial x^A},$$

it is sufficient to show that

$$(XY - YX)I^A{}_B = 0.$$

However, this follows immediately from the assumption that  $u^A$  and  $v^A$  are pseudo-analytic.

**3. Curvature tensors.**

**THEOREM 5.** (cf. [3])

(9) (i)  $R^A{}_{BCD} I^E{}_B = I^A{}_E R^E{}_{BCD}$ ,  
 (ii)  $R_{ABCD} = I^E{}_A I^F{}_B R_{EFCD}$ .

*Proof.* From (1) we get

$$0 = I^A{}_{B,CD} - I^A{}_{B,DC} = R^A{}_{BCD} I^E{}_B - R^E{}_{BCD} I^A{}_E.$$

Equation (9ii) follows immediately from (9i). The curvature tensor  $R^A{}_{BCD}$  is pseudo-unitary with respect to the first two indices.

**THEOREM 6.**

$$\begin{aligned}
 (10) \quad & \text{(i)} \quad R_{BC} = I^E{}_B I^F{}_C R_{EF}, \\
 & \text{(ii)} \quad R^A{}_B = -I^A{}_E I^F{}_B R^E{}_F, \\
 & \text{(iii)} \quad I^A{}_B R^B{}_C = R^A{}_B I^B{}_C, \\
 & \text{(iv)} \quad I^E{}_A R_{EB} = -I^E{}_B R_{EA}.
 \end{aligned}$$

$R^A{}_B$  is pseudo-unitary too.

**THEOREM 7.**

$$(11) \quad R_{ABCD} I^{AB} = 2I^E{}_C R_{ED} (= -2I^E{}_D R_{EC}).$$

Let us prove Theorems 6 and 7 at the same time. First

$$R_{ABCD} I^{AB} = -I^A{}_E R^E{}_{ACD} = I^A{}_E (R^E{}_{CDA} + R^E{}_{DAC}) = I^E{}_C R^A{}_{EDA} + I^E{}_D R^A{}_{EA}$$

Hence we get

$$(12) \quad R_{ABCD} I^{AB} = I^E{}_C R_{ED} - I^E{}_D R_{EC}.$$

Now, from (9) we see that

$$R_{BC} = I^E{}_A I^F{}_B R_{EFCD} g^{AD} = -I^{ED} I^F{}_B (R_{FCED} + R_{CEFD}).$$

By virtue of (12), the first term of the right hand side of equation becomes  $R_{BC} - R_{EF} I^E{}_B I^F{}_C$  and the second term can be easily to be  $-R_{BC}$ . Hence we get

$$R_{BC} = R_{EF} I^E{}_B I^F{}_C.$$

(10 ii, iii, iv) can be immediately seen to be equivalent to (10 i). use (10 iii), then (12) reduces to (11).

**4. Pseudo-analytic vector fields.**

**THEOREM 8.** *Let  $u_A$  be a pseudo-analytic covariant vector field on a pseudo-Kählerian manifold  $M^{2n}$ , then it satisfies the relation*

$$(13) \quad u_{A,BC} g^{BC} - R^B{}_A u_B = 0.$$

*Epecially, if  $M^{2n}$  is compact, then  $u_A$  is a harmonic vector. (cf. |*

*Proof.* By hypothesis

$$\begin{aligned}
 I^A{}_E u_{A,B} &= I^A{}_B u_{E,A}, \\
 I^A{}_E u_{A,BC} g^{BC} &= I^A{}_B u_{E,AC} g^{BC} = I^{AC} u_{E,AC} = -\frac{1}{2} R^F{}_{EAC} u_F I^{AC}.
 \end{aligned}$$

The last equation can be transformed by (12) into

$$I^A{}_E u_{A,BC} g^{BC} = -I^H{}_F R_{HE} g^{FB} u_B = I^A{}_E R^B{}_A u_B.$$

As  $\det(I^A_E) \neq 0$ , we see that (13) is true. Especially, if  $M^{2n}$  is compact and orientable, then by de Rham's theorem [2],  $u_A$  is a harmonic vector.

**THEOREM 9.** *Let  $u^A$  be a pseudo-analytic contravariant vector field over a pseudo-Kählerian manifold  $M^{2n}$ , then*

$$(14) \quad u^A_{,BC} g^{BC} + R^A_B u^B = 0.$$

*Especially, if  $u^A_{,A} = 0$  and  $M^{2n}$  is compact, then  $u^A$  is a Killing vector. (cf. [4], [6]).*

The proof is quite similar to that of Theorem 8. Instead of de Rham's theorem we use a theorem due to one of the authors (cf. [4], [6]).

**THEOREM 10.** *Suppose that  $u^A$  and  $v_A$  are contravariant and covariant pseudo-analytic vector field over a compact pseudo-Kählerian manifold  $M^{2n}$ . Then  $u^A v_A$  is a constant over the manifold  $M^{2n}$ .*

*Proof.* It is sufficient to show that  $u^A v_A$  is harmonic. We put

$$\phi = u^A v_A.$$

Then we get

$$\Delta\phi = (u^A_{,BC} v_A + 2u^A_{,B} v_{A,C} + u^A v_{A,BC}) g^{BC}.$$

Putting (13) and (14) into the right hand side of the last equation we get

$$\Delta\phi = 2v_{A,C} u^A_{,B} g^{BC}.$$

However, the right hand side can be transformed as follows:

$$\begin{aligned} 2v_{A,C} u^A_{,B} g^{BC} &= -2(u^A_{,E} I^E_D I^D_B) g^{BC} v_{A,C} = -2(I^A_B u^E_{,D} I^D_B) g^{BC} v_{A,C} = -2u^E_{,D} I^{D0} I^F_C v_{E,F} \\ &= -2u^E_{,D} (-I^F_C I^C_K g^{KD}) v_{E,F} = -2u^E_{,D} v_{E,F} g^{FD} = -\Delta\phi. \end{aligned}$$

Hence  $\Delta\phi = 0$ , and  $\phi$  is a harmonic function.

**THEOREM 11.** *Suppose that  $M^{2n}$  is a compact pseudo-Kählerian manifold. If the Ricci tensor  $R_{AB}$  is positive definite, then there exists no pseudo-analytic covariant tensors other than the zero vector. (If  $R_{AB}$  is positive semi-definite, then the covariant derivative of any pseudo-analytic covariant vector field vanishes). (cf. [1], [6]).*

*Proof.* We put

$$\phi = g^{AB} u_A u_B,$$

then we get

$$\Delta\phi=(g^{AB}u_Au_B)_{,CD}g^{CD}=2g^{AB}(u_{A,C}u_{B,D}+u_{A,CD}u_B)g^{CD}.$$

If we substitute (13) in the second term of the last equation we get

$$\Delta\phi=2g^{AB}g^{CD}u_{A,C}u_{B,D}+2R^{AB}u_Au_B.$$

Hence, by virtue of Bochner's lemma (cf. [1], [6]), we can see immediately that our assertion is true.

**THEOREM 12.** *Suppose that  $M^{2n}$  is a compact pseudo-Kählerian manifold. If the Ricci tensor  $R_{AB}$  is negative definite, then there exists no pseudo-analytic contravariant vector field other than the zero vector. (If  $R_{AB}$  is negative semi-definite, then the covariant derivative of any pseudo-analytic contravariant vector field vanishes). (cf [1], [6]).*

The proof is quite similar to that of Theorem 11.

**THEOREM 13.** *Suppose that  $M^{2n}$  is a compact pseudo-Kählerian manifold and  $u_B$  is a covariant vector field over  $M^{2n}$  such that  $u_B$  is expressible in a neighborhood of each point of  $M^{2n}$  as  $\phi_{,B}+I^A_B\psi_{,A}$  where  $\phi$  and  $\psi$  are harmonic functions in such neighborhood with respect to the pseudo-Kählerian metric. If the Ricci tensor  $R_{AB}$  is positive definite, then  $u_B=0$ , that is, the set of functions  $(\phi, \psi)$  is pseudo-analytic. (If  $R_{AB}$  is positive semi-definite, then the covariant derivative of  $u_B$  vanishes). (cf. [1], [6]).*

*Proof.* We put

$$T=g^{AB}u_Au_B, \quad u_A=\phi_{,A}+I^E_A\psi_{,E},$$

then we get

$$\Delta T=(2g^{AB}u_{A,C}u_{B,D}+2g^{AB}u_{A,CD}u_B)g^{CD}.$$

Now, the second term of the right hand side of the last equation is transformed in the following way:

$$\begin{aligned} II &= 2g^{AB}(\phi_{,CAD}+I^E_A\psi_{,CED})(\phi_{,B}+I^F_B\psi_{,F})g^{CD} \\ &= 2g^{AB}(-R^H_{CAD}\phi_{,H}-I^E_A R^H_{CED}\psi_{,H})(\phi_{,B}+I^F_B\psi_{,F})g^{CD} \\ &= 2g^{AB}(R^H_A\phi_{,H}+I^E_A R^H_E\psi_{,H})(\phi_{,B}+I^F_B\psi_{,F})g^{CD} \\ &= 2R^{HB}u_Hu_B. \end{aligned}$$

In the process of the transformation we used (10 ii) and the fact that  $\phi$  and  $\psi$  are harmonic functions. Hence

$$\Delta T=2g^{AB}g^{CD}u_{A,C}u_{B,D}+2R^{AB}u_Au_B,$$

so  $\Delta T \geq 0$ . Accordingly, by virtue of Bochner's lemma (cf. [1], [6]), we see that the theorem is true.

## REFERENCES

- S. Bochner, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc., **52** (1946), 763-797.
- G. de Rham, *Remarque au sujet de la théorie des formes différentielles harmoniques*, in. l'Univ. Grenoble, **23** (1947-48), 55-56.
- S. Sasaki, *On the real representation of spaces with Hermitian connexion*, Sci. Rep. Tohoku Univ. Ser. I, **33** (1949), 53-61.
- K. Yano, *On harmonic and Killing vector fields*, Ann. of Math., **55** (1952), 38-45.
- , *Groups of transformations in generalized spaces*, Tokyo (1949).
- K. Yano, and S. Bochner, *Curvature and Betti numbers*, Ann. of Math. Studies, **32** (1953).

INSTITUTE FOR ADVANCED STUDY, PRINCETON AND  
INSTITUTO NAZIONALE DI ALTA MATEMATICA, ROME





../../../../FrontMatter/paper.pdf

# Pacific Journal of Mathematics

Vol. 5, No. 6

, 1955

Nesmith Cornett Ankeny and Theodore Joseph Rivlin, <i>On a theorem of S. Bernstei</i> . . . . .	849
Louis Auslander, <i>The use of forms in variational calculation</i> . . . . .	853
Paul Civin, <i>Abstract Riemann sum</i> . . . . .	861
Paul Civin, <i>Some ergodic theorems involving two operator</i> . . . . .	869
Eckford Cohen, <i>The number of solutions of certain cubic congruence</i> . . . . .	877
Richard M. Cohn, <i>Specializations over difference field</i> . . . . .	887
Jean Dieudonné, <i>Pseudo-discriminant and Dickson invariant</i> . . . . .	907
Ky Fan, <i>A comparison theorem for eigenvalues of normal matrice</i> . . . . .	911
Richard P. Gosselin, <i>On the convergence behaviour of trigonometric interpolating polynomial</i> . . . . .	915
Peter K. Henrici, <i>On generating functions of the Jacobi polynomial</i> . . . . .	923
Meyer Jerison, <i>An algebra associated with a compact grou</i> . . . . .	933
Wilhelm Magnus, <i>Infinite determinants associated with Hill's equatio</i> . . . . .	941
G. Power and D. L. Scott-Hutton, <i>The slow steady motion of liquid past a semi-elliptical bos</i> . . . . .	953
Lyle E. Pursell, <i>An algebraic characterization of fixed ideals in certain function ring</i> . . . . .	963
C. T. Rajagopal, <i>Additional note on some Tauberian theorems of O. Szás</i> . . . . .	971
Louis Baker Rall, <i>Error bounds for iterative solutions of Fredholm integral equation</i> . . . . .	977
Shigeo Sasaki and Kentaro Yano, <i>Pseudo-analytic vectors on pseudo-Kählerian manifold</i> . . . . .	987
Eugene Schenkman, <i>On the tower theorem for finite group</i> . . . . .	995
P. Stein and John E. L. Peck, <i>On the numerical solution of Poisson's equation over a rectangl</i> . . . . .	999
Morgan Ward, <i>The mappings of the positive integers into themselves which preserve divisio</i> . . . . .	1013
Seth Warner, <i>Weak locally multiplicatively-convex algebra</i> . . . . .	1025
Louis Weisner, <i>Group-theoretic origin of certain generating function</i> . . . . .	1033