ESI The Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9
A-1090 Wien, Austria

Pseudo–Conformal Quaternionic CR Structure on (4n+3)–Dimensional Manifolds

Dmitri Alekseevsky
Yoshinobu Kamishima

Vienna, Preprint ESI 1780 (2006) February 20, 2006

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

PSEUDO-CONFORMAL QUATERNIONIC CR STRUCTURE ON $(4n+3)$-DIMENSIONAL MANIFOLDS

DMITRI ALEKSEEVSKY AND YOSHINOBU KAMISHIMA

Abstract. We study a geometric structure on a $(4n+3)$-dimensional smooth manifold M which is an integrable, nondegenerate codimension 3-subbundle $\mathcal{D}$ on M whose fiber supports the structure of $4n$-dimensional quaternionic vector space $\mathbb{H}^n$. It is thought of as a generalization of the quaternionic CR structure. In order to study this geometric structure on M, we single out an $\mathfrak{sp}(1)$-valued 1-form ω locally on a neighborhood U of M such that $\mathrm{Null}\omega = \mathcal{D}|U$. We shall construct the invariants on the pair (M, ω) whose vanishing implies that M is uniformized with respect to a finite dimensional flat quaternionic CR geometry. The invariants obtained on $(4n+3)$-manifold M have the same formula as the curvature tensor of quaternionic (indefinite) Kähler $4n$-manifolds. From this viewpoint, we exhibit a quaternionic analogue of Chern-Moser's CR structure.

Introduction

The Weyl curvature tensor is a conformal invariant of Riemannian manifolds and the Chern-Moser curvature tensor is a CR invariant on strictly pseudo-convex CR-manifolds. A geometric significance of the vanishing of these curvature tensors is the appearance of the finite dimensional Lie group $\mathcal{G}$ with homogeneous space X. The geometry $(\mathcal{G}, X)$ is known as conformally flat geometry $(\mathrm{PO}(n+1,1), S^n)$, spherical CR-geometry $(\mathrm{PU}(n+1,1), S^{2n+1})$ respectively. The complete simply connected quaternionic $(n+1)$-dimensional quaternionic hyperbolic space $\mathbb{H}_{\mathbb{H}}^{n+1}$ with the group of isometries $\mathrm{PSp}(n+1,1)$ has the natural compactification homeomorphic to a $(4n+4)$-ball endowed with an extended smooth action of $\mathrm{PSp}(n+1,1)$. When the boundary sphere S^{4n+3} of the ball is viewed as the real hypersurface in the quaternionic projective space $\mathbb{HP}^{n+1}$, the elements of $\mathrm{PSp}(n+1,1)$ act as quaternionic projective transformations of S^{4n+3}. Since the action of $\mathrm{PSp}(n+1,1)$ is transitive on S^{4n+3}, we obtain a flat (spherical) quaternionic CR geometry $(\mathrm{PSp}(n+1,1), S^{4n+3})$. (Compare [16].) Combined with the above two geometries, this exhibits *parabolic geometry* on the boundary of the compactification of rank-one symmetric space of noncompact type over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. (See [10],[12],[35],[17].)

This observation naturally leads us to the problems: (1) existence of geometric structure on a $(4n+3)$-dimensional manifold M and (2) existence of geometric invariant whose vanishing implies that M is locally equivalent to the flat quaternionic CR manifold S^{4n+3}.

Date: February 28, 2006.

1991 *Mathematics Subject Classification.* 53C55, 57S25.51M10 vin-cr06.tex.

Key words and phrases. Pseudo-conformal quaternionic structure, Quaternionic CR structuire, Quaternionic Kähler manifold, Sasakian 3-structure, G-structure, integrability, uniformization, transformation group.

For this purpose we shall introduce a notion of pseudo-conformal quaternionic CR (p-c qCR) structure $(\mathcal{D}, \{\omega_\alpha\}_{\alpha=1,2,3})$ on a $(4n+3)$-dimensional manifold M. First of all, in §1 we recall a pseudo-conformal quaternionic structure (p-c q structure) $\mathcal{D}$ which was discussed in [3]. Compare Remark 1.7 for the difference between CR structure. Contrary to the nondegenerate CR structure, the almost complex structure on $\mathcal{D}$ is not assumed to be integrable. However, by the requirement of structure equations defining the qCR-structure, we can prove the integrability of quaternionic structure in §2.1:

Theorem A. *Each almost complex structure $\bar{J}_\alpha$ of the quaternionic CR structure is integrable on the codimension-1 contact subbundle* $\mathrm{Null}\,\omega_\alpha$ $(\alpha = 1, 2, 3)$.

There exists a canonical pseudo-Riemannian metric g associated to the nondegenerate p-c qCR structure. In §4 we see that that *the integrability of three almost complex structures $\{\bar{J}_\alpha\}_{\alpha=1,2,3}$ is equivalent with the condition that (M, g) is a pseudo-Sasakian 3-structure.* Namely the notion is equivalent between nondegenerate quaternionic CR structure and pseudo-Sasakian 3-structure (cf. [4]). In particular, p-c qCR manifolds contain the class of pseudo 3-Sasakian manifolds. (Refer to [5],[8],[33],[34] for (positive definite) Sasakian 3-structure.) However, we emphasize that the converse is not true. There are two typical classes of compact (spherical) p-c qCR manifolds but not pseudo-Sasakian 3-manifolds [16]; one is a quaternionic Heisenberg manifold $\mathcal{M}/\Gamma$. Some finite cover of $\mathcal{M}/\Gamma$ is a Heisenberg nilmanifold which is a principal 3-torus bundle over the flat quaternionic n-torus $T^{p,q}_{\mathbb{H}}$ of signature (p, q) (p+q=n), see §7.3. Another manifold is a pseudo-Riemannian standard space form $\Sigma^{3,4n}_{\mathbb{H}}/\Gamma$ of constant negative curvature of type $(4n, 3)$. It is a compact quotient of the homogeneous space $\Sigma^{3,4n}_{\mathbb{H}} = \mathrm{Sp}(1, n)/\mathrm{Sp}(n)$. Some finite cover of $\Sigma^{3,4n}_{\mathbb{H}}/\Gamma$ is a principal S^3-bundle over the quaternionic hyperbolic space form $\mathbb{H}^n_{\mathbb{H}}/\Gamma^*$. Obviously those manifolds are not positive-definite compact 3-Sasakian manifolds. (cf. [16], [18] more generally.)

For the second problem, we shall try to construct the curvature tensor of p-c qCR structure. This is thought of as a quaternionic analogue of Chern-Moser's CR curvature tensor. When M is a $2n+1$-dimensional manifold equipped with a nondegenerate CR structure (H, J), it follows from the Cartan geometry that there is an $\mathfrak{su}(p+1, q+1)$-valued 1-form κ called a Cartan connection whose associated curvature form Π vanishes if and only if M is locally isomorphic to $\mathrm{PU}(p+1, q+1)/\mathrm{P}^+(\mathbb{C})$ where $\mathrm{P}^+(\mathbb{C})$ is the maximal parabolic subgroup $(p+q = n)$. The 4-th order Chern-Moser CR curvature tensor $S = (S_{\alpha}{}^{\beta}{}_{\rho\sigma})$ is the coefficient of the curvature component Φ^β_α of Π. By the observation of Webster (cf. [35], [36]) the other components are obtained from S by further covariant differentiation for $n > 1$. In the CR case, the Chern-Moser curvature tensor S vanishes on M if and only if so does the $\mathfrak{su}(p+1, q+1)$-valued Cartan curvature form Π.

On a $(4n+3)$-dimensional p-c q manifold $(M, \mathcal{D})$, there is also an $\mathfrak{sp}(p+1, q+1)$-valued Cartan form κ whose associated curvature form Π has zero curvature if and only if $(M, \mathcal{D})$ is locally isomorphic to $\mathrm{PSp}(p+1, q+1)/P^+(\mathbb{H})$. We don't know whether a curvature tensor on M could be derived only from the Cartan form Π on the p-c q structure $\mathcal{D}$ because $\mathcal{D}$ lacks the structure equations representing the integrability conditions but not the p-c qCR structure. However, with the aid of pseudo-Riemannian connection of the pseudo-Sasakian 3-structure which is locally equivalent to p-c qCR structure, we can define a quaternionic CR curvature tensor (cf. §5). Based on this curvature tensor, in §8 we shall establish a curvature tensor T which is invariant under the equivalence of p-c qCR structures. Remark that if T vanishes under the existence of p-c qCR structure, Π also vanishes. The explicit formula of T is described as follows (cf. Theorem 9.1 of §9).

Theorem B. *There exists a fourth-order curvature tensor* $T=(T^i_{jk\ell})$ *on a nondegenerate p-c qCR manifold M in dimension* $4n+3$ $(n\geq 0)$. *If* $n\geq 2$, *then* $T=(T^i_{jk\ell})\in\mathcal{R}_0(\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1))$ *which has the formula:*

$$T^i_{jk\ell}=R^i_{jk\ell}-\Big\{(g_{j\ell}\delta^i_k-g_{jk}\delta^i_\ell)+\Big[I_{j\ell}I^i_k-I_{jk}I^\ell_i+2I^i_jI_{k\ell} +J_{j\ell}J^i_k-J_{jk}J^\ell_i+2J^i_jJ_{k\ell}+K_{j\ell}K^i_k-K_{jk}K^i_\ell+2K^i_jK_{k\ell}\Big]\Big\}.$$

When $n=1$, $T=(W^i_{jk\ell})\in\mathcal{R}_0(\mathrm{SO}(4))$ *which has the same formula as the Weyl conformal curvature tensor. When* $n=0$, *there exists the fourth-order curvature tensor* TW *on* M *which has the same formula as the Weyl-Schouten tensor.*

In §7, we introduce the $(4n+3)$-dimensional manifold $S^{3+4p,4q}=\mathrm{Sp}(p+1,q+1)/P^+(\mathbb{H})$ which is a pc-qCR manifold with vanishing p-c qCR curvature tensor T. In particular, $S^{4n+3}=S^{3+4n,0}$ is the positive-definite flat (spherical) quaternionic CR manifold. As in CR geometry, we prove that the vanishing of T gives rise to a *uniformization* with respect to the flat (spherical) p-c qCR geometry, see Theorem 9.3 in §8.1. (Compare [23] for uniformization in general.)

Theorem C.

(i) *If M is a* $(4n+3)$*-dimensional nondegenerate p-c qCR manifold of type* $(3+4p,4q)$ $(p+q=n\geq 1)$ *whose curvature tensor T vanishes, then M is uniformized over* $S^{3+4p,4q}$ *with respect to the group* $\mathrm{PSp}(p+1,q+1)$.

(ii) *If M is a 3-dimensional p-c qCR manifold whose curvature tensor TW vanishes, then M is conformally flat (locally modelled on S^3 with respect to the group* $\mathrm{PSp}(1,1)$ *).*

In the positive definite case, our p-c qCR geometry presents spherical quaternionic CR geometry $(\mathrm{PSp}(n+1,1),S^{4n+3})$ as in the beginning of Introduction.

When a geometric structure is either contact structure or complex contact structure, it is known that the first Stiefel-Whitney class or the first Chern class is the obstruction to the existence of global 1-forms representing their strutures respectively. As a concluding remark to p-c q structure but not necessarily p-c qCR structure, we verify that the obstruction relates to the first Pontrjagin class $p_1(M)$ of a $(4n+3)$-dimensional p-c q manifold M $(n\geq 1)$. In §10, we prove that the following relation of the first Pontrjagin classes. (See Theorem 10.4.)

Theorem D. *Let* $(M,\mathcal{D})$ *be a* $(4n+3)$*-dimensional p-c q manifold. Then the first Pontrjagin classes of M and the bundle* $L=TM/\mathcal{D}$ *has the relation that* $2p_1(M)=(n+2)p_1(L)$. *Moreover, if M is simply connected, then the following are equivalent.*

(1) $2p_1(M)=0$. *In particular, the first rational Pontrjagin class vanishes.*

(2) *There exists a global* $\mathrm{Im}\mathbb{H}$*-valued 1-form ω on M which represents a p-c q structure $\mathcal{D}$. In particular, there exists a hypercomplex structure* $\{I,J,K\}$ *on* $\mathcal{D}$.

1. Pseudo-conformal quaternionic CR structure

When $\mathbb{H}$ denotes the field of quaternions, the Lie algebra $\mathfrak{sp}(1)$ of Sp(1) is identified with $\mathrm{Im}\mathbb{H} = \mathbb{R}\boldsymbol{i} + \mathbb{R}\boldsymbol{j} + \mathbb{R}\boldsymbol{k}$. Let M be a $(4n+3)$-dimensional smooth manifold M.

Definition 1.1. *A $4n$-dimensional orientable subbundle $\mathcal{D}$ equipped with a quaternionic structure Q is called a* pseudo-conformal quaternionic *structure (p-c q structure) on M if it satisfies that*

(i) $\mathcal{D} \cup [\mathcal{D}, \mathcal{D}] = TM$.
(ii) *The 3-dimensional quotient bundle $TM/\mathcal{D}$ at any point is isomorphic to the Lie algebra* $\mathrm{Im}\mathbb{H}$.
(iii) *There exists a* $\mathrm{Im}\mathbb{H}$*-valued 1-form $\omega = \omega_1\boldsymbol{i} + \omega_2\boldsymbol{j} + \omega_3\boldsymbol{k}$ locally defined on a neighborhood of M such that $\mathcal{D} = \mathrm{Null}\,\omega = \bigcap_{\alpha=1}^{3} \mathrm{Null}\,\omega_\alpha$ and $d\omega_\alpha|\mathcal{D}$ is nondegenerate. Here each ω_α is a real valued 1-form $(\alpha = 1, 2, 3)$.*
(iv) *The endomorphism $J_\gamma = (d\omega_\beta|\mathcal{D})^{-1} \circ (d\omega_\alpha|\mathcal{D}) : \mathcal{D} \rightarrow \mathcal{D}$ constitutes the quaternionic structure Q on $\mathcal{D}$: ${J_\gamma}^2 = -1$, $J_\alpha J_\beta = J_\gamma = -J_\beta J_\alpha$, $(\gamma = 1, 2, 3)$ etc.*

Lemma 1.2. *If we put $\sigma_\alpha = (d\omega_\alpha|\mathcal{D})$ on $\mathcal{D}$, then there is the following equality: $\sigma_1(J_1X, Y) = \sigma_2(J_2X, Y) = \sigma_3(J_3X, Y)$ $(\forall\ X, Y \in \mathcal{D})$. Moreover, the form*

$$g^{\mathcal{D}} = \sigma_\alpha \circ J_\alpha \tag{1.1}$$

is a nondegenerate Q-invariant symmetric bilinear form on $\mathcal{D}$; $g^{\mathcal{D}}(X, Y) = g^{\mathcal{D}}(J_\alpha X, J_\alpha Y)$, $g^{\mathcal{D}}(X, J_\alpha Y) = \sigma_\alpha(X, Y)$, $(\alpha = 1, 2, 3)$, etc.

Proof. By (iv) of Definition 1.1, it follows that

$$\begin{aligned} \sigma_\alpha(J_\alpha X, Y) &= \sigma_\alpha(J_\beta(J_\gamma X), Y) = \sigma_\gamma(J_\gamma X, Y) \\ &= \sigma_\gamma(J_\alpha(J_\beta X), Y) = \sigma_\beta(J_\beta X, Y). \end{aligned} \tag{1.2}$$

Put $g^{\mathcal{D}}(X, Y) = \sigma_\alpha(J_\alpha X, Y)$ for $X, Y \in \mathcal{D}$ $(\alpha = 1, 2, 3)$, which is nondegenerate by (iii). As $-J_\beta = \sigma_\gamma^{-1} \circ \sigma_\alpha$ by (iv), calculate that $g^{\mathcal{D}}(Y, X) = -\sigma_\alpha(X, J_\alpha Y) = \sigma_\gamma(J_\beta X, J_\alpha Y) = -\sigma_\beta(Y, J_\beta X) = g^{\mathcal{D}}(X, Y)$. It follows that $g^{\mathcal{D}}(X, Y) = \sigma_\alpha(J_\alpha X, Y) = \sigma_\alpha(J_\alpha(J_\alpha Y), J_\alpha X) = g^{\mathcal{D}}(J_\alpha Y, J_\alpha X)$. □

In general, there is no canonical choice of ω which annihilates $\mathcal{D}$. The fiber of the quotient bundle $TM/\mathcal{D}$ is isomorphic to Im $\mathbb{H}$ by ω on a neighborhood U by (ii). The coordinate change of the fiber $\mathbb{H}$ is described as $v \rightarrow \lambda \cdot v \cdot \mu$ for some nonzero elements $\lambda, \mu \in \mathbb{H}$. If ω' is another 1-form such that $\mathrm{Null}\,\omega' = \mathcal{D}$ on a neighborhood U', then it follows that $\omega' = \lambda \cdot \omega \cdot \mu$ for some $\mathbb{H}$-valued functions λ, μ locally defined on $U \cap U'$. This can be rewritten as $\omega' = u \cdot a \cdot \omega \cdot b$ where a, b are functions with valued in Sp(1) and u is a positive function. Since $\bar{\omega}' = -\omega'$, it follows that $a \cdot \omega \cdot b = \bar{b} \cdot \omega \cdot \bar{a}$, i.e. $(\bar{b}a) \cdot \omega \cdot (\bar{b}a) = \omega$. As $\omega : T(U \cap U') \rightarrow \mathrm{Im}\ \mathbb{H}$ is surjective, $\bar{b}a$ centralizes Im $\mathbb{H}$ so that $\bar{b}a \in \mathbb{R}$. Hence, $b = \pm\bar{a}$. As we may assume that $\mathcal{D}$ is orientable, ω' is uniquely determined by

$$\omega' = u \cdot a \cdot \omega \cdot \bar{a} \text{ for some functions } a \in \mathrm{Sp}(1), u > 0 \text{ on } U \cap U'. \tag{1.3}$$

We must show that Definition 1.1 does not depend on the choice of ω' satisfying (1.3).

Lemma 1.3. *Any form ω' locally conjugate to ω satisfies* (iii), (iv) *of Definition 1.1.*

Proof. First, if $A = (a_{ij}) \in \mathrm{SO}(3)$ is the matrix function determined by

$$\mathrm{Ad}_a \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} = a \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \bar{a} = A \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix}, \tag{1.4}$$

then a new quaternionic structure on $\mathcal{D}$ is introduced as

$$\begin{pmatrix} J'_1 \\ J'_2 \\ J'_3 \end{pmatrix} = {}^t A \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix}. \tag{1.5}$$

Then the formula of (1.3) is described as

$$({\omega'}_1, {\omega'}_2, {\omega'}_3) = (\omega_1, \omega_2, \omega_3)u \cdot A = u(\sum_{\beta=1}^{3} a_{\beta 1}\omega_\beta, \sum_{\beta=1}^{3} a_{\beta 2}\omega_\beta, \sum_{\beta=1}^{3} a_{\beta 3}\omega_\beta). \tag{1.6}$$

Differentiate (1.6) and restricting to $\mathcal{D}$, use Lemma 1.2 (note that $d\omega' = u \cdot a \cdot d\omega \cdot \bar{a}$ on $\mathcal{D}|U \cap U'$),

$$\begin{aligned} d{\omega'}_\alpha(X, Y) &= u \sum_\beta a_{\beta\alpha} d\omega_\beta(X, Y) = -u(a_{1\alpha} g^{\mathcal{D}}(J_1 X, Y) + a_{2\alpha} g^{\mathcal{D}}(J_2 X, Y) + a_{3\alpha} g^{\mathcal{D}}(J_3 X, Y)) \\ &= -ug^{\mathcal{D}}((a_{1\alpha} J_1 + a_{2\alpha} J_2 + a_{3\alpha} J_3)X, Y) = -ug^{\mathcal{D}}({J'}_\alpha X, Y), \end{aligned}$$

$$d{\omega'}_\alpha({J'}_\alpha X, Y) = ug^{\mathcal{D}}(X, Y) \ (\alpha = 1, 2, 3). \tag{1.7}$$

In particular, $d\omega'_\alpha|\mathcal{D}$ is nondegenerate, proving (iii). Put $\sigma'_\alpha = d\omega'_\alpha|\mathcal{D}$. As in (iv) of Definition 1.1, the endomorphism is defined by the rule: $I'_\gamma = (\sigma'_\beta|\mathcal{D})^{-1} \circ (\sigma'_\alpha|\mathcal{D})$, i.e. $\sigma'_\beta(I'_\gamma X, Y) = \sigma'_\alpha(X, Y)$ $(\forall X, Y \in \mathcal{D})$. Then we show that the quaternionic structure $\{I'_\alpha\}_{\alpha=1,2,3}$ coincides with $\{J'_\alpha\}_{\alpha=1,2,3}$ on $\mathcal{D}$. For this, as $\sigma'_\alpha(X, Y) = -ug^{\mathcal{D}}(J'_\alpha X, Y)$ by (1.7), it follows that $\sigma'_\beta(I'_\gamma X, Y) = -ug^{\mathcal{D}}(J'_\beta(I'_\gamma X), Y)$ and the above equality implies that $J'_\beta(I'_\gamma X) = J'_\alpha X$ $(\forall X \in \mathcal{D})$. Hence, $I'_\gamma = -J'_\beta J'_\alpha = J'_\gamma$. This proves (iv). □

By Lemma 1.2, we may assume that $g^{\mathcal{D}}$ locally defined on $\mathcal{D}|U$ has signature $(4p, 4q)$ with $4p$-times positive sign and $4q$-times negative sign $(p + q = n)$. As above put ${g'}^{\mathcal{D}}(X, Y) = d{\omega'}_\alpha({J'}_\alpha X, Y)$ $(X, Y \in \mathcal{D})$. We have

Corollary 1.4. *If $\omega' = u\bar{a} \cdot \omega \cdot a$ on $U \cap U'$, then ${g'}^{\mathcal{D}} = u \cdot g^{\mathcal{D}}$. As a consequence, the signature (p, q) is constant on $U \cap U'$ (and hence everywhere on M) under the change $\omega' = u\bar{a} \cdot \omega \cdot a$.*

We are now going to consider an integrability condition on the p-c q structure $\mathcal{D}$.

Definition 1.5. *Suppose that the following structure equation is locally given:*

$$\rho_\alpha = d\omega_\alpha + 2\omega_\beta \wedge \omega_\gamma \tag{1.8}$$

where $(\alpha, \beta, \gamma) \sim (1, 2, 3)$ up to cyclic permutation. If the skew symmetric 2-form ρ_α satisfies that

$$\mathrm{Null}\, \rho_1 = \mathrm{Null}\, \rho_2 = \mathrm{Null}\, \rho_3, \tag{1.9}$$

the pair (ω, Q) *is a* local quaternionic CR structure *(qCR structure) on* M*.*

See [6], [4]. If the (local) qCR structure has a Im$\mathbb{H}$-valued 1-form ω defined entirely on M, then it is noted that the global qCR structure coincides with the pseudo-Sasakian 3-structure of M, see §4.1. Using two Definitions 1.1, 1.5, we come to the following notion due to the manner of Libermann [27].

Definition 1.6. *The pair* $(\mathcal{D}, Q)$ *on* M *is said to be a* pseudo-conformal quaternionic CR structure *(p-c qCR structure) if there exists locally a 1-form* η *with* $\mathrm{Null}\,\eta = \mathcal{D}$ *on a neighborhood* U *of* M *such that* η *is conjugate to a qCR structure on* U*. Namely there exists a qCR structure* ω *on* U *for which* $\eta = \lambda \cdot \omega \cdot \bar{\lambda}$ *where* $\lambda : U \to \mathbb{H}$ *is a function and* $\bar{\lambda}$ *is the conjugate of the quaternion.*

Remark 1.7. *For the nondegenerate CR case, let* ω *be a 1-form which represents a CR structure* $(\mathrm{Null}\omega, J)$*. Since* $\sigma_\alpha(X, Y) = g^{\mathcal{D}}(X, J_\alpha Y)$ *by Lemma 1.2, the corresponding (complex) formula of the structure equation* (1.8) *of Definition 1.5 becomes (cf.* [35]*):*

$$d\omega = g_{i\bar{j}}\theta^i \wedge \theta^{\bar{j}},$$

where J *is assumed to be integrable although the CR structure has no such equation as* (1.9)*. In the p-c qCR case, however Theorem 2.7 shows that each almost complex structure* $\bar{J}_\alpha$ *is integrable (cf.* (2.9) *also). Moreover, each characteristic vector filed* ξ_α *is a CR vector field (cf.* (3) *of Lemma 2.3). In general, this never occurs from the structure equation to the nondegenerate CR structure.*

2. Quaternionic CR structure

Suppose that ω is a qCR structure on a neighborhood of M. Let $\rho_\alpha = d\omega_\alpha + 2\omega_\beta \wedge \omega_\gamma$ be as in (1.8). Put $V = \mathrm{Null}\,\rho_\alpha$ $(\alpha = 1, 2, 3)$ (cf. (1.9)). Since $\dim\mathcal{D} = 4n$, let $\{v_1, v_2, v_3\}$ be a basis of V. Put $\omega_i(v_j) = a_{ij}$. As $\omega_1 \wedge \omega_2 \wedge \omega_3 | V \neq 0$, the 3×3-matrix (a_{ij}) is nonsingular. Put $b_{ij} = {}^t(a_{ij})^{-1}$ and $\xi_j = \sum b_{jk} v_k$. Then $\omega_\alpha(\xi_\beta) = \delta_{\alpha\beta}$ and locally,

$$V = \{\xi_\alpha, \alpha = 1, 2, 3\} \tag{2.1}$$

Lemma 2.1. *Let* $\mathcal{L}$ *be the Lie derivative. Then,* $\mathcal{L}_{\xi_\alpha}(\mathcal{D}) = \mathcal{D}$ $(\alpha = 1, 2, 3)$*.*

Proof. For $X \in \mathcal{D}$, $\omega_\beta(\mathcal{L}_{\xi_\alpha}(X)) = \omega_\beta([\xi_\alpha, X])$. As

$$0 = \rho_\beta(\xi_\alpha, X) = d\omega_\beta(\xi_\alpha, X) + 2\omega_\gamma \wedge \omega_\alpha(\xi_\alpha, X) = \frac{1}{2}(-\omega_\beta([\xi_\alpha, X]),$$

we have $\omega_\beta([\xi_\alpha, X]) = 0$ for $\beta = 1, 2, 3$. Hence, $\mathcal{L}_{\xi_\alpha}(X) \in \mathcal{D} = \bigcap_{\beta=1}^{3} \mathrm{Null}\ \omega_\beta$.

□

We prove also that $\mathcal{L}_\xi V = V$ for $\xi \in V$.

Lemma 2.2. *The distribution* V *is integrable. The vector fields* ξ_α *determined by* (2.1) *generates the Lie algebra isomorphic to* $\mathfrak{so}(3)$*, i.e.* $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$*.* $(\alpha, \beta, \gamma) \sim (1, 2, 3)$*.*

Proof. By (2.1), note that

$$V = \{\xi \in TM \mid \rho_1(\xi, v) = \rho_2(\xi, v) = \rho_3(\xi, v) = 0,\ \forall v \in TM\} = \{\xi_\alpha\ ; \alpha = 1, 2, 3\}. \tag{2.2}$$

Since $0 = \rho_\alpha(\xi_\beta,\xi_\gamma) = \dfrac{1}{2}(-\omega_\alpha([\xi_\beta,\xi_\gamma]) + 2)$, it follows that $[\xi_\beta,\xi_\gamma] - 2\xi_\alpha \in \text{Null } \omega_\alpha$. Applying $\rho_\beta(\xi_\beta,\xi_\gamma) = \dfrac{1}{2}(-\omega_\beta([\xi_\beta,\xi_\gamma]) + 0) = 0$, it yields also that $[\xi_\beta,\xi_\gamma] - 2\xi_\alpha \in \text{Null } \omega_\beta$. Similarly as $\rho_\gamma(\xi_\beta,\xi_\gamma) = 0$, we obtain $[\xi_\beta,\xi_\gamma] - 2\xi_\alpha \in \bigcap_{\beta=1}^{3} \text{Null } \omega_\beta = \mathcal{D}$ for $\alpha = 1,2,3$. As $\rho_\alpha([\xi_\beta,\xi_\gamma] - 2\xi_\alpha, v) = \rho_\alpha([\xi_\beta,\xi_\gamma], v)$ for arbitrary $v \in \mathcal{D}$, By the definition of ρ_α, calculate

$$\begin{aligned}\rho_\alpha([\xi_\beta,\xi_\gamma], v) &= -\frac{1}{2}\omega_\beta([[\xi_\beta,\xi_\gamma], v])\\ &= \frac{1}{2}(\omega_\beta([[\xi_\gamma, v],\xi_\beta]) + \omega_\beta([[\xi_\beta, v],\xi_\gamma])) \text{ (by Jacobi identity)}\\ &= 0 \text{ (by Lemma 2.1)}.\end{aligned}$$

Since ρ_α is nondegenerate on $\mathcal{D}$ by (iii), $[\xi_\beta,\xi_\gamma] = 2\xi_\alpha$ $(\alpha = 1,2,3)$. Hence, such a Lie algebra V is locally isomorphic to the Lie algebra of SO(3). □

We collect the properties of $\omega_\alpha, \rho_\alpha, J_\alpha, g^{\mathcal{D}}$. (Compare [4].)

Lemma 2.3. *Up to cyclic permutation of* $(\alpha,\beta,\gamma) \sim (1,2,3)$, *the following properties hold.*

(1) $\mathcal{L}_{\xi_\alpha}\omega_\alpha = 0$, $\mathcal{L}_{\xi_\alpha}\omega_\beta = \omega_\gamma = -\mathcal{L}_{\xi_\beta}\omega_\alpha$.
(2) $\mathcal{L}_{\xi_\alpha}\rho_\alpha = 0$, $\mathcal{L}_{\xi_\alpha}\rho_\beta = \rho_\gamma = -\mathcal{L}_{\xi_\beta}\rho_\alpha$.
(3) $\mathcal{L}_{\xi_\alpha}J_\alpha = 0$, $\mathcal{L}_{\xi_\alpha}J_\beta = J_\gamma = -\mathcal{L}_{\xi_\beta}J_\alpha$.
(4) $\mathcal{L}_{\xi_\alpha}g^{\mathcal{D}} = 0$.

Proof. (1). First note that $\iota_{\xi_\alpha}\omega_\alpha(x) = \omega_\alpha(\xi_\alpha) = 1$ $(x \in M)$, $\iota_{\xi_\alpha}(\omega_\beta\wedge\omega_\gamma)(X) = \omega_\beta\wedge\omega_\gamma(\xi_\alpha, X) = 0$ $(\alpha \neq \beta,\gamma)$, and $\iota_{\xi_\alpha}\rho_\alpha(X) = \rho_\alpha(\xi_\alpha, X) = 0$ by (3.7).

$$\begin{aligned}\mathcal{L}_{\xi_\alpha}\omega_\alpha = (d\iota_{\xi_\alpha} + \iota_{\xi_\alpha}d)\omega_\alpha = \iota_{\xi_\alpha}d\omega_\alpha &= \iota_{\xi_\alpha}(-2\omega_\beta\wedge\omega_\gamma + \rho_\alpha) \text{ by (1.8)}\\ &= -2\iota_{\xi_\alpha}(\omega_\beta\wedge\omega_\gamma) + \iota_{\xi_\alpha}\rho_\alpha = 0,\end{aligned} \tag{2.3}$$

Next,

$$\mathcal{L}_{\xi_\alpha}\omega_\beta = \iota_{\xi_\alpha}d\omega_\beta = \iota_{\xi_\alpha}(-2\omega_\gamma\wedge\omega_\alpha + \rho_\beta) = -2\iota_{\xi_\alpha}(\omega_\gamma\wedge\omega_\alpha), \text{ while}$$

$-2\iota_{\xi_\alpha}(\omega_\gamma\wedge\omega_\alpha)(v) = 0$ for $v \notin \text{Null } \omega_\gamma$ and $-2\iota_{\xi_\alpha}(\omega_\gamma\wedge\omega_\alpha)(\xi_\gamma) = 1$. Hence $\mathcal{L}_{\xi_\alpha}\omega_\beta = \omega_\gamma$.
(2).

$$\begin{aligned}\mathcal{L}_{\xi_\alpha}\rho_\beta &= \mathcal{L}_{\xi_\alpha}(d\omega_\beta + 2\omega_\gamma\wedge\omega_\alpha)\\ &= (d\iota_{\xi_\alpha} + \iota_{\xi_\alpha}d)d\omega_\beta + 2\mathcal{L}_{\xi_\alpha}(\omega_\gamma\wedge\omega_\alpha)\\ &= d\iota_{\xi_\alpha}d\omega_\beta + 2\mathcal{L}_{\xi_\alpha}\omega_\gamma\wedge\omega_\alpha + 2\omega_\gamma\wedge\mathcal{L}_{\xi_\alpha}\omega_\alpha\\ &= d(\mathcal{L}_{\xi_\alpha} - d\iota_{\xi_\alpha})\omega_\beta + 2\mathcal{L}_{\xi_\alpha}\omega_\gamma\wedge\omega_\alpha \quad \text{(by (1))}\\ &= d(\mathcal{L}_{\xi_\alpha}\omega_\beta) - 2\mathcal{L}_{\xi_\gamma}\omega_\alpha\wedge\omega_\alpha = d\omega_\gamma - 2\omega_\beta\wedge\omega_\alpha\\ &= d\omega_\gamma + 2\omega_\alpha\wedge\omega_\beta = \rho_\gamma.\end{aligned} \tag{2.4}$$

Similarly,

$$
\begin{aligned}
\mathcal{L}_{\xi_\alpha}\rho_\alpha &= \mathcal{L}_{\xi_\alpha}(d\omega_\alpha + 2\omega_\beta \wedge \omega_\gamma) \\
&= d\iota_{\xi_\alpha} d\omega_\alpha + 2\mathcal{L}_{\xi_\alpha}\omega_\beta \wedge \omega_\gamma + 2\omega_\beta \wedge \mathcal{L}_{\xi_\alpha}\omega_\gamma \\
&= d(\mathcal{L}_{\xi_\alpha} - d\iota_{\xi_\alpha})\omega_\alpha + 2\omega_\gamma \wedge \omega_\gamma + 2\omega_\beta \wedge (-\omega_\beta) \\
&= d\mathcal{L}_{\xi_\alpha}\omega_\alpha = 0 \quad \text{(by (1))}.
\end{aligned} \tag{2.5}
$$

(3). As $\mathcal{L}_{\xi_\alpha}\rho_\alpha = 0$ by property (2),

$$
\begin{aligned}
0 &= (\mathcal{L}_{\xi_\alpha}\rho_\alpha)(J_\beta X, Y) \\
&= \mathcal{L}_{\xi_\alpha}(\sigma_\alpha(J_\beta X, Y)) - \sigma_\alpha(\mathcal{L}_{\xi_\alpha}(J_\beta X), Y) - \sigma_\alpha(J_\beta X, \mathcal{L}_{\xi_\alpha} Y).
\end{aligned}
$$

Noting that $J_\beta = \sigma_\alpha^{-1} \circ \sigma_\gamma$ by Lemma 1.2, we have

$$
\begin{aligned}
&\sigma_\alpha((\mathcal{L}_{\xi_\alpha} J_\beta)X, Y) = \sigma_\alpha(\mathcal{L}_{\xi_\alpha}(J_\beta X), Y) - \sigma_\alpha(J_\beta \mathcal{L}_{\xi_\alpha}(X), Y) \\
&= \mathcal{L}_{\xi_\alpha}(\sigma_\alpha(J_\beta X, Y)) - \sigma_\alpha(J_\beta X, \mathcal{L}_{\xi_\alpha} Y) - \sigma_\alpha(J_\beta \mathcal{L}_{\xi_\alpha} X, Y) \\
&= (\mathcal{L}_{\xi_\alpha}\sigma_\gamma)(X, Y) = -\sigma_\beta(X, Y) \quad \text{(by property (2))} \\
&= \sigma_\alpha(J_\gamma X, Y)
\end{aligned} \tag{2.6}
$$

As σ_α is nondegenerate on $\mathcal{D}$, $\mathcal{L}_{\xi_\alpha} J_\beta = J_\gamma$. Similarly,

$$
\begin{aligned}
&\sigma_\gamma((\mathcal{L}_{\xi_\alpha} J_\alpha)X, Y) = \sigma_\gamma(\mathcal{L}_{\xi_\alpha}(J_\alpha X), Y) - \sigma_\gamma(J_\alpha \mathcal{L}_{\xi_\alpha}(X), Y) \\
&= -(\mathcal{L}_{\xi_\alpha}\sigma_\gamma)(J_\alpha X, Y) + \mathcal{L}_{\xi_\alpha}(\sigma_\gamma(J_\alpha X, Y)) \\
&\quad - \sigma_\gamma(J_\alpha X, \mathcal{L}_{\xi_\alpha} Y) - \sigma_\gamma(J_\alpha \mathcal{L}_{\xi_\alpha} X, Y) \\
&= \sigma_\beta(J_\alpha X, Y) + \mathcal{L}_{\xi_\alpha}(\sigma_\beta(X, Y)) - \sigma_\beta(X, \mathcal{L}_{\xi_\alpha} Y) - \sigma_\beta(\mathcal{L}_{\xi_\alpha} X, Y) \\
&= \sigma_\beta(J_\alpha X, Y) + (\mathcal{L}_{\xi_\alpha}\sigma_\beta)(X, Y) \\
&= -\sigma_\gamma(X, Y) + \sigma_\gamma(X, Y) = 0,
\end{aligned} \tag{2.7}
$$

it follows that $\mathcal{L}_{\xi_\alpha} J_\alpha = 0$.

(4). Recall from Lemma 1.2 that $g^{\mathcal{D}}(X, Y) = \sigma_\alpha(J_\alpha X, Y) = \rho_\alpha(J_\alpha X, Y)$ $(X, Y \in \mathcal{D})$ for each α. Then

$$
\begin{aligned}
(\mathcal{L}_{\xi_\alpha} g^{\mathcal{D}})(X, Y) &= \xi_\alpha(g^{\mathcal{D}}(X, Y)) - g^{\mathcal{D}}(\mathcal{L}_{\xi_\alpha} X, Y) - g^{\mathcal{D}}(X, \mathcal{L}_{\xi_\alpha} Y) \\
&= \xi_\alpha(\rho_\beta(J_\beta X, Y)) - \rho_\beta(J_\beta \mathcal{L}_{\xi_\alpha} X, Y) - \rho_\beta(J_\beta X, \mathcal{L}_{\xi_\alpha} Y).
\end{aligned} \tag{2.8}
$$

On the other hand, $\mathcal{L}_{\xi_\alpha}\rho_\beta = \rho_\gamma$ by property (2) and so

$$
\xi_\alpha(\rho_\beta(J_\beta X, Y)) = \rho_\beta(\mathcal{L}_{\xi_\alpha} J_\beta X, Y) + \rho_\beta(J_\beta X, \mathcal{L}_{\xi_\alpha} Y) + \rho_\gamma(J_\beta X, Y).
$$

Substitute this into the equation (2.8).

$$
\begin{aligned}
(\mathcal{L}_{\xi_\alpha} g^{\mathcal{D}})(X, Y) &= \rho_\beta(\mathcal{L}_{\xi_\alpha} J_\beta X, Y) + \rho_\beta(J_\beta X, \mathcal{L}_{\xi_\alpha} Y) \\
&\quad + \rho_\gamma(J_\beta X, Y) - \rho_\beta(J_\beta \mathcal{L}_{\xi_\alpha} X, Y) - \rho_\beta(J_\beta X, \mathcal{L}_{\xi_\alpha} Y) \\
&= \rho_\beta((\mathcal{L}_{\xi_\alpha} J_\beta)X, Y) + \rho_\gamma(J_\beta X, Y) \quad \text{(by property (3))} \\
&= \rho_\beta(J_\gamma X, Y) + \rho_\gamma(J_\beta X, Y) = 0,
\end{aligned}
$$

hence, $\mathcal{L}_{\xi_\alpha} g^{\mathcal{D}} = 0$.

$\square$

2.1. **Three CR structures.** Let $(\{\omega_\alpha\}, \{J_\alpha\}, \{\xi_\alpha\};\ \alpha = 1, 2, 3)$ be a nondegenerate qCR structure on $U \subset M$ such that $\mathcal{D}|U = \bigcap_{\alpha=1}^{3} \mathrm{Null}\,\omega_\alpha$. We can extend the almost complex structure J_α to an almost complex structure $\bar{J}_\alpha$ on $\mathrm{Null}\,\omega_\alpha = \mathcal{D} \oplus \{\xi_\beta, \xi_\gamma\}$ by setting:

$$\begin{aligned} &\bar{J}_\alpha|\mathcal{D} = J_\alpha, \\ &\bar{J}_\alpha \xi_\beta = \xi_\gamma, \bar{J}_\alpha \xi_\gamma = -\xi_\beta. \end{aligned} \tag{2.9}$$

(α, β, γ) is a cyclic permutation of $(1, 2, 3)$. First of all, note the following formula (cf. [21]):

$$\mathcal{L}_X(\iota_Y d\omega_a) = \iota_{(\mathcal{L}_X Y)} d\omega_a + \iota_Y \mathcal{L}_X d\omega_a = \iota_{[X,Y]} d\omega_a + \iota_Y \mathcal{L}_X d\omega_a \quad (\forall X, Y \in TU). \tag{2.10}$$

Secondly, we remark the following.

Lemma 2.4. *For $X \in \mathcal{D}$,*

$$\iota_X d\omega_a = \iota_{J_c X} d\omega_b \quad (a, b, c) \sim (1, 2, 3).$$

Proof. Let $TU = \mathcal{D} \oplus V$ where $V = \{\xi_1, \xi_2, \xi_3\}$. If $X \in \mathcal{D}$, then $d\omega_a(X, \xi) = 0$ for $\forall\ \xi \in V$. As $d\omega_b(J_c X, \xi) = 0$ similarly, it follows that $\iota_X d\omega_a = \iota_{J_c X} d\omega_b = 0$ on V. If $Y \in \mathcal{D}$, calculate

$$\begin{aligned} d\omega_a(X, Y) &= -d\omega_a(J_a(J_a X), Y) = -d\omega_b(J_b(J_a X), Y) \text{ (from Lemma 1.2)} \\ &= d\omega_b(J_c X, Y), \text{ hence } \iota_X d\omega_a = \iota_{J_c X} d\omega_b \text{ on } U. \end{aligned}$$

□

In particular, we have

$$\iota_X d\omega_2 = \iota_{J_1 X} d\omega_3 \text{ for } \forall\ X \in \mathcal{D}. \tag{2.11}$$

There is the decomposition with respect to the almost complex structure $\bar{J}_1$:

$$\mathrm{Null}\ \omega_1 \otimes \mathbb{C} = \mathcal{D} \otimes \mathbb{C} \oplus \{\xi_2, \xi_3\} \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1} \tag{2.12}$$

where $T^{1,0} = \mathcal{D}^{1,0} \oplus \{\xi_2 - \boldsymbol{i}\xi_3\}$. We shall observe that the same formula as in Lemma 6.8 of Hitchin [14] can be also obtained for $\mathcal{D}$. (We found Lemma 6.8 when we saw a key lemma to the Kashiwada's theorem [19].)

Lemma 2.5. *If $X, Y \in \mathcal{D}^{1,0}$, then $\iota_{[X,Y]} d\omega_2 = \boldsymbol{i}\iota_{[X,Y]} d\omega_3$.*

Proof. Let $X \in \mathcal{D}^{1,0}$ so that $J_1 X = \boldsymbol{i}X$, then

$$\begin{aligned} \mathcal{L}_X d\omega_2 &= (d\iota_X + \iota_X d) d\omega_2 = d(\iota_X d\omega_2) = d(\iota_{J_1 X} d\omega_3) \text{ (by (2.11))} \\ &= \boldsymbol{i}(d\iota_X) d\omega_3 = \boldsymbol{i}(\mathcal{L}_X - \iota_X d) d\omega_3 = \boldsymbol{i}\mathcal{L}_X d\omega_3. \end{aligned} \tag{2.13}$$

Applying $Y \in \mathcal{D}^{1,0}$ to the equation (2.11) and using (2.10) (extended to a $\mathbb{C}$-valued one),

$$\begin{aligned} \mathcal{L}_X(\iota_Y d\omega_2) &= \mathcal{L}_X(\iota_{J_1 Y} d\omega_3) = \boldsymbol{i}\mathcal{L}_X(\iota_Y d\omega_3) \text{ (from (2.11))} \\ &= \boldsymbol{i}\iota_{[X,Y]} d\omega_3 + \iota_Y \boldsymbol{i}\mathcal{L}_X d\omega_3 \\ &= \boldsymbol{i}\iota_{[X,Y]} d\omega_3 + \iota_Y \mathcal{L}_X d\omega_2 \text{ (by (2.13))}. \end{aligned}$$

Compared this with (2.10) for $\omega_a = \omega_2$, we obtain $\boldsymbol{i}\iota_{[X,Y]} d\omega_3 = \iota_{[X,Y]} d\omega_2$.

□

We prove the following equation (which is used to show the existence of a complex contact structure on the quotient of the quaternionic CR manifold by S^1 [2].)

Proposition 2.6. *For any $X, Y \in \mathcal{D}^{1,0}$, there exist $a \in \mathbb{R}$ and $u \in \mathcal{D}^{1,0}$ such that*

$$[X, Y] = a(\xi_2 - \boldsymbol{i}\xi_3) + u.$$

Conversely, given an arbitrary $a \in \mathbb{R}$, we can choose such $X, Y \in \mathcal{D}^{1,0}$ and some $u \in \mathcal{D}^{1,0}$.

Proof. As $g(J_\alpha \cdot, J_\alpha \cdot) = g(\cdot, \cdot)$ (cf. Lemma 1.2), we note that $d\omega_1|(\mathcal{D}^{1,0}, \mathcal{D}^{0,1})$, $d\omega_2|(\mathcal{D}^{1,0}, \mathcal{D}^{1,0})$, $d\omega_3|(\mathcal{D}^{1,0}, \mathcal{D}^{1,0})$ are nondegenerate. Given $X, Y \in \mathcal{D}^{1,0}$, put $d\omega_2(X, Y) = g(X, J_2Y) = -\frac{1}{2}a$ for some $a \in \mathbb{R}$. (Note that conversely for any $a \in \mathbb{R}$, we can choose $X, Y \in \mathcal{D}^{1,0}$ such that $d\omega_2(X, Y) = g(X, J_2Y) = -\frac{1}{2}a$.) Then $\omega_2([X, Y]) = a$ so that there is an element $v \in \mathrm{Null}\,\omega_2 \otimes \mathbb{C}$ such that $[X, Y] - a \cdot \xi_2 = v$. As $d\omega_3(X, Y) = g(X, J_1J_2Y) = -g(X, J_2(J_1Y)) = -\boldsymbol{i}g(X, J_2Y) = -\frac{\boldsymbol{i}}{2}a$, it follows that $\omega_3([X, Y]) = -\boldsymbol{i}a$. Since $\omega_3(v) = \omega_3([X, Y] - \xi_2) = \omega_3([X, Y])$, $v = -\boldsymbol{i}a \cdot \xi_3 + u$ for some $u \in \mathrm{Null}\,\omega_3 \otimes \mathbb{C}$. Then we have that $[X, Y] = a(\xi_2 - \boldsymbol{i}\xi_3) + u$. Obviously, $\omega_2(u) = 0$. As $X, Y \in \mathcal{D}^{1,0}$, $\omega_1(u) = \omega_1([X, Y]) = -2d\omega_1(X, Y) = 0$ for which $u \in \mathcal{D} \otimes \mathbb{C}$. We now prove that $u \in \mathcal{D}^{1,0}$. First we note that

$$\iota_{[X,Y]}d\omega_2 = a\iota_{(\xi_2 - \boldsymbol{i}\xi_3)}d\omega_2 + \iota_u d\omega_2. \tag{2.14}$$

As ξ_2 (respectively ξ_3) is characteristic for ω_2 (respectively ω_3) from Lemma 2.3, $\iota_{\xi_2}d\omega_2 = 0$ (respectively $\iota_{\xi_3}d\omega_3 = 0$). Using (3.7), the function satisfies $d\iota_{\xi_3}\omega_2 = 0$ (respectively $d\iota_{\xi_2}\omega_3 = 0$). It follows that $\iota_{\xi_3}d\omega_2 = (\mathcal{L}_{\xi_3} - d\iota_{\xi_3})\omega_2 = \mathcal{L}_{\xi_3}\omega_2 = -\omega_1$. Then $\iota_{(\xi_2 - \boldsymbol{i}\xi_3)}d\omega_2 = (\iota_{\xi_2}d\omega_2 - \boldsymbol{i}\iota_{\xi_3}d\omega_2) = \boldsymbol{i}\omega_1$ so (2.14) becomes

$$\iota_{[X,Y]}d\omega_2 = a\boldsymbol{i}\omega_1 + \iota_u d\omega_2. \tag{2.15}$$

As $\mathcal{L}_{\xi_2}\omega_3 = \omega_1$, it follows $\iota_{\xi_2}d\omega_3 = \omega_1$. Similarly

$$\iota_{[X,Y]}d\omega_3 = a\iota_{(\xi_2 - \boldsymbol{i}\xi_3)}d\omega_3 + \iota_u d\omega_3 = a\omega_1 + \iota_u d\omega_3. \tag{2.16}$$

Substitute (2.15), (2.16) into the equlaity $\iota_{[X,Y]}d\omega_2 = \boldsymbol{i}\iota_{[X,Y]}d\omega_3$ of Lemma 2.5, which concludes that

$$\iota_u d\omega_2 = \boldsymbol{i}\iota_u d\omega_3. \tag{2.17}$$

Since $d\omega_2(u, X) = d\omega_3(J_1u, X)$ for any $X \in \mathcal{D} \otimes \mathbb{C}$, (2.17) implies that $d\omega_3(J_1u, X) = \iota_u d\omega_2(X) = d\omega_3(\boldsymbol{i}u, X)$. As $d\omega_3$ is nondegenerate on $\mathcal{D} \otimes \mathbb{C}$, we obtain that $J_1u = \boldsymbol{i}u$. Hence, $u \in \mathcal{D}^{1,0}$. □

Recall that a nondegenerate CR structure on an odd dimensional manifold consists of the pair $(\mathrm{Null}\,\omega, J)$ where ω is a contact structure and J is a complex structure on the contact subbundle $\mathrm{Null}\,\omega$ (i.e. J is integrable). In addition, the characteristic (Reeb) vector field ξ for ω is said to be a *characteristic CR-vector field* if $\mathcal{L}_\xi J = 0$. Consider $(\mathrm{Null}\,\omega_a, \bar{J}_a)$ on U $(a = 1, 2, 3)$. By Lemma 2.3, each ξ_a is a characteristic vector field for ω_a on U. From (3) of Lemma 2.3, $\mathcal{L}_{\xi_\alpha}J_\alpha = 0$. It is easy to check that $\mathcal{L}_{\xi_a}\bar{J}_a = 0$.

Theorem 2.7. *Each $\bar{J}_\alpha$ is integrable on $\mathrm{Null}\omega_\alpha$. As a consequence, a nondegenerate qCR structure $\{\omega_\alpha, J_\alpha\}_{\alpha=1,2,3}$ on a neighborhood U of M^{4n+3} induces three nondegenerate*

CR structures $(\mathrm{Null}\,\omega_\alpha, \bar{J}_\alpha)$ *equipped with characteristic CR-vector field* ξ_α *for each* ω_α $(\alpha = 1,2,3)$. *In fact,* $\omega_\alpha(\xi_\alpha) = 1$ *and* $d\omega_\alpha(\xi_\alpha, X) = 0$ $(\forall\ X \in TM)$ $(\alpha = 1,2,3)$.

Proof. Consider the case for $(\mathrm{Null}\,\omega_1, \bar{J}_1)$. Let $\mathrm{Null}\,\omega_1 \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ where $T^{1,0} = \mathcal{D}^{1,0} \oplus \{\xi_2 - \boldsymbol{i}\xi_3\}$. By Proposition 2.6, if $X, Y \in \mathcal{D}^{1,0}$, then $[X,Y] = a(\xi_2 - \boldsymbol{i}\xi_3) + u$ for some $a \in \mathbb{R}$ and $u \in \mathcal{D}^{1,0}$. By definition,

$$\bar{J}_1[X,Y] = a\bar{J}_1(\xi_2 - \boldsymbol{i}\xi_3) + J_1 u = a\boldsymbol{i}(\xi_2 - \boldsymbol{i}\xi_3) + \boldsymbol{i}u = \boldsymbol{i}[X,Y],$$

it follows $[X,Y] \in T^{1,0}$. It suffices to show that the element $[\xi_2 - \boldsymbol{i}\xi_3, v] \in T^{1,0}$ for $v \in \mathcal{D}^{1,0}$. As $\mathcal{L}_{\xi_2} J_1 = -J_3$ and $-J_3 v = (\mathcal{L}_{\xi_2} J_1)v = \mathcal{L}_{\xi_2}(J_1 v) - J_1(\mathcal{L}_{\xi_2} v)$,

$$J_1(\mathcal{L}_{\xi_2} v) = J_3 v + \boldsymbol{i}\mathcal{L}_{\xi_2} v. \tag{2.18}$$

Note that $[\xi_2 - \boldsymbol{i}\xi_3, v] = \mathcal{L}_{\xi_2} v - \boldsymbol{i}\mathcal{L}_{\xi_3} v \in \mathcal{D}\otimes\mathbb{C}$ on which $\bar{J}_a = J_a$. Then $\bar{J}_1[\xi_2 - \boldsymbol{i}\xi_3, v] = J_1(\mathcal{L}_{\xi_2} v) - \boldsymbol{i}J_1(\mathcal{L}_{\xi_3} v)$. Moreover, as $J_2 v = (\mathcal{L}_{\xi_3} J_1)v = \boldsymbol{i}\mathcal{L}_{\xi_3}(v) - J_1(\mathcal{L}_{\xi_3} v)$ and $J_2 v = J_3 J_1 v = \boldsymbol{i}J_3 v$, it follows that $J_1(\mathcal{L}_{\xi_3} v) = -\boldsymbol{i}J_3 v + \boldsymbol{i}\mathcal{L}_{\xi_3} v$. Using this equality and (2.18), it follows that

$$\begin{aligned}\bar{J}_1[\xi_2 - \boldsymbol{i}\xi_3, v] &= J_1(\mathcal{L}_{\xi_2} v) - \boldsymbol{i}J_1(\mathcal{L}_{\xi_3} v) = \boldsymbol{i}\mathcal{L}_{\xi_2} v + \mathcal{L}_{\xi_3} v\\ &= \boldsymbol{i}(\mathcal{L}_{\xi_2} v - \boldsymbol{i}\mathcal{L}_{\xi_3} v) = \boldsymbol{i}[\xi_2 - \boldsymbol{i}\xi_3, v].\end{aligned}$$

Therefore, $[T^{1,0}, T^{1,0}] \subset T^{1,0}$ so that $\bar{J}_1$ is a complex structure on $\mathrm{Null}\,\omega_1$, i.e. $(\mathrm{Null}\,\omega_1, \bar{J}_1)$ is a CR structure on U. The same holds for $(\mathrm{Null}\,\omega_b, \bar{J}_b)$ $(b = 2,3)$. ☐

3. Model of qCR space forms with type $(4p+3, 4q)$

Suppose that $p + q = n$. Let $\mathbb{H}^{n+1}$ be the quaternionc number space in quaternionic dimension $n+1$ with nondegenerate quaternionic Hermitian form

$$\langle x, y\rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_{p+1} y_{p+1} - \bar{x}_{p+2} y_{p+2} - \cdots - \bar{x}_{n+1} y_{n+1}. \tag{3.1}$$

If we denote $\mathrm{Re}\langle x, y\rangle$ the real part of $\langle x, y\rangle$, then it is noted that $\mathrm{Re}\langle\ ,\ \rangle$ is a nondegenerate symmetric bilinear form on $\mathbb{H}^{n+1}$. In the quaternion case, the group of all invertible matrices $\mathrm{GL}(n+1,\mathbb{H})$ is acting from the left and $\mathbb{H}^* = \mathrm{GL}(1,\mathbb{H})$ acting as the scalar multiplications from the right on $\mathbb{H}^{n+1}$, which forms the group $\mathrm{GL}(n+1,\mathbb{H})\cdot\mathrm{GL}(1,\mathbb{H}) = \mathrm{GL}(n+1,\mathbb{H}) \underset{\mathbb{R}^*}{\times} \mathrm{GL}(1,\mathbb{H})$. Let $\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1)$ be the subgroup of $\mathrm{GL}(n+1,\mathbb{H})\cdot\mathrm{GL}(1,\mathbb{H})$ whose elements preserve the nondegenerate bilinear form $\mathrm{Re}\langle\ ,\ \rangle$. Denote by $\Sigma_{\mathbb{H}}^{3+4p,4q}$ the $(4n+3)$-dimensional quadric space:

$$\{(z_1, \cdots, z_{p+1}, w_1, \cdots, w_q) \in \mathbb{H}^{n+1} \mid |z_1|^2 + \cdots + |z_{p+1}|^2 - |w_1|^2 - \cdots - |w_q|^2 = 1\}.$$

In particular, the group $\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1)$ leaves $\Sigma_{\mathbb{H}}^{3+4p,4q}$ invariant. Let $\langle\ ,\ \rangle_x$ be the nondegenerate quaternionic inner product on the tangent space $T_x\mathbb{H}^{n+1}$ obtained from the parallel translation of $\langle\ ,\ \rangle$ to the point $x \in \mathbb{H}^{n+1}$. Recall that $\{I,\ J,\ K\}$ is the standard quaternionic structure on $\mathbb{H}^{n+1}$ which operates as $Iz = z\boldsymbol{i}$, $Jz = z\boldsymbol{j}$, or $Kz = z\boldsymbol{k}$. As usual, $\{I_x,\ J_x,\ K_x\}$ acts on $T_x\mathbb{H}^{n+1}$ at each point x. Then it is easy to see that $g_x^{\mathbb{H}}(X,Y) = \mathrm{Re}\langle X, Y\rangle_x$ $(\forall\ X, Y \in T_x\mathbb{H}^{n+1})$ is the standard pseudo-euclidean metric of type $(p+1, q)$ on $\mathbb{H}^{n+1}$ which is invariant under $\{I,\ J,\ K\}$. Restricted $g^{\mathbb{H}}$ to the quadric $\Sigma_{\mathbb{H}}^{3+4p,4q}$ in $\mathbb{H}^{n+1}$, we obtain a nondegenerate pseudo-Riemannian metric g of type $(3+4p, 4q)$ where $p+q = n$. Compare [38], [24] for the following definition.

Definition 3.1. *The quadric* $\Sigma_{\mathbb{H}}^{3+4p,4q}$ *is referred to the quaternionic pseudo-Riemannian space form of type* $(3+4p,4q)$ *with constant curvature* 1 *endowed with a transitive group of isometries* $\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1)$ *for which* $\Sigma_{\mathbb{H}}^{3+4p,4q}=\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1)/\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ *where* $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ *is the stabilizer at* $(1,0,\cdots,0)$.

When $(\Sigma_{\mathbb{H}}^{3+4p,4q},g^{\mathbb{H}})$ is viewed as a real pseudo-Riemannian space form, the full group of isometries is $\mathrm{O}(4p{+}4,4q)$. It is noted that the intersection of $\mathrm{O}(4p{+}4,4q)$ with $\mathrm{GL}(n{+}1,\mathbb{H})\cdot\mathrm{GL}(1,\mathbb{H})$ is $\mathrm{Sp}(p{+}1,q)\cdot\mathrm{Sp}(1)$. When N_x is the normal vector at $x\in\Sigma_{\mathbb{H}}^{3+4p,4q}$, $T_x\Sigma_{\mathbb{H}}^{3+4p,4q}=N_x^{\perp}$ with respect to $g^{\mathbb{H}}$. If N is a normal vector field on $\Sigma_{\mathbb{H}}^{3+4p,4q}$, then $IN,JN,KN\in T\Sigma_{\mathbb{H}}^{3+4p,4q}$ such that there is the decomposition $T\Sigma_{\mathbb{H}}^{3+4p,4q}=\{IN,JN,KN\}\oplus\{IN,JN,KN\}^{\perp}$. Let $\mathcal{D}=\{IN,JN,KN\}^{\perp}$ which is the $4n$-dimensional subbundle. As $g^{\mathbb{H}}$ is a $\{I,\ J,\ K\}$ -invariant metric, $(\mathcal{D},g|\mathcal{D})$ is also invariant under $\{I,\ J,\ K\}$. Now, $\mathrm{Sp}(1)$ acts freely on $\Sigma_{\mathbb{H}}^{3+4p,4q}$ as right translations:

$$(\lambda,(z_1,\cdots,z_{p+1},w_1,\cdots,w_q))=(z_1\cdot\bar\lambda,\cdots,z_{p+1}\cdot\bar\lambda,w_1\cdot\bar\lambda,\cdots,w_q\cdot\bar\lambda)\quad(\lambda\in\mathrm{Sp}(1)).$$

Definition 3.2. *The orbit space* $\Sigma_{\mathbb{H}}^{3+4p,4q}/\mathrm{Sp}(1)$ *is said to be the quaternionic pseudo-Kähler projective space* $\mathbb{HP}^{p,q}$ *of type* $(4p,4q)$.

For the definition of quaternionic pseudo-Kähler manifold in general, see Definition 4.5. Note that $\mathbb{HP}^{p,q}$ is a quaternionic pseudo-Kähler manifold by Theorem 4.6 provided that $4n\geq 8$. When $p=n,q=0$, $\mathbb{HP}^{n,0}$ is the standard quaternionic projective space $\mathbb{HP}^{n}$. When $p=0,q=n$, $\mathbb{HP}^{0,n}$ is the quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{H}}$. It is easy to see that $\mathbb{HP}^{p,q}$ is *homotopic* to the canonical quaternionic line bundle over the quaternionic Kähler projective space $\mathbb{HP}^{p}$. There is the equivariant principal bundle:

$$\mathrm{Sp}(1)\to(\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1),\Sigma_{\mathbb{H}}^{3+4p,4q})\xrightarrow{\ \pi\ }(\mathrm{PSp}(p+1,q),\mathbb{HP}^{p,q})\tag{3.2}$$

On the other hand, let

$$\omega_0=-(\bar z_1dz_1+\cdots+\bar z_{p+1}dz_{p+1}-\bar w_1dw_1-\cdots-\bar w_qdw_q).\tag{3.3}$$

Then it is easy to check that ω_0 is an $\mathfrak{sp}(1)$-valued 1-form on $\Sigma_{\mathbb{H}}^{3+4p,4q}$. Let ξ_1,ξ_2,ξ_3 be the vector fields on $\Sigma_{\mathbb{H}}^{3+4p,4q}$ induced by the one-parameter subgroups $\{e^{\boldsymbol{i}\theta}\}_{\theta\in\mathbb{R}}$, $\{e^{\boldsymbol{j}\theta}\}_{\theta\in\mathbb{R}}$, $\{e^{\boldsymbol{k}\theta}\}_{\theta\in\mathbb{R}}$ respectively, which is equivalent to that $\xi_1=IN,\xi_2=JN,\xi_3=KN$. A calculation shows that

$$\omega_0(\xi_1)=\boldsymbol{i},\ \omega_0(\xi_2)=\boldsymbol{j},\ \omega_0(\xi_3)=\boldsymbol{k}.\tag{3.4}$$

By the formula of ω_0, if $a\in\mathrm{Sp}(1)$, then the right translation R_a on $\Sigma_{\mathbb{H}}^{3+4p,4q}$ satisfies that

$$\mathrm{R}_a^*\omega_0=a\cdot\omega_0\cdot\bar a.\tag{3.5}$$

Therefore, ω_0 is a connection form of the above bundle (3.2). Note that $\mathrm{Sp}(p+1,q)$ leaves ω_0 invariant. We shall check the conditions (i), (ii), (iii), (iv) of Definition 1.1 and (1.9) so that $(\Sigma_{\mathbb{H}}^{3+4p,4q},\{I,J,K\},g,\omega_0)$ will be a quaternionic CR manifold. First of all, it follows that

$$\omega_0\wedge\omega_0\wedge\omega_0\wedge\overbrace{(d\omega_0\wedge d\omega_0)\wedge\cdots\wedge(d\omega_0\wedge d\omega_0)}^{n\ \text{times}}\neq 0\ \ \text{at any point of}\ \Sigma_{\mathbb{H}}^{3+4p,4q}.$$

(Compare [16],[31] for example). In fact, letting $\omega_0=\omega_1\boldsymbol{i}+\omega_2\boldsymbol{j}+\omega_3\boldsymbol{k}$ as before,

$$\omega_0{}^3\wedge d\omega_0{}^{2n}=6\omega_1\wedge\omega_2\wedge\omega_3\wedge(d\omega_1{}^2+d\omega_2{}^2+d\omega_3{}^2)^n.$$

This calculation shows (iii). In particular, each ω_a is a nondegenerate contact form on $\Sigma_{\mathbb{H}}^{3+4p,4q}$. Using (3.5) and as ξ_1 generates $\{e^{\boldsymbol{i}\theta}\}_{\theta\in\mathbb{R}}\subset \mathrm{Sp}(1)$, $\mathcal{L}_{\xi_1}\omega_1=0$. (Similarly we have $\mathcal{L}_{\xi_2}\omega_2=\mathcal{L}_{\xi_3}\omega_3=0$.) Noting that $\omega_a(\xi_a)=1$ and $0=\mathcal{L}_{\xi_a}\omega_a=\iota_{\xi_a}d\omega_a$ from (3.4), each ξ_a is the characteristic vector field for ω_a. Moreover, note that $\{\xi_1,\xi_2,\xi_3\}$ generates the fields of Lie algebra of Sp(1). It follows that $\mathcal{D}=\bigcap_{a=1}^{3}\mathrm{Null}\omega_a$ for which there is the decomposition $T\Sigma_{\mathbb{H}}^{3+4p,4q}=\{\xi_1,\xi_2,\xi_3\}\oplus\mathcal{D}$. If $\{e_i\}_{i=1,\cdots,4n}$ is the orthonormal basis of $\mathcal{D}$, then the dual frame θ^i is obtained as $\theta^i(e_j)=\delta^i_j$ and $\theta^i(\xi_1)=\theta^i(\xi_2)=\theta^i(\xi_3)=0$. In order to prove that the distribution uniquely determined by (1.9) are $\{\xi_1,\xi_2,\xi_3\}$ (cf. (4.3) also), we need the following lemma.

Lemma 3.3.

$$d\omega_1(X,Y)=g(X,IY),\ d\omega_2(X,Y)=g(X,JY),\ d\omega_3(X,Y)=g(X,KY)$$

where $X,Y\in\mathcal{D}$.

Proof. Given $X,Y\in\mathcal{D}_x$, let u,v be the vectors at the origin by parallel translation of X,Y at $x\in\Sigma_{\mathbb{H}}^{3+4p,4q}$ respectively. Then by definition, $g(X,Y)=\mathrm{Re}\langle u,v\rangle$. Furthermore,

$$g(X,IY)=\mathrm{Re}(\langle u,v\cdot\boldsymbol{i}\rangle)=\mathrm{Re}(\langle u,v\rangle\cdot\boldsymbol{i}). \tag{3.6}$$

From (3.3), if $X,Y\in\mathcal{D}_x$, then

$$d\omega_0(X,Y)=-(d\bar z_1\wedge dz_1+\cdots+d\bar z_{p+1}\wedge dz_{p+1}-d\bar w_1\wedge dw_1-\cdots-d\bar w_q\wedge dw_q)(u,v).$$

Then a calculation shows that $d\omega_0(X,Y)=-\dfrac{1}{2}(\langle u,v\rangle-\overline{\langle u,v\rangle})$. It is easy to check that the $\boldsymbol{i}$-part of $-\dfrac{1}{2}(\langle u,v\rangle-\overline{\langle u,v\rangle})$ is $\mathrm{Re}(\langle u,v\rangle\cdot\boldsymbol{i})$. Since $d\omega_1(X,Y)$ is the $\boldsymbol{i}$-part of $d\omega(X,Y)$ and by (3.6), we obtain the equality $g(X,IY)=d\omega_1(X,Y)$. Similarly, we have that $g(X,JY)=d\omega_2(X,Y),\ g(X,KY)=d\omega_3(X,Y)$. □

From this lemma, $d\omega_a(e_i,e_j)=g(e_i,J_ae_j)=-\mathbf{J}^a_{ij}$. Since $\{\xi_1,\xi_2,\xi_3\}$ generates Sp(1) of the bundle (3.2), we obtain $d\omega_a+2\omega_b\wedge\omega_c=-\mathbf{J}^a_{ij}\theta^i\wedge\theta^j$. Applying to J,K similarly, we obtain the following structure equation of the bundle (3.2):

$$d\omega_0+\omega_0\wedge\omega_0=-(\mathbf{I}_{ij}\boldsymbol{i}+\mathbf{J}_{ij}\boldsymbol{j}+\mathbf{K}_{ij}\boldsymbol{k})\theta^i\wedge\theta^j. \tag{3.7}$$

From this equation, the condition (1.9) is easily checked so that $\mathrm{Null}\,\omega_\alpha=\{\xi_1,\xi_2,\xi_3\}$. We summarize that

Theorem 3.4. $(\Sigma_{\mathbb{H}}^{3+4p,4q},\{\omega_a\}_{a=1,2,3},\{I,J,K\},g)$ *is a* $(4n+3)$*-dimensional homogeneous qCR manifold of type* $(3+4p,4q)$ *where* $p+q=n\geq 0$*. Moreover, there exists the equivariant principal bundle of the pseudo-Riemannian submersion over the homogeneous quaternionic pseudo-Kähler projective space* $\mathbb{HP}^{p,q}$ *of type* $(4p,4q)$*:* $\mathrm{Sp}(1)\to(\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1),\Sigma_{\mathbb{H}}^{3+4p,4q},g)\xrightarrow{\pi}(\mathrm{PSp}(p+1,q),\mathbb{HP}^{p,q},\hat g)$.

We shall prove more generally in Theorem 4.6 that $(\mathrm{PSp}(p+1,q),\mathbb{HP}^{4p,4q})$ supports an invariant quaternionic pseudo-Kähler metric $\hat g$ of type $(4p,4q)$.

Remark 3.5. (**a**) *In* [2], *it is shown that* $(\Sigma_{\mathbb{H}}^{3+4p,4q}, \{I, J, K\}, g)$ *is a pseudo-Sasakian space form of constant positive curvature with type* $(4p+3, 4q)$.

(**b**) *When* $q = 0$ *or* $p = 0$, *we can find discrete cocompact subgroups from* $\mathrm{Sp}(n+1)\cdot\mathrm{Sp}(1)$ *or* $\mathrm{Sp}(1,n)\cdot\mathrm{Sp}(1)$ *that act properly and freely on* $\Sigma_{\mathbb{H}}^{3+4n,0} = S^{4n+3}$ *or* $\Sigma_{\mathbb{H}}^{3,4n} = V_{-1}^{4n+3}$ *respectively. Thus, we obtain compact nondegenerate qCR manifolds. In fact,* (**i**) *The spherical space form* S^{4n+3}/F *which is* $\mathrm{Sp}(1)$ *or* $\mathrm{SO}(3)$*-bundle over the quaternionic Kähler projective orbifold* $\mathbb{HP}^n/F^*$ *of positive scalar curvature.* ($F \subset \mathrm{Sp}(n+1)\cdot\mathrm{Sp}(1)$ *is a finite group.*) (**ii**) *The pseudo-Riemannian standard space form* V_{-1}^{4n+3}/Γ *of type* $(4n, 3)$ *with constant sectional curvature* -1 *which is an* $\mathrm{Sp}(1)$*-bundle over the quaternionic Kähler hyperbolic orbifold* $\mathbb{H}_{\mathbb{H}}^n/\Gamma^*$ *of negative scalar curvature.* ($\Gamma^* \subset \mathrm{PSp}(1,n)$ *is a discrete subgroup.*) *As we know, there exists no compact pseudo-Sasakian manifold (or qCR manifold) whose pseudo-Kähler orbifold has zero Ricci curvature. However in our case, an indefinite Heisenberg nilmanifold is a compact* p-c qCR *manifold whose pseudo-Kähler orbifold is the complex euclidean orbifold (i.e. zero Ricci curvature), see* §7.3.

4. Local Principal bundle

Let $\{e_i\}_{i=1,\cdots,4n}$ be the basis of $\mathcal{D}|U$ such that $g^{\mathcal{D}}(e_i, e_j) = g_{ij}$. We choose a local coframe θ^i for which

$$\theta^i|V = 0 \quad \text{and} \quad \theta^i(e_j) = \delta_{ij}. \tag{4.1}$$

As usual the quaternionic structure $\{J_\alpha\}_{\alpha=1,2,3}$ can be represented locally by the matrix $\mathbf{J}^{\alpha}{}_{i}^{j}$ such as $J_\alpha e_i = \mathbf{J}^{\alpha}{}_{i}^{j} e_j$. Note that $\rho_\alpha(e_j, e_i) = \mathbf{J}^{\alpha}{}_{i}^{k} g_{jk} = \mathbf{J}^{\alpha}{}_{ij}$ by (1.1). Here the matrix (g_{ij}) lowers and raises the indices. Using θ^i we can write the structure equation (1.8):

$$d\omega_\alpha + 2\omega_\beta \wedge \omega_\gamma = -\mathbf{J}^{\alpha}{}_{ij}\theta^i \wedge \theta^j \quad (\alpha = 1, 2, 3). \tag{4.2}$$

If we use ω of Definition 1.1, the above formula is equivalent to the following:

$$d\omega + \omega \wedge \omega = -(\mathbf{J}^{\mathbf{1}}{}_{ij}\boldsymbol{i} + \mathbf{J}^{\mathbf{2}}{}_{ij}\boldsymbol{j} + \mathbf{J}^{\mathbf{3}}{}_{ij}\boldsymbol{k})\theta^i \wedge \theta^j. \tag{4.3}$$

Denote by $\mathcal{E}$ the local transformation groups generated by V acting on a small neighborhood U' of U. As $\mathcal{E}$ is locally isomorphic to the compact Lie group SO(3) by Lemma 2.2, it acts properly on U'. (See for example [30].) If we note that each ξ_a is a nonzero vector field everywhere on U, then the stabilizer of $\mathcal{E}$ is finite at every point. By the slice theorem of compact Lie groups [9], choosing a sufficiently small neighborhood $\mathcal{E}'$ of the identity from $\mathcal{E}$, $\mathcal{E}'$ acts properly and freely on U'. We choose such U' (respectively $\mathcal{E}'$) from the beginning and replace it by U (respectively $\mathcal{E}$). Then there is a principal local fibration:

$$\mathcal{E} \to U \xrightarrow{\pi} U/\mathcal{E}. \tag{4.4}$$

If we note that $V \oplus \mathcal{D} = TM|U$, π maps $\mathcal{D}$ isomorphically onto $T(U/\mathcal{E})$ at each point of U. So $\{\pi_* e_i \mid i = 1, \cdots, 4n\}$ is a basis of $T(U/\mathcal{E})$ at each point of $U/\mathcal{E}$. Let $\hat\theta^i$ be the dual frame on $U/\mathcal{E}$ such that

$$\hat\theta^i(\pi_* e_j) = \delta_{ij} \quad \text{on } U/\mathcal{E}. \tag{4.5}$$

Since θ^i is the coframe of $\{e_i\}$ and $\pi^*\hat{\theta}^i|V = \theta^i|V = 0$, it follows that

$$\pi^*\hat{\theta}^i = \theta^i \text{ on } U \ (i = 1, \cdots, 4n). \tag{4.6}$$

Lemma 4.1. *Put $J_1 = I$, $J_2 = J$, $J_3 = K$ respectively. Let $\{\varphi_\theta\}_{-\varepsilon<\theta<\varepsilon}$ be a local one-parameter subgroup of the local group $\mathcal{E}$. Then there exists an element $G_\theta \in \mathrm{SO}(3)$ satisfying the following:*

$$(1) \quad (\varphi_\theta)_* \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = G_\theta \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}.$$

(4.7)

$$(2) \quad \begin{pmatrix} I_{\varphi_\theta y} \\ J_{\varphi_\theta y} \\ K_{\varphi_\theta y} \end{pmatrix} \circ \varphi_{\theta *} = \varphi_{\theta *} \circ {}^tG(\theta) \begin{pmatrix} I_y \\ J_y \\ K_y \end{pmatrix}.$$

Proof. Since every leaf of V is locally isomorphic to $\mathrm{SO}(3)$, ξ_a is viewed as the fundamental vector field to the principal fibration $\pi : U \to U/\mathcal{E}$. Thus we may assume that ξ_1, ξ_2, ξ_3 correspond to $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ respectively so that $\varphi^1_\theta = e^{\boldsymbol{i}\theta}$, $\varphi^2_\theta = e^{\boldsymbol{j}\theta}$, $\varphi^3_\theta = e^{\boldsymbol{k}\theta}$ up to conjugacy by an element of $\mathrm{SO}(3)$, A calculation shows that $(\varphi^1_\theta)_*((\xi_2)_x) = \cos 2\theta \cdot (\xi_2)_{\varphi^1_\theta x} + \sin 2\theta \cdot (\xi_3)_{\varphi^1_\theta x}$. Similarly, $(\varphi^1_\theta)_*((\xi_3)_x) = -\sin 2\theta \cdot (\xi_2)_{\varphi^1_\theta x} + \cos 2\theta \cdot (\xi_3)_{\varphi^1_\theta x}$, $(\varphi^1_\theta)_*((\xi_1)_x) = (\xi_1)_{\varphi^1_\theta x}$. This holds similarly for $\varphi^1_\theta, \varphi^2_\theta$. It turns out that if $\varphi_\theta \in \mathcal{E}$, then there exists an element $G_\theta \in \mathrm{SO}(3)$ which shows the above formula (1). Since φ_t preserves $\mathcal{D}$ $(-\varepsilon < t < \varepsilon)$, using (1) we see that

$$\varphi_t^*(\omega_1, \omega_2, \omega_3) = (\omega_1, \omega_2, \omega_3)G_t. \tag{4.8}$$

Since there exists an element $g_t \in \mathrm{Sp}(1)$ such that $g_t \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \bar{g}_t = G_t \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix}$ ($\bar{g}_t$ is the quaternion conjugate of g_t), (4.8) is equivalent with

$$\varphi_t^*\omega = g_t \cdot \omega \cdot \bar{g}_t. \tag{4.9}$$

Differentiate this equation which yields that

$$\varphi_t^*(d\omega + \omega \wedge \omega) \equiv g_t(d\omega + \omega \wedge \omega)\bar{g}_t \mod \omega. \tag{4.10}$$

Using the equation (4.2), it follows that

$$\begin{aligned} \varphi_t^*((I_{ij}, J_{ij}, K_{ij}) \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \theta^i \wedge \theta^j) &\equiv (I_{ij}, J_{ij}, K_{ij}) g_t \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \bar{g}_t \theta^i \wedge \theta^j \\ &= (I_{ij}, J_{ij}, K_{ij}) G_t \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \theta^i \wedge \theta^j. \end{aligned}$$

Noting that $\varphi_t^*\theta^i = \varphi_t^*(\pi^*\hat{\theta}^i) = \theta^i$, the above equation implies that

$$(I_{ij}(\varphi_t(x)), J_{ij}(\varphi_t(x)), K_{ij}(\varphi_t(x))) \equiv (I_{ij}(x), J_{ij}(x), K_{ij}(x))G_t(x) \mod \omega. \tag{4.11}$$

Since $\pi_*\varphi_{t*}((e_i)_x) = \pi_*((e_i)_{\varphi_t x})$ $(x \in U)$, it follows $\varphi_{t*}((e_i)_x) = (e_i)_{\varphi_t x}$. Letting $G_t = (s_{ij}) \in \mathrm{SO}(3)$ and using (4.11),

$$\begin{aligned} &I_{\varphi_t x}(\varphi_t)_*((e_i)_x) = I_{\varphi_t x}((e_i)_{\varphi_t x}) = I_i^j(\varphi_t x)((e_j)_{\varphi_t x}) \\ &= (I_i^j(x)\cdot s_{11} + J_i^j(x)\cdot s_{21} + K_i^j(x)\cdot s_{31}))((\varphi_t)_*((e_j)_x)) \\ &= (\varphi_t)_*(s_{11}\cdot I_x((e_i)_x) + s_{21}\cdot J_x((e_i)_x) + s_{31}\cdot K_x((e_i)_x)) \\ &= (\varphi_t)_*((s_{11}, s_{21}, s_{31})\begin{pmatrix} I_x \\ J_x \\ K_x \end{pmatrix}(e_i)_x). \end{aligned}$$

The same argument applies to $J_{\varphi_t x}, K_{\varphi_t x}$ to conclude that $\begin{pmatrix} I_{\varphi_t x} \\ J_{\varphi_t x} \\ K_{\varphi_t x} \end{pmatrix} \circ \varphi_{t*} = \varphi_{t*} \circ {}^tG_t \begin{pmatrix} I_x \\ J_x \\ K_x \end{pmatrix}$. This proves (2). □

Lemma 4.2. *The quaternionic structure* $\{I, J, K\}$ *on* $\mathcal{D}|U$ *induces a family of quaternionic structures* $\{\hat{I}_i, \hat{J}_i, \hat{K}_i\}_{i\in\Lambda}$ *on* $U/\mathcal{E}$.

Proof. Choose a small neighborhood $V_i \subset U/\mathcal{E}$ and a section $s_i : V_i \to U$ for the principal bundle $\pi : U \to U/\mathcal{E}$. Let $\hat{x} \in V_i$ and a vector $\hat{X}_{\hat{x}} \in TV_i$. Choose a vector $X_{s_i(\hat{x})} \in \mathcal{D}_{s_i(\hat{x})}$ such that $\pi_*(X_{s_i(\hat{x})}) = \hat{X}_{\hat{x}}$. Define endomorphisms $\hat{I}_i, \hat{J}_i, \hat{K}_i$ on V_i to be

$$\begin{aligned} (\hat{I}_i)_{\hat{x}}(\hat{X}_{\hat{x}}) &= \pi_* I_{s_i(\hat{x})} X_{s_i(\hat{x})}, \\ (\hat{J}_i)_{\hat{x}}(\hat{X}_{\hat{x}}) &= \pi_* J_{s_i(\hat{x})} X_{s_i(\hat{x})}, \\ (\hat{K}_i)_{\hat{x}}(\hat{X}_{\hat{x}}) &= \pi_* K_{s_i(\hat{x})} X_{s_i(\hat{x})}. \end{aligned} \tag{4.12}$$

Since $\pi_* : \mathcal{D}_{s_i(\hat{x})} \to T_{\hat{x}}(U/\mathcal{E})$ is an isomorphism, $\hat{I}_i, \hat{J}_i, \hat{K}_i$ are well-defined almost complex structures on V_i. So we have a family $\{\hat{I}_i, \hat{J}_i, \hat{K}_i\}_{i\in\Lambda}$ of almost complex structures associated to an open cover $\{V_i\}_{i\in\Lambda}$ of $U/\mathcal{E}$. Suppose that $V_i \cap V_j \neq \emptyset$. If $\hat{x} \in V_i \cap V_j$, then there is an element $\varphi_\theta \in \mathcal{E}$ such that $s_j(\hat{x}) = \varphi_\theta \cdot s_i(\hat{x})$. As φ_θ preserves $\mathcal{D}$, $\varphi_{\theta*} X_{s_i(\hat{x})} \in \mathcal{D}_{s_j(\hat{x})}$ and $\pi_*(\varphi_{\theta*} X_{s_i(\hat{x})}) = \hat{X}_{\hat{x}}$. Then

$$X_{s_j(\hat{x})} = \varphi_{\theta*} X_{s_i(\hat{x})}. \tag{4.13}$$

Let $\{\hat{I}_j, \hat{J}_j, \hat{K}_j\}$ be almost complex structures on V_j obtained from (4.12). Using Lemma 4.1 and (4.13), calculate at $s_j(\hat{x})$ ($\hat{x} \in V_i \cap V_j$),

$$\begin{pmatrix} (\hat{I}_j)_{\hat{x}} \\ (\hat{J}_j)_{\hat{x}} \\ (\hat{K}_j)_{\hat{x}} \end{pmatrix} \hat{X}_{\hat{x}} = \pi_* \begin{pmatrix} I_{s_j(\hat{x})} \\ J_{s_j(\hat{x})} \\ K_{s_j(\hat{x})} \end{pmatrix} X_{s_j(\hat{x})} = \pi_* \begin{pmatrix} I_{\varphi_\theta \cdot s_i(\hat{x})} \\ J_{\varphi_\theta \cdot s_i(\hat{x})} \\ K_{\varphi_\theta \cdot s_i(\hat{x})} \end{pmatrix} \varphi_{\theta *} X_{s_i(\hat{x})}$$
$$= \pi_* \varphi_{\theta *} \circ {}^tG_\theta \begin{pmatrix} I_{s_i(\hat{x})} \\ J_{s_i(\hat{x})} \\ K_{s_i(\hat{x})} \end{pmatrix} X_{s_i(\hat{x})}$$
$$= {}^tG(\theta) \pi_* \begin{pmatrix} I_{s_i(\hat{x})} \\ J_{s_i(\hat{x})} \\ K_{s_i(\hat{x})} \end{pmatrix} X_{s_i(\hat{x})} = {}^tG_\theta \begin{pmatrix} (\hat{I}_i)_{\hat{x}} \\ (\hat{J}_i)_{\hat{x}} \\ (\hat{K}_i)_{\hat{x}} \end{pmatrix} \hat{X}_{\hat{x}},$$

hence $\begin{pmatrix} (\hat{I}_j)_{\hat{x}} \\ (\hat{J}_j)_{\hat{x}} \\ (\hat{K}_j)_{\hat{x}} \end{pmatrix} = {}^tG_\theta \begin{pmatrix} (\hat{I}_i)_{\hat{x}} \\ (\hat{J}_i)_{\hat{x}} \\ (\hat{K}_i)_{\hat{x}} \end{pmatrix}$ on $\hat{x} \in V_i \cap V_j$. Thus, $\{\hat{I}_i, \hat{J}_i, \hat{K}_i\}_{i \in \Lambda}$ defines a quaternionic structure on $U/\mathcal{E}$. □

4.1. **Pseudo-Sasakian 3-structure and Pseudo-Kähler structure.** We now take $\{e_i\}_{i=1,\cdots,4n}$ of $\mathcal{D}|U$ as the orthonormal basis, i.e. $g_{ij} = \delta_{ij}$. Then the bilinear form $g^{\mathcal{D}} = \sum_{i=1}^{4p} \theta^i \cdot \theta^i - \sum_{i=4p+1}^{4n} \theta^i \cdot \theta^i$ defined on $\mathcal{D}$ induces a pseudo-Riemannian metric on $U/\mathcal{E}$:

$$\hat{g} = \sum_{i=1}^{4p} \hat{\theta}^i \cdot \hat{\theta}^i - \sum_{i=4p+1}^{4n} \hat{\theta}^i \cdot \hat{\theta}^i \tag{4.14}$$

such that $g^{\mathcal{D}} = \pi^* \hat{g}$. Let $\hat{\nabla}$ be the covariant derivative on $U/\mathcal{E}$. If $\hat{\omega}^i_j$ is the Levi-Civita connection with respect to $\hat{g}$, then $\hat{\nabla} \hat{e}_i = \hat{\omega}^j_i \hat{e}_j$ for which $\hat{\omega}^i_j$ satisfies that

$$d\hat{\theta}^i = \hat{\theta}^j \wedge \hat{\omega}^i_j, \quad \hat{\omega}_{ij} + \hat{\omega}_{ji} = 0. \tag{4.15}$$

Put

$$\hat{\Omega}^i_j = d\hat{\omega}^i_j - \hat{\omega}^\sigma_j \wedge \hat{\omega}^i_\sigma = \frac{1}{2} \hat{R}^i_{jkl} \hat{\theta}^k \wedge \hat{\theta}^\ell. \tag{4.16}$$

Consider the following pseudo-Riemannian metric on U:

$$\begin{aligned} &\tilde{g}_x(X, Y) = \sum_{a=1}^{3} \omega_a(X) \cdot \omega_a(Y) + \hat{g}_{\pi(x)}(\pi_* X, \pi_* Y) \quad (X, Y \in T_x U). \\ &(\text{Equivalently } \tilde{g} = \sum_{a=1}^{3} \omega_a \cdot \omega_a + \sum_{i=1}^{4p} \theta^i \cdot \theta^i - \sum_{i=4p+1}^{4n} \theta^i \cdot \theta^i.) \end{aligned} \tag{4.17}$$

Then we have shown in [4] that the local principal fibration $\mathcal{E} \to (U, \tilde{g}) \xrightarrow{\pi} (U/\mathcal{E}, \hat{g})$ is a pseudo-Sasakian 3-structure. In fact the next equation (4.18) is equivalent with the normality condition of the pseudo-Sasakian 3-structure. (Compare [33], [5].)

Proposition 4.3. *Let* $(\{\omega_\alpha\}, \{J_\alpha\}, \{\xi_\alpha\})_{\alpha=1,2,3}$ *be a nondegenerate quaternionic CR structure on* U *of a* $(4n+3)$*-manifold* M*. If* ∇ *is the Levi-Civita connection on* $(U, \tilde{g})$*, then,*

$$(\nabla_X \bar{J}_\alpha)Y = \tilde{g}(X,Y)\xi_\alpha - \omega_\alpha(Y)X \quad (\alpha = 1,2,3). \tag{4.18}$$

Proof. For $X, Y \in TU$, consider the following tensor

$$N^{\omega_\alpha}(X,Y) = N(X,Y) + (X\omega_\alpha(Y) - Y\omega_\alpha(X))\xi_\alpha \tag{4.19}$$

where $N(X,Y) = [\bar{J}_\alpha X, \bar{J}_\alpha Y] - [X,Y] - \bar{J}_\alpha[\bar{J}_\alpha X, Y] - \bar{J}_\alpha[X, \bar{J}_\alpha Y]$ is the Nijenhuis torsion of $\bar{J}_\alpha$ $(\alpha = 1,2,3)$. A direct calculation for a contact metric structure $\tilde{g}$ (cf. [5]) shows that

$$\begin{aligned} 2\tilde{g}((\nabla_X \bar{J}_\alpha)Y, Z) = \tilde{g}(N^{\omega_\alpha}(Y,Z), \bar{J}_\alpha X) + (\mathcal{L}_{\bar{J}_\alpha X}\omega_\alpha)(Y) \\ - (\mathcal{L}_{\bar{J}_\alpha Y}\omega_\alpha)(X) + 2\tilde{g}(X,Y)\omega_\alpha(Z) - 2\tilde{g}(X,Z)\omega_\alpha(Y). \end{aligned}$$

Since each $\bar{J}_\alpha$ is integrable on $\mathrm{Null}\omega_\alpha$ from Theorem 2.7, it follows that the Nijenhuis torsion of $\bar{J}_\alpha$, $N(X,Y) = 0$ $(\forall\, X, Y \in \mathrm{Null}\,\omega_\alpha)$. By the formula (4.19), $N^{\omega_\alpha}(X,Y) = 0$ for $\forall\, X, Y \in \mathrm{Null}\,\omega_\alpha$. Noting the decomposition $TU = \{\xi_1\} \oplus \mathrm{Null}\,\omega_1$, to obtain (4.18), it suffices to show that $N^{\omega_1}(\xi_1, X) = 0$ (similarly for $\alpha = 2,3$). As ξ_α is a characteristic CR-vector field for $(\omega_\alpha, \bar{J}_\alpha)$ $(\alpha = 1,2,3)$, i.e. $\mathcal{L}_{\xi_1}\bar{J}_1 = 0$, it follows that $\bar{J}_1[\xi_1, Y] = [\xi_1, \bar{J}_1 Y]$ $(\forall\, Y \in \mathrm{Null}\,\omega_1)$. In particular, $\bar{J}_1[\xi_1, \bar{J}_1 X] = -[\xi_1, X]$. Hence, $N^{\omega_\alpha}(\xi_1, X) = 0$. As a consequence, we see that $N^{\omega_\alpha}(X,Y) = 0$ $(\forall\, X, Y \in TU)$. On the other hand, if $N^{\omega_\alpha}(X,Y) = 0$ $(\forall\, X, Y \in TU)$, then it is easy to see that $(\mathcal{L}_{\bar{J}_\alpha X}\omega_\alpha)(Y) - (\mathcal{L}_{\bar{J}_\alpha Y}\omega_\alpha)(X) = 0$. (See [5].) From (4.17), note that $\omega_\alpha(X) = \tilde{g}(\xi_\alpha, X)$. The above equation (4.18) follows. □

As $\{\omega_\alpha, \theta^i\}_{\alpha=1,2,3; i=1\cdots 4n}$ are orthonormal coframes for the pseudo-Sasakian metric $\tilde{g}$ (cf. (4.17)), the structure equation says that there exist unique 1-forms φ^i_j, τ^i_α $(i,j = 1,\cdots,4n; \alpha = 1,2,3)$ satisfying:

$$d\theta^i = \theta^j \wedge \varphi^i_j + \sum_{\alpha=1}^{3} \omega_\alpha \wedge \tau^i_\alpha \quad (\varphi_{ij} + \varphi_{ji} = 0). \tag{4.20}$$

Then the normality condition for the pseudo-Sasakian 3-structure is reinterpreted as the following structure equation.

Theorem 4.4. *There exsists a connection form* $\{\omega^i_j\}$ *such that*

$$d\bar{\mathbf{J}}^a_{ij} - \omega^\sigma_i \bar{\mathbf{J}}^a_{\sigma j} - \bar{\mathbf{J}}^a_{i\sigma}\omega^\sigma_j = 2\bar{\mathbf{J}}^b_{ij} \cdot \omega_c - 2\bar{\mathbf{J}}^c_{ij} \cdot \omega_b \quad ((a,b,c) \sim (1,2,3)). \tag{4.21}$$

Proof. It follows from Proposition 4.3 that $(\nabla_X \bar{J}_a)e_i = \tilde{g}(X, e_i)\xi_a$ for $\{e_i\} = \mathcal{D}$ at a point. From (4.20), let $\nabla_X e_i = \varphi^j_i(X)e_j + \sum_{b=1}^{3}(\tau_b)_i \xi_b$ which is substituted into the equality $(\nabla_X \bar{J}_a)e_i = \nabla_X(\bar{J}_a e_i) - \bar{J}_a(\nabla_X e_i)$:

$$\begin{aligned} (\nabla_X \bar{J}_a)e_i &= (d(\bar{\mathbf{J}}^a)^\ell_i(X) - \varphi^\sigma_i(X)(\bar{\mathbf{J}}^a)^\ell_\sigma + (\bar{\mathbf{J}}^a)^\sigma_i \varphi^\ell_\sigma(X))e_\ell \\ &\quad + \sum_{b=1}^{3}(\bar{\mathbf{J}}^a)^\ell_i (\tau_b)_\ell(X)\xi_b - \sum_{c\neq a}(\tau_b)_i(X)\xi_c \ (\text{Here } \bar{J}_a\xi_b = \xi_c) \\ &= \tilde{g}(X, e_i)\xi_a \quad ((4.18)). \end{aligned} \tag{4.22}$$

As $\tilde{g}(X, e_i) = \tilde{g}_{ki}\theta^k(X)$ (cf. (4.17)), this implies that $d(\bar{\mathbf{J}}^a)^\ell_i - \varphi^\sigma_i(\bar{\mathbf{J}}^a)^\ell_\sigma + (\bar{\mathbf{J}}^a)^\sigma_i\varphi^\ell_\sigma = 0$ and $(\bar{\mathbf{J}}^a)^\ell_i(\tau_a)_\ell(X)\xi_a = \tilde{g}_{ki}\theta^k(X)\xi_a$. It follows that $-(\tau_a)_i = (\bar{\mathbf{J}}^a)_{ij}\theta^j$. Then $(\tau_a)_i\tilde{g}^{ik} = -(\bar{\mathbf{J}}^a)_{ij}\tilde{g}^{ik}\theta^j = (\bar{\mathbf{J}}^a)_{ji}\tilde{g}^{ik}\theta^j$, so that $(\tau_a)^i = (\bar{\mathbf{J}}^a)^i_j\theta^j$. As $\tilde{g}_{ij} = \pm\delta_{ij}$, use $\tilde{g}^{ij}$ to lower the above equations:

$$\begin{aligned} & d(\bar{\mathbf{J}}^a)_{ij} - \varphi^\sigma_i(\bar{\mathbf{J}}^a)_{\sigma j} - (\bar{\mathbf{J}}^a)_{i\sigma}\varphi^\sigma_j = 0. \\ & (\tau_a)^i = (\bar{\mathbf{J}}^a)^i_j\theta^j. \end{aligned} \tag{4.23}$$

Putting

$$\omega^i_j = \varphi^i_j - \sum_{a=1}^{3}(\bar{\mathbf{J}}^a)^i_j\omega_a, \tag{4.24}$$

the equation (4.20) reduces to

$$d\theta^i = \theta^j \wedge \omega^i_j \quad (\omega_{ij} + \omega_{ji} = 0). \tag{4.25}$$

Differentiate our equation (4.2) $d\omega_a + 2\omega_b \wedge \omega_c = -\bar{\mathbf{J}}^a_{ij}\theta^i \wedge \theta^j$ $((a, b, c) \sim (1, 2, 3))$ and substitute (4.25). It becomes (after alternation):

$$(d\bar{\mathbf{J}}^a_{ij} - \omega^\sigma_i\bar{\mathbf{J}}^a_{\sigma j} - \bar{\mathbf{J}}^a_{i\sigma}\omega^\sigma_j + \omega_b \cdot 2\bar{\mathbf{J}}^c_{ij} - \omega_c \cdot 2\bar{\mathbf{J}}^b_{ij}) \wedge \theta^i \wedge \theta^j = 0.$$

Since $d\bar{\mathbf{J}}^a_{ij} - \omega^\sigma_i\bar{\mathbf{J}}^a_{\sigma j} - \bar{\mathbf{J}}^a_{i\sigma}\omega^\sigma_j \equiv 0 \bmod \omega_1, \omega_2, \omega_3$ from (4.23), (4.24) and the forms $\omega_a \wedge \theta^i \wedge \theta^j$ $(a = 1, 2, 3)$ are linearly independent, the result follows. □

Definition 4.5. *Let $\hat{\nabla}$ be the Levi-Civita connection on an almost quaternionic pseudo-Riemannian manifold $(X, \hat{g})$ of type $(4p, 4q)$ $(p+q = n)$. Then X is said to be a quaternionic pseudo-Kähler manifold if for each quaternionic structure $\{\hat{J}_a; a = 1, 2, 3\}$ defined locally on a neighborhood of X, there exists a smooth local function $A \in \mathfrak{so}(3)$ such that*

$$\hat{\nabla}\begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{pmatrix} = A \cdot \begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{pmatrix}$$

provided that $\dim X = 4n \geq 8$. *Equivalently if $\hat{\Omega}$ is the fundamental 4-form globally defined on X, then $\hat{\nabla}\hat{\Omega} = 0$.*

We have shown the following result in [2] when $\dim U/\mathcal{E} = 4n \geq 12$ by Swann's method.

Theorem 4.6. *The set $(U/\mathcal{E}, \hat{g}, \{\hat{I}_i, \hat{J}_i, \hat{K}_i\}_{i\in\Lambda})$ is a quaternionic pseudo-Kähler manifold of type $(4p, 4q)$ provided that* $\dim U/\mathcal{E} = 4n \geq 8$. *Moreover, $(U/\mathcal{E}, \hat{g})$ is an Einstein manifold of positive scalar curvature $(4n \geq 4)$ such that*

$$\hat{R}_{j\ell} = 4(n+2)\hat{g}_{j\ell}. \tag{4.26}$$

Proof. As we put $\theta^i = \pi^*\hat{\theta}^i$, (4.15) implies that $d\theta^i = \theta^j \wedge \pi^*\hat{\omega}^i_j$, $\pi^*\hat{\omega}_{ij} + \pi^*\hat{\omega}_{ji} = 0$. Compared this with (4.25) and by skew-symmetry, it is easy to check that

$$\pi^*\hat{\omega}^i_j = \omega^i_j. \tag{4.27}$$

Put $\hat{V} = V_i$ and $\hat{J}_1 = \hat{I}_i$, $\hat{J}_2 = \hat{J}_i$, $\hat{J}_3 = \hat{K}_i$ on $\hat{V}$. Let $s = s_i : \hat{V} \to U$ be the section as before. Since $\pi_*s_*((\hat{e}_j)_x) = (\hat{e}_j)_x = \pi_*((e_j)_{s(\hat{x})})$, $s_*((\hat{e}_j)_x) - (e_j)_{s(\hat{x})} \in V = \{\xi_1, \xi_2, \xi_3\}$. Then

$\theta^i(s_*((\hat{e}_j)_x)) = \theta^i((e_j)_{s(\hat{x})})$ from (4.1). A calculation shows that $(\hat{J}_a)_{\hat{x}}\hat{e}_i = \pi_*(J_a)_{s(\hat{x})}e_i = \pi_*((\bar{\mathbf{J}}^a)_i^j(s(\hat{x}))e_j) = (\bar{\mathbf{J}}^a)_i^j(s(\hat{x}))\hat{e}_j$ (cf. (4.12)). As we put $\hat{J}^a_{\hat{x}}\hat{e}_i = (\hat{\mathbf{J}}^a)_i^j(\hat{x})\hat{e}_j$, note that

$$\bar{\mathbf{J}}^a_{ij}(s(\hat{x})) = \hat{\mathbf{J}}^a_{ij}(\hat{x}) \quad (a = 1, 2, 3). \tag{4.28}$$

In particular,

$$d(\bar{\mathbf{J}}^a)_{ij} \circ s_*(\hat{X}_{\hat{x}}) = d(\hat{\mathbf{J}}^a)_{ij}(\hat{X}_{\hat{x}}) \quad (\forall\ \hat{X}_{\hat{x}} \in T_{\hat{x}}(\hat{V}))\ (a = 1, 2, 3). \tag{4.29}$$

Since $\pi_* s_*(\hat{X}_{\hat{x}}) = \hat{X}_{\hat{x}}$ $(\hat{X}_{\hat{x}} \in T_{\hat{x}}(\hat{V}))$, (4.27) implies that $\hat{\omega}^\sigma_j(\hat{X}_{\hat{x}}) = \omega^\sigma_j(s_*(\hat{X}_{\hat{x}}))$. Plug this equation and (4.28), (4.29) into (4.21):

$$\begin{aligned}
&d(\bar{\mathbf{J}}^a)_{ij}(s_*\hat{X}) - \omega^\sigma_i(s_*\hat{X}) \cdot (\bar{\mathbf{J}}^a)_{\sigma j}(s(\hat{x})) - (\bar{\mathbf{J}}^a)_{i\sigma}(s(\hat{x})) \cdot \omega^\sigma_j(s_*\hat{X}) \\
&= d((\hat{\mathbf{J}}^a)_{ij})_{\hat{x}}(\hat{X}) - \hat{\omega}^\sigma_i(\hat{X}) \cdot (\hat{\mathbf{J}}^a)_{\sigma j}(\hat{x}) - (\hat{\mathbf{J}}^a)_{i\sigma}(\hat{x}) \cdot \hat{\omega}^\sigma_j(\hat{X}) \\
&= 2(\bar{\mathbf{J}}^b)_{ij}(s(\hat{x})) \cdot \omega_c(s_*\hat{X}) - 2(\bar{\mathbf{J}}^c)_{ij}(s(\hat{x})) \cdot \omega_b(s_*\hat{X}) \\
&= 2(\hat{\mathbf{J}}^b)_{ij}(\hat{x}) \cdot \omega_c(s_*\hat{X}) - 2(\hat{\mathbf{J}}^c)_{ij}(\hat{x}) \cdot \omega_b(s_*\hat{X}).
\end{aligned}$$

Using these,

$$\begin{aligned}
&(\hat{\nabla}_{\hat{X}}(\hat{J}_a)((\hat{e}_i)_{\hat{x}}) = \hat{\nabla}_{\hat{X}}(\hat{J}_a)\hat{e}_i - (\hat{J}_a)(\hat{\nabla}_{\hat{X}}\hat{e}_i) \\
&= (d(\hat{\mathbf{J}}^a)_{ij}(\hat{X}) - (\hat{\mathbf{J}}^a)_{i\sigma}(\hat{x}) \cdot \hat{\omega}^\sigma_j(\hat{X}) - \hat{\omega}^\sigma_i(\hat{X}) \cdot (\hat{\mathbf{J}}^a)_{\sigma j}(\hat{x}))(\hat{e}_j)_{\hat{x}} \\
&= 2(\hat{\mathbf{J}}^b)_{ij}(\hat{x})(\hat{e}_j)_{\hat{x}} \cdot s^*\omega_c(\hat{X}) - 2(\hat{\mathbf{J}}^c)_{ij}(\hat{x})(\hat{e}_j)_{\hat{x}} \cdot s^*\omega_b(\hat{X}) \\
&= \Big(2(\hat{J}_b)_{\hat{x}} \cdot s^*\omega_c(\hat{X}) - 2(\hat{J}_c)_{\hat{x}} \cdot s^*\omega_b(\hat{X})\Big)(\hat{e}_i)_{\hat{x}}.
\end{aligned}$$

Therefore, $\hat{\nabla}_{\hat{X}}(\hat{J}_a) = 2(\hat{J}_b)_{\hat{x}} \cdot s^*\omega_c(\hat{X}) - 2(\hat{J}_c)_{\hat{x}} \cdot s^*\omega_b(\hat{X})$. This concludes that

$$\hat{\nabla} \begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{pmatrix} = 2 \begin{pmatrix} 0 & s^*\omega_3 & -s^*\omega_2 \\ -s^*\omega_3 & 0 & s^*\omega_1 \\ s^*\omega_2 & -s^*\omega_1 & 0 \end{pmatrix} \begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{pmatrix}. \tag{4.30}$$

As we put $\hat{J}_1 = \hat{I}_i,\ \hat{J}_2 = \hat{J}_i,\ \hat{J}_3 = \hat{K}_i$ on $\hat{V}$, $(U/\mathcal{E}, \hat{g}, \{\hat{I}_i, \hat{J}_i, \hat{K}_i\}_{i\in\Lambda})$ is a quaternionic pseudo-Kähler manifold for $\dim U/\mathcal{E} \geq 8$. Using the Ricci identity (cf. (2.11), (2.12) of [15], [34]), a calculation shows that

$$(n > 1)$$

$$\begin{aligned}
\hat{R}_{jl} &= -4(n+2)\Big(s^*(d\omega_1 + 2\omega_2 \wedge \omega_3)\Big)(\hat{e}_j, \hat{e}_k)\hat{I}^k_\ell(\hat{x}). \\
\hat{R}_{jl} &= -4(n+2)\Big(s^*(d\omega_2 + 2\omega_3 \wedge \omega_1)\Big)(\hat{e}_j, \hat{e}_k)\hat{J}^k_\ell(\hat{x}). \\
\hat{R}_{jl} &= -4(n+2)\Big(s^*(d\omega_3 + 2\omega_1 \wedge \omega_2)\Big)(\hat{e}_j, \hat{e}_k)\hat{K}^k_\ell(\hat{x}).
\end{aligned} \tag{4.31}$$

$$(n = 1)$$

$$\begin{aligned}
\hat{R}_{jl} = &-4\Big(s^*(d\omega_1 + 2\omega_2 \wedge \omega_3)\Big)(\hat{e}_j, \hat{e}_k)\hat{I}^k_\ell(\hat{x}) - 4\Big(s^*(d\omega_2 + 2\omega_3 \wedge \omega_1)\Big)(\hat{e}_j, \hat{e}_k)\hat{J}^k_\ell(\hat{x}) \\
&- 4\Big(s^*(d\omega_3 + 2\omega_1 \wedge \omega_2)\Big)(\hat{e}_j, \hat{e}_k)\hat{K}^k_\ell(\hat{x}).
\end{aligned} \tag{4.32}$$

Using $d\omega_a+2\omega_b\wedge\omega_c=-\mathbf{J}^a_{ij}\theta^i\wedge\theta^j$ and (4.28), it follows that $\Big(s^*(d\omega_a+2\omega_b\wedge\omega_c)\Big)(\hat e_j,\hat e_k)=-\mathbf{J}^a_{jk}(s(\hat x))=-\hat{\mathbf{J}}^a_{jk}(\hat x)$. Since $(\hat{\mathbf{J}}^a)^j_i\cdot(\hat{\mathbf{J}}^a)^k_j=-\delta^k_i$, $\hat R_{jl}=+4(n+2)(\hat{\mathbf{J}}^a)_{jk}(\hat x)\cdot(\hat{\mathbf{J}}^a)^k_\ell(\hat x)=4(n+2)g_{j\ell}$ when $n>1$ and $\hat R_{jl}=+4(\hat I_{jk}(\hat x)\cdot\hat I^k_\ell(\hat x)+\hat J_{jk}(\hat x)\cdot\hat J^k_\ell(\hat x)+\hat K_{jk}(\hat x)\cdot\hat K^k_\ell(\hat x))=4\cdot 3g_{j\ell}$ when $n=1$. □

5. Quaternionic CR curvature tensor

Recall from (4.25) that $d\theta^i=\theta^j\wedge\omega^i_j,\ \ \omega_{ij}+\omega_{ji}=0$ where $\pi^*\hat\omega^i_j=\omega^i_j$, $\pi^*\hat\theta^i=\theta^i$ from (4.20), (4.6) respectively $(i,j=1,\cdots,4n)$. Define the fourth-order tensor $R^i_{jk\ell}$ on U by putting

$$d\omega^i_j-\omega^\sigma_j\wedge\omega^i_\sigma\equiv\frac{1}{2}R^i_{jk\ell}\theta^k\wedge\theta^\ell \mod \omega_1,\omega_2,\omega_3. \tag{5.1}$$

By (4.16), it follows that

$$R^i_{jk\ell}=\pi^*\hat R^i_{jk\ell}. \tag{5.2}$$

The equality (4.26) implies that

$$R_{j\ell}=4(n+2)g_{j\ell}. \tag{5.3}$$

Differentiate the structure equation (4.20).

$$0=d\theta^j\wedge\varphi^i_j-\theta^j\wedge d\varphi^i_j+\sum_a d\omega_a\wedge\tau^i_a-\sum_a\omega_a\wedge d\tau^i_a. \tag{5.4}$$

Substitute (4.2) and (4.20) into (5.4);

$$\begin{aligned}&\theta^j\wedge(d\varphi^i_j-\varphi^k_j\wedge\varphi^i_k-\sum_a\mathbf{J}^a_{kj}\theta^k\wedge\tau^i_a)+\sum_a\omega_a\wedge(d\tau^i_a-\tau^k_a\wedge\varphi^i_k)\\&+2\omega_2\wedge\omega_3\wedge\tau^i_1+2\omega_3\wedge\omega_1\wedge\tau^i_2+2\omega_1\wedge\omega_2\wedge\tau^i_3=0.\end{aligned}$$

This implies that

$$\theta^j\wedge(d\varphi^i_j-\varphi^k_j\wedge\varphi^i_k-\sum_a\mathbf{J}^a_{kj}\theta^k\wedge\tau^i_a)\equiv 0 \mod \ \omega_1,\omega_2,\omega_3. \tag{5.5}$$

We use (5.5) to define the curvature form:

$$\Phi^i_j=d\varphi^i_j-\varphi^k_j\wedge\varphi^i_k+\sum_{a=1}^{3}\theta^k\wedge\mathbf{J}^a_{jk}\tau^i_a-\theta^i\wedge\theta_j. \tag{5.6}$$

Set

$$\begin{aligned}{}_1\Phi^i&=d\tau^i_1-\tau^k_1\wedge\varphi^i_k+\omega_2\wedge\tau^i_3-\omega_3\wedge\tau^i_2,\\ {}_2\Phi^i&=d\tau^i_2-\tau^k_2\wedge\varphi^i_k+\omega_3\wedge\tau^i_1-\omega_1\wedge\tau^i_3,\\ {}_3\Phi^i&=d\tau^i_3-\tau^k_3\wedge\varphi^i_k+\omega_1\wedge\tau^i_2-\omega_2\wedge\tau^i_1\end{aligned} \tag{5.7}$$

which satisfy the following relation.

$$\theta^j\wedge\Phi^i_j+\omega_1\wedge{}_1\Phi^i+\omega_2\wedge{}_2\Phi^i+\omega_3\wedge{}_3\Phi^i=0. \tag{5.8}$$

We may define the fourth-order curvature tensor T^i_{jkl} from Φ^i_j:

$$\Phi^i_j \equiv \frac{1}{2}T^i_{jkl}\theta^k \wedge \theta^\ell \mod \omega_1, \omega_2, \omega_3. \tag{5.9}$$

Remark 5.1. In view of (5.9), there exist the fourth-order curvature tensors W^i_{jka} $(a = 1, 2, 3)$ and V^i_{jbc} $(1 \le b < c \le 3)$ for which we can describe:

$$\Phi^i_j = \frac{1}{2}T^i_{jkl}\theta^k \wedge \theta^\ell + \frac{1}{2}\sum_a W^i_{jka}\theta^k \wedge \omega_a + \frac{1}{2}\sum_{b<c} V^i_{jbc}\omega_b \wedge \omega_c. \tag{5.10}$$

6. Transformation of p-c qCR structure

6.1. G**-structure.** When $\{\theta^i\}_{i=1,\cdots,4n}$ are the 1-forms locally defined on a neighborhood U of M, we form the $\mathbb{H}$-valued 1-form $\{\omega^i\}_{i=1,\cdots,n}$ such as

$$\omega^i = \theta^i + \theta^{n+i}\boldsymbol{i} + \theta^{2n+i}\boldsymbol{j} + \theta^{3n+i}\boldsymbol{k}. \tag{6.1}$$

We shall consider the transformations $f : U \to U$ of the following form:

$$\begin{aligned} f^*\omega &= \lambda \cdot \omega \cdot \bar{\lambda} \ (= u^2 a \cdot \omega \cdot \bar{a}), \\ f^*(\omega^j) &= U'^j_\ell \omega^\ell \cdot \bar{\lambda} + \lambda \tilde{v}^j \omega \bar{\lambda} \end{aligned} \tag{6.2}$$

such that $\lambda = u \cdot a$ for some smooth functions $u > 0$, $a \in \mathrm{Sp}(1)$ and $U' \in \mathrm{Sp}(p, q)$ with $p + q = n$. Let G be the subgroup of $\mathrm{GL}(n + 1, \mathbb{H}) \cdot \mathbb{H}^*$ consisting of matrices

$$\left(\begin{array}{c|c} \lambda & 0 \\ \hline \lambda \cdot \tilde{v}^i & U' \end{array}\right) \cdot \lambda. \tag{6.3}$$

Recall that $\mathrm{Sim}(\mathbb{H}^n) = \mathbb{H}^n \rtimes (\mathrm{Sp}(p, q) \cdot \mathbb{H}^*)$ is the quaternionic affine similarity group of the quaternionic vector space $\mathbb{H}^n$ where $\mathbb{H}^* = \mathrm{Sp}(1) \times \mathbb{R}^+$. Then note that G is anti-isomorphic to $\mathrm{Sim}(\mathbb{H}^n)$ given by the map

$$^t\left(\begin{array}{c|c} \lambda & x^j \\ \hline 0 & X \end{array}\right) \cdot \lambda \longrightarrow (X x^{j*}, X \cdot \lambda) \in \mathbb{H}^n \rtimes (\mathrm{Sp}(p, q) \cdot \mathbb{H}^*). \tag{6.4}$$

(Here $x^* = {}^t\bar{x}$.) We represent G as the real matrices. Let $\tilde{v} \in \mathbb{H}^n$ be a vector. The group $\mathrm{Sp}(p, q) \cdot \mathbb{H}^*$ is the subgroup of $\mathrm{GL}(4n, \mathbb{R})$ acting on $\mathbb{H}^n$ by

$$(U' \cdot \lambda)\tilde{v} = U'\tilde{v} \cdot \bar{\lambda} \tag{6.5}$$

where $U' \in \mathrm{Sp}(p, q)$, $\lambda \in \mathbb{H}^*$. Write $\lambda = u \cdot a \in \mathbb{R}^+ \times \mathrm{Sp}(1)$ so that $\mathrm{Sp}(p, q) \cdot \mathbb{H}^*$ is embedded into $\mathbb{R}^+ \times \mathrm{SO}(4p, 4q)$ in the following manner:

$$U' \cdot \lambda(\tilde{v}) = uU'\tilde{v}\bar{a} = uU'\bar{a} \circ (a\tilde{v}\bar{a}) = u(U'\bar{a}) \circ \mathrm{Ad}_a(\tilde{v}) = u \cdot U\tilde{v} \quad (\tilde{v} \in \mathbb{H}^n = \mathbb{R}^{4n}) \tag{6.6}$$

in which

$$U = U'\bar{a} \circ \mathrm{Ad}_a \in \mathrm{SO}(4p, 4q), \tag{6.7}$$

$$\mathrm{Ad}_a \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} = a \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \bar{a} = A \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \text{ for some } A \in \mathrm{SO}(3). \tag{6.8}$$

We put the vector $\tilde{v}^j \in \mathbb{H}^n$ in such a way that $\tilde{v}^j = v^j + v^{n+j}\boldsymbol{i} + v^{2n+j}\boldsymbol{j} + v^{3n+j}\boldsymbol{k}$ $(j = 1, \cdots, n)$. Form the real (4×3)-matrix

$$V^j = \begin{pmatrix} -v^{j+n} & -v^{j+2n} & -v^{j+3n} \\ v^j & -v^{j+3n} & v^{j+2n} \\ v^{j+3n} & v^j & -v^{j+n} \\ -v^{j+2n} & v^{j+n} & v^j \end{pmatrix}. \tag{6.9}$$

It is easy to check that

$$\lambda \tilde{v}^j \cdot \omega \bar{\lambda} = \lambda((1\ \boldsymbol{i}\ \boldsymbol{j}\ \boldsymbol{k}) V^j \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}) \bar{\lambda} = (1\ \boldsymbol{i}\ \boldsymbol{j}\ \boldsymbol{k}) u^2 \begin{pmatrix} 1 & 0 \\ 0 & {}^tA \end{pmatrix} V^j \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}. \tag{6.10}$$

Then G is isomorphic to the subgroup of $\mathrm{GL}(4n+3, \mathbb{R})$ consisting of matrices

$$\left(\begin{array}{c|c} u^2 \cdot {}^tA & 0 \\ \hline u^2 \begin{pmatrix} 1 & 0 \\ 0 & {}^tA \end{pmatrix} V^1 & \\ \vdots & u \cdot U \\ u^2 \begin{pmatrix} 1 & 0 \\ 0 & {}^tA \end{pmatrix} V^n & \end{array}\right). \tag{6.11}$$

Here $A \in \mathrm{SO}(3), U = (U^i_j) \in \mathrm{SO}(4p, 4q)$.
Using the coframe field $\{\omega_1, \omega_2, \omega_3, \theta^1, \cdots, \theta^{4n}\}$, f is represented by

$$\begin{aligned} & f^*(\omega_1, \omega_2, \omega_3) = u^2(\omega_1, \omega_2, \omega_3)A, \\ & f^*\theta^i = u\theta^k U^i_k + \sum_{\alpha=1}^{3} \omega_\alpha v^i_\alpha, \\ & \text{where } \begin{pmatrix} v_1^{4j-3} & v_2^{4j-3} & v_3^{4j-3} \\ v_1^{4j-2} & v_2^{4j-2} & v_3^{4j-2} \\ v_1^{4j-1} & v_2^{4j-1} & v_3^{4j-1} \\ v_1^{4j} & v_2^{4j} & v_3^{4j} \end{pmatrix} = u^2 \begin{pmatrix} 1 & 0 \\ 0 & {}^tA \end{pmatrix} V^j \quad (j = 1, \cdots, n). \end{aligned} \tag{6.12}$$

Let $\mathcal{F}(M)$ be the principal coframe bundle over M. A subbundle P of $\mathcal{F}(M)$ is said to be a *bundle of the nondegenerate integrable G-structure* if P is the total space of the principal bundle $G \to P \to M$ whose points consist of such coframe fields $\{\omega_1, \omega_2, \omega_3, \theta^1, \cdots, \theta^{4n}\}$ satisfying the conditions of Definition 1.1, (1.8), (1.9). A diffeomorphism $f : M \to M$ is a G-automorphism if the derivative $f^* : \mathcal{F}(M) \to \mathcal{F}(M)$ induces a bundle map $f^* : P \to P$ in which f^* has the form locally as in (6.2) (equivalently (6.12)).

Definition 6.1. *Let* $\mathrm{Aut}_{qCR}(M)$ *be the group of all* G*-automorphisms of* M*.*

6.2. **Automorphism group** $\mathrm{Aut}(M)$**.** Let W be the $(n+2)$-dimensional arithmetic vector space $\mathbb{H}^{p+1,q+1}$ over $\mathbb{H}$ equipped with the standard Hermitian metric $\mathcal{B}$ of signature $(p+1, q+1)$ where $p+q=n$. Then note that the isometry group $\mathrm{Sp}(W) = \mathrm{Aut}(W,\mathcal{B}) = \mathrm{Sp}(p+1,q+1)$ and W has the gradation $W = W^{-1} + W^0 + W^{+1}$, where $W^{\pm 1}$ are dual 1-dimensional isotropic subspaces and W^0 is ($\mathcal{B}$-non-degenerate) orthogonal complement to $W^{-1}+W^{+1}$. The gradation W induces the gradation of the Lie algebra $\mathfrak{g}$ of depth two, i.e.

$$\mathfrak{g} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^{0} + \mathfrak{g}^{1} + \mathfrak{g}^{2}.$$

Here $\mathfrak{g}^0 = \mathbb{R} + \mathfrak{sp}(1) + \mathfrak{sp}(n)$.

In [3] we introduced a notion of *p-c q* structure. This geometry is defined by a codimension three distribution $\mathcal{H}$ on a $(4n+3)$-dimensional manifold M, which satisfies the only one condition that the associated graded tangent space ${}^{gr}T_xM = T_xM/\mathcal{H}_x + \mathcal{H}_x$ at any point is isomorphic to the quaternionic Heisenberg Lie algebra $\mathfrak{M}(p,q) \cong \mathfrak{g}^- = \mathfrak{g}^{-2} + \mathfrak{g}^{-1}$, i.e. the Iwasawa subalgebra of $\mathrm{Sp}(p+1, q+1)$. We proved that such a geometry is a parabolic geometry so that it admits a canonical Cartan connection and its automorphism group $\mathrm{Aut}(M)$ is a Lie group. More precisely, if $P^+(\mathbb{H})$ is the parabolic connected subgroup of the symplectic group $\mathrm{Sp}(W)$ corresponding to the dual parabolic subalgebra $\mathfrak{p}^+(\mathbb{H}) = \mathfrak{g}^+ + \mathfrak{g}^0$ of $\mathfrak{sp}(W)$, then there is a $P^+(\mathbb{H})$-principal bundle $\pi : B \to M$ with a normal Cartan connection $\kappa : TB \to \mathfrak{sp}(W)$ of type $\mathrm{Sp}(W)/P^+(\mathbb{H})$. There exists a canonical p-c q structure $\mathcal{H}^{\mathrm{can}}$ on $\mathrm{Sp}(p+1,q+1)/P^+(\mathbb{H})$ with all vanishing curvature tensors (cf. §7.2). A p-c q manifold $(M,\mathcal{H})$ is locally isomorphic to a $(\mathrm{Sp}(p+1,q+1)/P^+(\mathbb{H}), \mathcal{H}^{\mathrm{can}})$ if and only if the associated Cartan connection κ is flat (i.e. has zero curvature). Put $S^{4p+3,4q} = \mathrm{Sp}(p+1,q+1)/P^+(\mathbb{H})$. Then $S^{4p+3,4q}$ is the flat homogeneous model diffeomorphic to $S^{4p+3} \times S^{4q+3}/\mathrm{Sp}(1)$ where the product of spheres $S^{4p+3} \times S^{4q+3} = \{(z^+, z^-) \in \mathbb{H}^{p+1,q+1} \mid \mathcal{B}(z^+,z^+) = 1, \mathcal{B}(z^-,z^-) = -1\}$ is the subspace of $W = \mathbb{H}^{p+1,q+1}$ and the action of $\mathrm{Sp}(1)$ is induced by the diagonal right action on W. The group of all automorphisms $\mathrm{Aut}(S^{4p+3,4q})$ preserving this flat structure is $\mathrm{PSp}(p+1,q+1)$. Suppose that M is a p-c qCR manifold. By definition, $T_xM \cong T_xM/\mathcal{D}_x + \mathcal{D}_x = \mathrm{Im}\mathbb{H} + \mathbb{H}^n \cong \mathfrak{M}(p,q)$ at $\forall\, x \in M$. Then each G-automorphism of $\mathrm{Aut}_{qCR}(M)$ preserves $\mathfrak{M}(p,q)$ by the above formula (6.12). Since a p-c qCR structure is a refinement of a p-c q structure by Definition 1.6, note that $\mathrm{Aut}_{qCR}(M)$ is a closed subgroup of $\mathrm{Aut}(M)$ which is a Lie group as above.

Corollary 6.2. *The group* $\mathrm{Aut}_{qCR}(M)$ *is a finite dimensional Lie group for a p-c q*CR *manifold* M*.*

7. Pseudo-conformal qCR structure on $S^{3+4p,4q}$

We shall prove that the qCR homogeneous model $\Sigma_{\mathbb{H}}^{3+4p,4q}$ induces a p-c qCR structure on $S^{3+4p,4q}$ which coincides with the flat p-c q structure.

7.1. **Quaternionic pseudo-hyperbolic geometry.** Let

$$\mathcal{B}(z,w) = \bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_{p+1} w_{p+1} - \bar{z}_{p+2} w_{p+2} - \cdots - \bar{z}_{n+2} w_{n+2} \tag{7.1}$$

be the above Hermitian form on $\mathbb{H}^{n+2}=\mathbb{H}^{p+1,q+1}$ $(p+q=n)$. We consider the following subspaces in $\mathbb{H}^{n+2}-\{0\}$:

$$V_0^{4n+7}=\{z\in\mathbb{H}^{n+2}|\ \mathcal{B}(z,z)=0\},$$
$$V_-^{4n+8}=\{z\in\mathbb{H}^{n+2}|\ \mathcal{B}(z,z)<0\}.$$

Let $\mathbb{H}^*\to((\mathrm{Sp}(p+1,q+1)\cdot\mathbb{H}^*,\mathbb{H}^{n+2}-\{0\})\stackrel{P}{\longrightarrow}(\mathrm{PSp}(p+1,q+1),\mathbb{HP}^{n+1})$ be the equivariant projection. The quaternionic pseudo-hyperbolic space $\mathbb{H}_{\mathbb{H}}^{p+1,q}$ is defined to be $P(V_-^{4n+8})$ (cf. [11]). Let $\mathrm{GL}(n+2,\mathbb{H})$ be the group of all invertible $(n+2)\times(n+2)$-matrices with quaternion entries. Denote by $\mathrm{Sp}(p+1,q+1)$ the subgroup consisting of

$$\{A\in\mathrm{GL}(n+2,\mathbb{H})\mid\mathcal{B}(Az,Aw)=\mathcal{B}(z,w),z,w\in\mathbb{H}^{n+2}\}.$$

The action $\mathrm{Sp}(p+1,q+1)$ on V_-^{4n+8} induces an action on $\mathbb{H}\ _{\mathbb{H}}^{p+1,q}$. The kernel of this action is the center $\mathbb{Z}/2=\{\pm1\}$ whose quotient is the pseudo-quaternionic hyperbolic group $\mathrm{PSp}(p+1,q+1)$. It is known that $\mathbb{H}_{\mathbb{H}}^{p+1,q}$ is a complete simply connected pseudo-Riemannian manifold of negative sectional curvature from -1 to $-\frac{1}{4}$, and with the group of isometries $\mathrm{PSp}(p+1,q+1)$ (cf. [21]). Remark that when $q=0,p=n$, $P(V_-^{4n+8})=\mathbb{H}_{\mathbb{H}}^{n+1}$ is the quaternionic Kähler hyperbolic space with the group of isometries $\mathrm{PSp}(n+1,1)$. The projective compactification of $\mathbb{H}_{\mathbb{H}}^{p+1,q}$ is obtained by taking the closure $\bar{\mathbb{H}}_{\mathbb{H}}^{p+1,q}$ in $\mathbb{HP}^{n+1}$. Then it is easy to check that $\bar{\mathbb{H}}_{\mathbb{H}}^{p+1,q}=\mathbb{H}_{\mathbb{H}}^{p+1,q}\cup P(V_0^{4n+7})$. The boundary $P(V_0^{4n+7})$ of $\mathbb{H}_{\mathbb{H}}^{p+1,q}$ is identified with the quadric $S^{3+4p,4q}$ by the correspondence:

$$[z_+,z_-]\mapsto\left[\frac{z_+}{||z_-||},\frac{z_-}{||z_-||}\right]. \tag{7.2}$$

Since the pseudo-hyperbolic action of $\mathrm{PSp}(p+1,q+1)$ on $\mathbb{H}_{\mathbb{H}}^{p+1,q}$ extends to a smooth action on $S^{3+4p,4q}=P(V_0^{4n+7})$ as projective transformations because the projective compactification $\bar{\mathbb{H}}_{\mathbb{H}}^{p+1,q}$ is an invariant domain of $\mathbb{HP}^{n+1}$.

7.2. **Existence of p-c qCR structure on $S^{3+4p,4q}$.** Recall that $\Sigma_{\mathbb{H}}^{3+4p,4q}=\{(z_1,\cdots,z_{p+1},w_1,\cdots,w_q)\in\mathbb{H}^{n+1}\mid|z_1|^2+\cdots+|z_{p+1}|^2-|w_1|^2-\cdots-|w_q|^2=1\}$ equipped with qCR structure ω_0 (cf. §3). The embedding ι of $\Sigma_{\mathbb{H}}^{3+4p,4q}$ into $S^{4p+3,4q}$ is defined by $(z_1,\cdots,z_{p+1},w_1,\cdots,w_q)\mapsto[(z_1,\cdots,z_{p+1},w_1,\cdots,w_q,1)]$. Then $\iota(\Sigma_{\mathbb{H}}^{3+4p,4q})$ is an open dense submanifold of $S^{4p+3,4q}$ because it misses $S^{4p+3,4(q-1)}=S^{4p+3}\times S^{4q-1}/\mathrm{Sp}(1)$ in $S^{4p+3,4q}$. We know that $\Sigma_{\mathbb{H}}^{3+4p,4q}$ has the transitive isometry group $\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1)$ (cf. Definition 3.1). Then this embedding implies that $\mathrm{Sp}(p+1,q)\cdot\mathrm{Sp}(1)$ is identified with the subgroup $P(\mathrm{Sp}(p+1,q)\times\mathrm{Sp}(1))$ of $\mathrm{PSp}(p+1,q+1)$ leaving the last component z_{n+2} invariant in $V_0^{4n+7}\subset\mathbb{H}^{n+2}$.

By pullback, each element h of $\mathrm{PSp}(p+1,q+1)$ gives a qCR structure $h^{-1*}\omega_0$ on the open subset $h(\Sigma_{\mathbb{H}}^{3+4p,4q})$ of $S^{3+4p,4q}$. Noting that $h^{-1*}\mathcal{H}^{can}=\mathcal{H}^{can}$ and Definition 1.6, we shall prove that $(S^{3+4p,4q},\mathcal{H}^{can})$ admits a p-c qCR structure by showing that $\mathrm{Null}h^{-1*}\omega_0$ coincides with the restriction of $\mathcal{H}^{can}|h(\Sigma_{\mathbb{H}}^{3+4p,4q})$.

Theorem 7.1. *The* $(4n+3)$*-dimensional p-c q manifold* $(S^{4p+3,4q}, \mathcal{H}^{can})$ *supports a p-c qCR structure, i.e. there exists locally a qCR strucrure* ω *on a neighborhood* U *such that*

$$\mathcal{H}^{can}|U = \mathrm{Null}\omega.$$

Moreover, the automorphism group $\mathrm{Aut}_{qCR}(S^{4p+3,4q})$ *with respect to this p-c qCR structure is* $\mathrm{PSp}(p+1, q+1)$*.*

Proof. First we describe the canonical p-c q structure $\mathcal{H}^{can}$ on $S^{3+4p,4q}$ explicitly. Choose isotropic vectors $x, y \in V_0$ such that $\mathcal{B}(x, y) = 1$ and denote by V the orthogonal complement to $\{x, y\}$ in $\mathbb{H}^{p+1,q+1}$. Then it follows that $T_x V_0 = \mathfrak{sp}(W)x = y\mathrm{Im}\mathbb{H} + V + x\mathbb{H}$ where $T_x(x\mathbb{H}^*) = x\mathbb{H}$. Then

$$T_{[x]} S^{4k+3,4q} = P_*(T_x V_0) = (y\mathrm{Im}\mathbb{H} + V + x\mathbb{H})/x\mathbb{H}.$$

We associate to each $[x] \in S^{4k+3,4q}$ the orthogonal complement $x^\perp = V + x\mathbb{H}$. It does not depend on the choice of points from $[x]$. In fact, if $x' \in [x]$, then $x' = x \cdot \lambda$ for some $\lambda \in \mathbb{H}^*$. By the definition choosing y' such that $T_{x'} V_0 = y'\mathrm{Im}\mathbb{H} + V' + x'\mathbb{H}$ where the orthogonal complement V' to $\{x', y'\}$ in $\mathbb{H}^{p+1,q+1}$ is uniquely determined. Let v' be any vector of V' which is described as $v' = y \cdot a + v + x \cdot b$ for some $a, b \in \mathbb{H}$. Then

$$\begin{aligned} 0 = \mathcal{B}(x', v') &= \mathcal{B}(x', y)a + \mathcal{B}(x', v) + \mathcal{B}(x', x)b \\ &= \bar{\lambda}\mathcal{B}(x, y)a + \bar{\lambda}\mathcal{B}(x, v) + \bar{\lambda}\mathcal{B}(x, x)b = \bar{\lambda} \cdot a. \end{aligned}$$

Since $\lambda \neq 0$, $a = 0$ and so $v' = v + x \cdot b$. Hence $x'^\perp = V' + x'\mathbb{H} = V + x\mathbb{H}$. Therefore the orthogonal complement $x^\perp = V + x\mathbb{H}$ in $\mathbb{H}^{p+1,q+1}$ determines a codimension three subbundle

$$\begin{aligned} \mathcal{H}^{can} &= \bigcup_{[x] \in S^{4p+3,4q}} P_*(x^\perp). \\ P_*(x^\perp) &= V + x\mathbb{H}/x\mathbb{H} \subset TS^{4p+3,4q}. \end{aligned} \tag{7.3}$$

On the other hand, recall that if N_p is the normal vector at $p \in \Sigma_{\mathbb{H}}^{3+4p,4q}$, then $(\mathrm{Null}\,\omega_0)_p = \mathcal{D}_p = \{IN_p, JN_p, KN_p\}^\perp$ by the definition (cf. § 3). Since $T_p\Sigma_{\mathbb{H}}^{3+4p,4q} = N_p^\perp$ with respect to $g^{\mathbb{H}}$, it follows that $T_p\mathbb{H}^{n+1}|\Sigma_{\mathbb{H}}^{3+4p,4q} = \{N_p, IN_p, JN_p, KN_p\} \oplus \mathcal{D}_p$. If we note that $\{N_p, IN_p, JN_p, KN_p\} = p\mathbb{H}$, then we have $\mathcal{D}_p = p\mathbb{H}^\perp$. It is easy to see that the orthogonal complement to $p\mathbb{H}$ with respect to $g^{\mathbb{H}}$ coincides with the orthogonal complement to p with respect to the inner product $\mathcal{B}$. Hence, $\mathcal{D}_p = p^\perp$. As the tangent subspace $\iota_*(\mathcal{D}_p)$ at $\iota(p)$ in $T_{\iota(p)}V_0$ is $(\mathcal{D}_p, 0)$ which is parallel to $\mathcal{D}_p$ in T_pV_0, it implies that $\mathcal{B}(\iota_*(\mathcal{D}_p), \iota(p)) = \mathcal{B}((\mathcal{D}_p, 0), (p, 1)) = \langle \mathcal{D}_p, p\rangle - \langle 0, 1\rangle = 0$. Hence $\iota_*(\mathcal{D}_p) \subset \iota(p)^\perp$ (with respect to $\mathcal{B}$). As $\iota(p)^\perp = V + \iota(p)\mathbb{H}$, $\iota_*(\mathcal{D}_p) \subset V + \iota(p)\mathbb{H}$. As above $\iota_*(\mathcal{D}_p) = (\mathcal{D}_p, 0)$ at $\iota(p)$, but $\iota(p)\mathbb{H} = (p, 1) \cdot H$. The intersection $\iota_*(\mathcal{D}_p) \cap \iota(p)\mathbb{H} = \{0\}$. It implies that $\iota_*(\mathcal{D}_p) = \iota_*(\mathcal{D}_p)/\iota(p)\mathbb{H} \subset V + \iota(p)\mathbb{H}/\iota(p)\mathbb{H}$. By (7.3), $\iota_*((\mathrm{Null}\,\omega_0)_p) = P_*(\iota(p)^\perp) = \mathcal{H}^{can}_{\iota(p)}$. Therefore $S^{4p+3,4q}$ admits a p-c qCR structure. Then $\mathrm{Aut}_{qCR}(S^{4p+3,4q})$ is a subgroup of $\mathrm{Aut}(S^{4p+3,4q}) = \mathrm{PSp}(p+1, q+1)$ from §6.2. ☐

7.3. **Pseudo-conformal quaternionic Heisenberg geometry.**
To prove $\mathrm{Aut}_{QCR}(S^{4p+3,4q}) = \mathrm{PSp}(p+1, q+1)$, we recall the quaternionic Heisenberg Lie group. Let $\mathrm{PSp}(p+1, q+1)$ be the group of all automorphisms preserving the flat p-c q structure of $S^{4p+3,4q} = \mathrm{PSp}(p+1, q+1)/P^+(\mathbb{H})$ (cf. § 6.2.) We consider the stabilizer of the point at infinity $\{\infty\} = [1, 0, \cdots, 0, 1] \in \Sigma_{\mathbb{H}}^{3+4p,4q} \subset S^{4p+3,4q}$. Recall the (indefinite) Heisenberg nilpotent Lie group $\mathcal{M} = \mathcal{M}(p, q)$ from [16]. It is the product $\mathbb{R}^3 \times \mathbb{H}^n$ with group law:

$$(a, y) \cdot (b, z) = (a + b - \mathrm{Im}\langle y, z\rangle, y + z).$$

Here $\langle\ \rangle$ is the Hermitian inner product of signature (p, q) on $\mathbb{H}^n$ as in (7.1) and $\mathrm{Im}\langle\ \rangle$ is the imaginary part $(p + q = n)$. It is nilpotent because the commutator subgroup $[\mathcal{M}, \mathcal{M}] = \mathbb{R}^3$ which is the center consisting of the form $(a, 0)$. In particular, there is the central extension:

$$1 \to \mathbb{R}^3 \to \mathcal{M} \longrightarrow \mathbb{H}^n \to 1. \tag{7.4}$$

Denote by $\mathrm{Sim}(\mathcal{M})$ the semidirect product $\mathcal{M} \rtimes (\mathrm{Sp}(p, q) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+)$ where the action $(A \cdot g, t) \in \mathrm{Sp}(p, q) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+$ on $(a, y) \in \mathcal{M}$ is given by:

$$(A \cdot g, t) \circ (a, y) = (t^2 \cdot gag^{-1},\ t \cdot Ayg^{-1}). \tag{7.5}$$

Denote the origin by $O = [1, 0, \cdots, 0, -1] \in \Sigma_{\mathbb{H}}^{3+4p,4q} - \{\infty\}$. The stabilizer $\mathrm{Aut}(S^{3+4p,4q})_\infty$ is isomorphic to $\mathrm{Sim}(\mathcal{M})$ (cf. [18]). The orbit $\mathcal{M} \cdot O$ is a dense open subset of $S^{4p+3,4q}$. The embedding ι is defined by:

$$((a, b, c), (z_+, z_-)) \in \mathcal{M} \xrightarrow{\iota} \begin{bmatrix} \frac{||z_+||^2 - ||z_-||^2}{2} - 1 + \mathbf{i}a + \mathbf{j}b + \mathbf{k}c \\ \sqrt{2}z_+ \\ \sqrt{2}z_- \\ \frac{||z_+||^2 - ||z_-||^2}{2} + 1 + \mathbf{i}a + \mathbf{j}b + \mathbf{k}c \end{bmatrix} \tag{7.6}$$

Then the pair $(\mathrm{Sim}(\mathcal{M}), \mathcal{M})$ is said to be *p-c q Heisenberg geometry* which is a subgeometry of flat p-c q geometry $(\mathrm{Aut}(S^{3+4p,4q}), S^{3+4p,4q})$. We prove the rest of Theorem 7.1.

Proposition 7.2. $\mathrm{Aut}_{qCR}(S^{4p+3,4q}) = \mathrm{PSp}(p + 1, q + 1)$.

Proof. First note that $\mathrm{PSp}(p+1, q+1)$ decomposes into $\mathrm{Sim}(\mathcal{M}) \cdot (\mathrm{Sp}(p+1, q) \cdot \mathrm{Sp}(1))$. We know (cf. §3) that each element $f = (A, a) \in \mathrm{Sp}(p+1, q) \cdot \mathrm{Sp}(1)$ satisfies that $f^*\omega_0 = a\omega_0\bar{a}$, obviously $f \in \mathrm{Aut}_{qCR}(S^{4p+3,4q})$. On the other hand, it is shown that an element h of $\mathrm{Sim}(\mathcal{M})$ satisfy that $h^*\omega_0 = \lambda\omega_0\bar{\lambda}$ for some function $\lambda \in \mathbb{H}^*$ by using the explicit formula of ω_0. (See [16].) When $h \in \mathrm{Sim}(\mathcal{M})$, note that $h(\infty) = \infty$. Let $\tau : \mathrm{PSp}(p + 1, q + 1)_\infty \to \mathrm{Aut}(\mathrm{T}_{\{\infty\}}(S^{3+4p,4q}))$ be the tangential representation at $\{\infty\}$. Since the elements of the center $\mathbb{R}^3$ of $\mathcal{M}$ are tangentially identity maps at $\mathrm{T}_{\{\infty\}}(S^{3+4p,4q})$, $\tau(\mathrm{PSp}(p + 1, q + 1)_\infty) = \mathbb{H}^n \rtimes (\mathrm{Sp}(p, q)\cdot\mathrm{Sp}(1)\times\mathbb{R}^+)$ which is isomorphic to the structure group G (cf. (6.11)). As $\tau(h) = h_*$, $h \in \mathrm{Aut}_{qCR}(S^{3+4p,4q})$ by Definition 6.1. We have $\mathrm{PSp}(p + 1, q + 1) \subset \mathrm{Aut}_{qCR}(S^{3+4p,4q})$. □

8. Pseudo-conformal quaternionic CR invariant

We shall consider the equivalence problem of p-c qCR structure. Let $d\omega + \omega \wedge \omega = -(I_{ij}\boldsymbol{i} + J_{ij}\boldsymbol{j} + K_{ij}\boldsymbol{k})\theta^i \wedge \theta^j$ be the equation (4.3) as before. We examine how this equation behaves under the change of transformation $f \in \mathrm{Aut}_{qCR}(M)$; $f^*\omega = \lambda \cdot \omega \cdot \bar{\lambda}$. Put $\omega' = f^*\omega$. By (6.12),

$$
\begin{aligned}
d\omega' + \omega' \wedge \omega' &= f^*(d\omega + \omega \wedge \omega) = -(I_{ij}\boldsymbol{i} + J_{ij}\boldsymbol{j} + K_{ij}\boldsymbol{k}) f^*\theta^i \wedge f^*\theta^j \\
&= -(I_{ij}\boldsymbol{i} + J_{ij}\boldsymbol{j} + K_{ij}\boldsymbol{k})(u\theta^k U^i_k + \sum_a \omega_a v^i_a) \wedge (u\theta^\ell U^j_\ell + \sum_b \omega_b v^j_b) \\
&= -(I_{ij}\boldsymbol{i} + J_{ij}\boldsymbol{j} + K_{ij}\boldsymbol{k})\Big(u^2 U^i_k U^j_\ell \theta^k \wedge \theta^\ell + \\
&\quad \sum_a \omega_a \wedge (u v^i_a U^j_\ell \theta^\ell - u v^j_a U^i_\ell \theta^\ell) + \sum_{a<b} \omega_a \wedge \omega_b (v^i_a v^j_b - v^i_b v^j_a)\Big) \\
&= -(I_{ij}\boldsymbol{i} + J_{ij}\boldsymbol{j} + K_{ij}\boldsymbol{k})\Big(u^2 U^i_k U^j_\ell \theta^k \wedge \theta^\ell + \sum_a \omega_a \wedge 2u v^i_a U^j_\ell \theta^\ell \\
&\quad + \sum_{a<b} \omega_a \wedge \omega_b (2 v^i_a v^j_b)\Big).
\end{aligned}
$$

Choosing w^k_a $(a = 1, 2, 3)$ such that $U^i_k w^k_a = v^i_a$, the above equation becomes

$$
\begin{aligned}
d\omega' + \omega' \wedge \omega' = -(I_{ij}\boldsymbol{i} + J_{ij}\boldsymbol{j} + K_{ij}\boldsymbol{k})\Big(u^2 U^i_k U^j_\ell \theta^k \wedge \theta^\ell + \\
\sum_a \omega_a \wedge 2u w^k_a U^i_k U^j_\ell \theta^\ell + \sum_{a<b} \omega_a \wedge \omega_b (2 U^i_k U^j_\ell w^k_a w^\ell_b)\Big).
\end{aligned}
$$

Let $U = U'\bar{a} \circ \mathrm{Ad}_a \in \mathrm{SO}(4p, 4q)$ be the matrix as in (6.7) so that $Uz = U'z\bar{a}$ $(z \in \mathbb{H}^n)$ (cf. (6.6)). If $\{I, J, K\}$ is the set of the standard quaternionic structure, then

$$
\begin{aligned}
IU(z) &= I(U'z\bar{a}) = U'z\bar{a}\boldsymbol{i} = U'z(\bar{a}\boldsymbol{i}a)\bar{a} \\
&= U'z(a_{11}\boldsymbol{i} + a_{21}\boldsymbol{j} + a_{31}\boldsymbol{k})\bar{a} = a_{11}U'z\boldsymbol{i}\bar{a} + a_{21}U'z\boldsymbol{j}\bar{a} + a_{31}U'z\boldsymbol{k}\bar{a} \\
&= a_{11}U(z\boldsymbol{i}) + a_{21}U(z\boldsymbol{j}) + a_{31}U(z\boldsymbol{k}) = a_{11}UI(z) + a_{21}UJ(z) + a_{31}UK(z).
\end{aligned}
$$

This follows that $IU = a_{11}UI + a_{21}UJ + a_{31}UK$. Since $IU(e_i) = U^j_j I^\ell_j e_\ell$, a calculation shows that $U^j_i I^\ell_j = a_{11} I^j_i U^\ell_j + a_{21} J^j_i U^\ell_j + a_{31} K^j_i U^\ell_j$, similarly for J, K. As

$$
(8.1) \qquad \begin{pmatrix} I' \\ J' \\ K' \end{pmatrix} = {}^tA \begin{pmatrix} I \\ J \\ K \end{pmatrix}
$$

is a new quaternionic structure (cf. (1.5)), it follows that

$$
(8.2) \qquad
\begin{aligned}
I_{ij}U^i_k U^j_\ell &= a_{11}I_{k\ell} + a_{21}J_{k\ell} + a_{31}K_{k\ell} = {I'}_{k\ell}. \\
J_{ij}U^i_k U^j_\ell &= a_{12}I_{k\ell} + a_{22}J_{k\ell} + a_{32}K_{k\ell} = {J'}_{k\ell}. \\
K_{ij}U^i_k U^j_\ell &= a_{13}I_{k\ell} + a_{23}J_{k\ell} + a_{33}K_{k\ell} = {K'}_{k\ell}.
\end{aligned}
$$

Then we obtain that

$$d\omega' + \omega' \wedge \omega' = -(I'_{ij}\boldsymbol{i} + J'_{ij}\boldsymbol{j} + K'_{ij}\boldsymbol{k})\Big(u^2\theta^i \wedge \theta^j + \sum_a \omega_a \wedge 2uw^i_a\theta^j + \sum_{a<b}\omega_a \wedge \omega_b(2w^i_a w^j_b)\Big). \tag{8.3}$$

We shall derive an invariant under the change $\omega' = \lambda \cdot \omega \cdot \bar{\lambda}$. Recall from (6.12) that

$$(\omega'_1, \omega'_2, \omega'_3) = (\omega_1, \omega_2, \omega_3)u^2 \cdot A. \tag{8.4}$$

Let $d\theta^i = \theta^j \wedge \varphi^i_j + \sum_a \omega_a \wedge \tau^i_a$ be the structure equation (4.20). We define 1-forms ${\nu'}^i_a$ by setting

$$\begin{pmatrix} {\nu'}^i_1 \\ {\nu'}^i_2 \\ {\nu'}^i_3 \end{pmatrix} = u^{-2} \cdot {}^tA \begin{pmatrix} \tau^i_1 \\ \tau^i_2 \\ \tau^i_3 \end{pmatrix}. \tag{8.5}$$

Since $\tau^i_a \equiv 0 \mod \theta^k$ $(k = 1, \cdots 4n)$ by (4.23), note that

$${\nu'}^i_a \equiv 0 \mod \theta^k. \tag{8.6}$$

Using (8.4) and (8.5),

$$\sum_a \omega_a \wedge \tau^i_a = ({\omega'}_1, {\omega'}_2, {\omega'}_3) \wedge \begin{pmatrix} {\nu'}^i_1 \\ {\nu'}^i_2 \\ {\nu'}^i_3 \end{pmatrix} = \sum_a {\omega'}_a \wedge {\nu'}^i_a,$$

the equation (4.20) becomes

$$d\theta^i = \theta^j \wedge \varphi^i_j + \sum_a {\omega'}_a \wedge {\nu'}^i_a. \tag{8.7}$$

Differentiate (8.7), and then substitute (8.3), we obtain that

$$\theta^j \wedge (d\varphi^i_j - \varphi^\sigma_j \wedge \varphi^i_\sigma + u^2{I'}_{jk}\theta^k \wedge {\nu'}^i_1 + u^2{J'}_{jk}\theta^k \wedge {\nu'}^i_2 + u^2{K'}_{jk}\theta^k \wedge {\nu'}^i_3) \equiv 0 \mod \omega_\alpha.$$

Taking into account this equation (which corresponds to (5.5)), we have the fourth-order tensor up to the terms $\omega_1, \omega_2, \omega_3$:

$$\frac{1}{2}{T'}^i_{jk\ell}\theta^k \wedge \theta^\ell \equiv d\varphi^i_j - \varphi^\sigma_j \wedge \varphi^i_\sigma + \sum_a u^2 \cdot {\mathbf{J}'}^a_{jk}\theta^k \wedge {\nu'}^i_a - \theta^i \wedge \theta_j. \tag{8.8}$$

Here we put $({I'}_{ij}, {J'}_{ij}, {K'}_{ij}) = ({\mathbf{J}'}^1_{ij}, {\mathbf{J}'}^2_{ij}, {\mathbf{J}'}^3_{ij})$. Since $({I'}_{ij}, {J'}_{ij}, {K'}_{ij}) = (I_{ij}, J_{ij}, K_{ij})A$ from (8.1) and (8.5),

$$\sum_a u^2 \cdot {\mathbf{J}'}^a_{jk}\theta^k \wedge {\nu'}^i_a = \theta^k \wedge (I_{jk}, J_{jk}, K_{jk}) \begin{pmatrix} \tau^i_1 \\ \tau^i_2 \\ \tau^i_3 \end{pmatrix} = \theta^k \wedge \sum_a \mathbf{J}^a_{jk}\tau^i_a.$$

The equation (8.8) can be reduced to the following:

$${T'}^i_{jk\ell}\theta^k \wedge \theta^\ell \equiv d\varphi^i_j - \varphi^\sigma_j \wedge \varphi^i_\sigma + \theta^k \wedge \sum_a \mathbf{J}^a_{jk}\tau^i_a - \theta^i \wedge \theta_j. \tag{8.9}$$

From (5.9) and (5.6), we have shown

Proposition 8.1. *If $\omega' = \lambda \cdot \omega \cdot \bar{\lambda}$ for which ω is a qCR structure, then the curvature tensor T' satisfies that ${T'}^i_{jk\ell} = T^i_{jk\ell}$. In particular, $T = (T^i_{jk\ell})$ is an invariant tensor under the p-c qCR structure.*

Remark 8.2. 1. Similarly, the quaternionic structures $\{I', J', K'\}$ extends to almost complex structures $\{\bar{I}', \bar{J}', \bar{K}'\}$ respectively.

2. Let $f \in \mathrm{Aut}_{qCR}(M)$ be an element satisfying (6.12). Then, $f_* e_i = uU^k_i e_k$. Using (8.2),

$$\begin{aligned} If_*e_i &= uU^k_i I^j_k e_j = u(a_{11}I^m_i + a_{21}J^m_i + a_{31}K^m_i)U^j_m e_j \\ &= f_*((a_{11}I^m_i + a_{21}J^m_i + a_{31}K^m_i)e_m) \\ &= f_*((a_{11}I + a_{21}J + a_{31}K)e_i). \end{aligned}$$

The similar argument to J, K yields that

$$\begin{pmatrix} f_*^{-1}If_* \\ f_*^{-1}Jf_* \\ f_*^{-1}Kf_* \end{pmatrix} = {}^tA \begin{pmatrix} I \\ J \\ K \end{pmatrix} \quad \text{on } \mathcal{D}. \tag{8.10}$$

8.1. **Formula of curvature tensor.** We shall find the formula of T. Substitute (4.24), (4.23) into (8.9):

$$\begin{aligned} T^i_{jk\ell}\theta^k \wedge \theta^\ell &= d(\omega^i_j + \sum_a (\mathbf{J}^a)^i_j \omega_a) - (\omega^\sigma_j + \sum_a (\mathbf{J}^a)^\sigma_j \omega_a) \wedge (\omega^i_\sigma + \sum_a (\mathbf{J}^a)^i_\sigma \omega_a) \\ &\quad + \theta^k \wedge (I_{jk} \cdot I^i_\ell \theta^\ell + J_{jk} \cdot J^i_\ell \theta^\ell + K_{jk} \cdot K^i_\ell \theta^\ell) - \theta^i \wedge \theta_j \mod \omega_a \\ &= d\omega^i_j + \sum_a (\mathbf{J}^a)^i_j d\omega_a - \omega^\sigma_j \wedge \omega^i_\sigma + \sum_a (\mathbf{J}^a_{jk}(\mathbf{J}^a)^i_\ell)\theta^k \wedge \theta^\ell - \theta^i \wedge \theta_j \mod \omega_a \\ &= (d\omega^i_j - \omega^\sigma_j \wedge \omega^i_\sigma) \\ &\quad + \sum_a (\mathbf{J}^a)^i_j(-\mathbf{J}^a_{k\ell})\theta^k \wedge \theta^\ell + \sum_a (\mathbf{J}^a_{jk}(\mathbf{J}^a)^i_\ell)\theta^k \wedge \theta^\ell - \theta^i \wedge \theta_j \mod \omega_a \\ &= \left(\frac{1}{2}R^i_{jk\ell} - \sum_a (\mathbf{J}^a)^i_j \mathbf{J}^a_{k\ell} + \sum_a \mathbf{J}^a_{jk}(\mathbf{J}^a)^i_\ell - g_{j\ell} \cdot \delta^i_k\right)\theta^k \wedge \theta^\ell \mod \omega_a. \end{aligned}$$

By alternation, we have

$$T^i_{jk\ell} = R^i_{jk\ell} - \left(2\sum_a (\mathbf{J}^a)^i_j \mathbf{J}^a_{k\ell} - \sum_a \mathbf{J}^a_{jk}(\mathbf{J}^a)^i_\ell + \sum_a \mathbf{J}^a_{j\ell}(\mathbf{J}^a)^i_k + (g_{j\ell}\delta^i_k - g_{jk}\delta^i_\ell)\right). \tag{8.11}$$

Recall the space of all curvature tensors $\mathcal{R}(\mathrm{Sp}(p,q)\cdot \mathrm{Sp}(1))$. (See [1] for example.) It decomposes into the direct sum $\mathcal{R}_0(\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)) \oplus \mathcal{R}_{\mathbb{HP}}(\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1))$ $(n \geq 2)$. Here $\mathcal{R}_0$ is the space of those curvatures with zero Ricci forms and $\mathcal{R}_{\mathbb{HP}} \approx \mathbb{R}$ is the space of curvature tensors of the quaternionic pseudo-Kähler projective space $\mathbb{HP}^{p,q}$ (cf. Definition 3.2).

Case n ≥ 2. Since we know that $R^i_{ji\ell} = R_{j\ell} = (4n+8)g_{j\ell}$ from (5.3), the curvature tensor $T = (T^i_{jk\ell})$ satisfies the *tracefree* condition:

$$T_{j\ell} = (T^i_{ji\ell}) = (4n+8)g_{j\ell} - \Big(3\cdot 3g_{j\ell} + (4n-1)g_{j\ell}\Big) = 0.$$

This implies that our curvature tensor T belongs to $\mathcal{R}_0(\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1))$ when $n \geq 2$.

Case n = 1. When dim $M = 7$, either $p = 1, q = 0$ or $p = 0, q = 1$. Choose the orthonormal basis $\{e_i\}_{i=1,2,3,4}$ with $e_1 = e, e_2 = Ie, e_3 = Je, e_4 = Ke$. Form another curvature tensor:

$$\begin{aligned} R'^i_{jk\ell} &= (g_{j\ell}\delta^i_k - g_{jk}\delta^i_\ell) + \Big[I_{j\ell}I^i_k - I_{jk}I^i_\ell + 2I^i_jI_{k\ell} \\ &+J_{j\ell}J^i_k - J_{jk}J^i_\ell + 2J^i_jJ_{k\ell} + K_{j\ell}K^i_k - K_{jk}K^i_\ell + 2K^i_jK_{k\ell}\Big]. \end{aligned} \tag{8.12}$$

For any two distinct e_i, e_j,

$$\begin{aligned} R'^i_{jij} &= (g_{jj}\delta^i_i - g_{ji}\delta^i_j) + \Big[I_{ji}I^i_i - I_{ji}I^i_j + 2I_{ij}I^i_j + J_{ji}J^i_i - J_{ji}J^i_j + 2J^i_jJ_{ij} \\ &+K_{ji}K^i_i - K_{ji}K^i_j + 2K^i_jK_{ij}\Big] = g_{jj} + 3\Big[I_{ij}I^i_j + J_{ij}J^i_j + K_{ij}K^i_j\Big]. \end{aligned}$$

Since $i \neq j$ and e_j is either one of $\pm Ie_i, \pm Je_i, \pm Ke_i$, $I^2_{ij} + J^2_{ij} + K^2_{ij} = 1$ (for example, if $e_j = Ie_i$, then $I^{i2}_j = 1$, $J^i_j = 0, K^i_j = 0$ so that $I_{ij}I^i_j = g_{jj}$.) Thus, $R'^i_{jij} = 4g_{jj}$. It follows from the Schur's theorem (cf. [21] for example) that

$$R'^i_{jk\ell} = 4(g_{j\ell}\delta^i_k - g_{jk}\delta^i_\ell). \tag{8.13}$$

When $n = 1$, we conclude that

$$T^i_{jk\ell} = R^i_{jk\ell} - R'^i_{jk\ell} = R^i_{jk\ell} - 4(g_{j\ell}\delta^i_k - g_{jk}\delta^i_\ell). \tag{8.14}$$

As the curvature $R^i_{jk\ell}$ satisfies the Einstein property from (5.3); $R_{j\ell} = 4\cdot 3g_{j\ell}$, the scalar curvature $\sigma = 4\cdot 12$. On the other hand, the curvature tensor $R^i_{jk\ell}$ has the decomposition:

$$R^i_{jk\ell} = W^i_{jk\ell} + \frac{4\cdot 12}{4\cdot 3}(g_{j\ell}\delta^i_k - g_{jk}\delta^i_\ell)$$

in the space $\mathcal{R}(\mathrm{SO}(4))$ where $\mathrm{SO}(4) = \mathrm{Sp}(1)\cdot\mathrm{Sp}(1)$. Hence,

$$T^i_{jk\ell} = W^i_{jk\ell} \in \mathcal{R}_0(\mathrm{SO}(4)) \tag{8.15}$$

for which $W^i_{jk\ell}$ corresponds to the Weyl curvature tensor (of $(U/\mathcal{E}, \hat{g})$).

Case n = 0. If dim $M = 3$, then the above tensor is empty, so we simply set $T = 0$. Define the Riemannian metric on a neighborhood U of a 3-dimensional p-c qCR manifold M:

$$g_x(X,Y) = \omega_1(X)\cdot\omega_1(Y) + \omega_2(X)\cdot\omega_2(Y) + \omega_3(X)\cdot\omega_3(Y) \tag{8.16}$$

($\forall\ X, Y \in T_xU$). Suppose that $\omega' = \lambda\cdot\omega\cdot\bar{\lambda}$. Since $(\omega'_1, \omega'_2, \omega'_3) = u^2\cdot(\omega_1,\omega_2,\omega_3)A$ for $A \in \mathrm{SO}(3)$, the metric g changes into $g' = \omega'_1\cdot\omega'_1 + \omega'_2\cdot\omega'_2 + \omega'_3\cdot\omega'_3$ satisfying that

$$g'_x(X,Y) = u^4\cdot g_x(X,Y) \quad (\forall\ X, Y \in T_xU). \tag{8.17}$$

Then g' is conformal to g on U. Define $TW(\omega)$ to be the Weyl-Schouten tensor $TW(g)$ of the Riemannian metric g on U. Then, it turns out that

$$TW(\omega') = TW(\omega). \tag{8.18}$$

As a consequence, $TW(\omega)$ is an invariant tensor of U under the change $\omega' = \lambda \cdot \omega \cdot \bar{\lambda}$.

9. Uniformization of p-c qCR structure

If $\{\omega^{(\alpha)}, (I^{(\alpha)}, J^{(\alpha)}, K^{(\alpha)}), g_{(\alpha)}, U_\alpha\}_{\alpha\in\Lambda}$ is a p-c qCR structure on M where $\underset{\alpha\in\Lambda}{\cup} U_\alpha = M$, then we have the curvature tensor $T^{(\alpha)} = ({}^{(\alpha)}T^{i}_{jk\ell})$ on each $(U_\alpha, \omega^{(\alpha)})$ $(n \geq 1)$. Similarly, $TW^{(\alpha)} = TW(\omega^{(\alpha)})$ on $(U_\alpha, \omega^{(\alpha)})$ for 3-dimensional case $(n = 0)$. Then it follows from Proposition 8.1 and (8.18) that if $\omega^{(\beta)} = \lambda_{\alpha\beta} \cdot \omega^{(\alpha)} \cdot \bar{\lambda}_{\alpha\beta}$ on $U_\alpha \cap U_\beta$, then $T^{(\alpha)} = T^{(\beta)}$, $TW^{(\alpha)} = TW^{(\beta)}$. By setting $T|U_\alpha = T^{(\alpha)}$ (respectively $TW|U_\alpha = TW^{(\alpha)}$), the curvature T (respectively TW) is globally defined on a $(4n+3)$-dimensional p-c qCR manifold M $(n \geq 0)$. This concludes that

Theorem 9.1. *Let M be a p-c qCR manifold of dimension $4n+3$ $(n \geq 0)$. If $n \geq 1$, there exists the fourth-order curvature tensor $T = (T^i_{jk\ell})$ on M satisfying that:*

(i) *When $n \geq 2$, $T = (T^i_{jk\ell}) \in \mathcal{R}_0(\mathrm{Sp}(p,q)\cdot \mathrm{Sp}(1))$ which has the formula:*

$$T^i_{jk\ell} = R^i_{jk\ell} - \Big\{(g_{j\ell}\delta^i_k - g_{jk}\delta^i_\ell) + \Big[I_{j\ell}I^i_k - I_{jk}I^i_\ell + 2I^i_jI_{k\ell} \\ +J_{j\ell}J^i_k - J_{jk}J^i_\ell + 2J^i_jJ_{k\ell} + K_{j\ell}K^i_k - K_{jk}K^i_\ell + 2K^i_jK_{k\ell}\Big]\Big\}.$$

(ii) *When $n = 1$, $T = (W^i_{jk\ell}) \in \mathcal{R}_0(\mathrm{SO}(4))$ which has the same formula as the Weyl conformal curvature tensor.*

(iii) *If $n = 0$, there exists the fourth-order curvature tensor TW on M which has the same formula as the Weyl-Schouten curvature tensor.*

We have associated to a p-c qCR structure $(\{\omega_a\}, \{J_a\}, \{\xi_a\})_{a=1,2,3}$ the pseudo-Sasakian metric $g = \sum_{a=1}^{3} \omega_a \cdot \omega_a + \pi^*\hat{g}$ on U for which $\mathcal{E} \to (U, g) \xrightarrow{\pi} (U/\mathcal{E}, \hat{g})$ is a pseudo-Riemannian submersion and the quotient $(U/\mathcal{E}, \hat{g}, \{\hat{I}_i, \hat{J}_i, \hat{K}_i\}_{i\in\Lambda})$ is a quaternionic pseudo-Kähler manifold by Theorem 4.6. Let ${}^{(g)}R^i_{jk\ell}$ (respectively $\hat{R}^i_{jk\ell}$) denote the curvature tensor of g (respectively $\hat{g}$). If $R_{\mathbb{HP}}$ is the generator of $\mathcal{R}_{\mathbb{HP}}(\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)) \approx \mathbb{R}$ $(n \geq 2)$, then it can be described as (cf. [1]):

$$R_{\mathbb{HP}} = (g_{j\ell}g_{ik} - g_{jk}g_{i\ell}) + \sum_{a=1}^{3} \mathbf{J}^a_{j\ell}\mathbf{J}^a_{ik} - \sum_{a=1}^{3} \mathbf{J}^a_{jk}\mathbf{J}^a_{i\ell} + 2\sum_{a=1}^{3} \mathbf{J}^a_{ij}\mathbf{J}^a_{k\ell} \tag{9.1}$$

where i, j, k, ℓ run over $\{1, \cdots, 4n\}$. Then the formula (12.8) of curvature tensor of g [33] $(n \geq 1)$ shows the following.

Lemma 9.2.

$$\begin{aligned} \pi^*\hat{R}_{ijk\ell} &= {}^{(g)}R_{ijk\ell} + \left(\sum_{a=1}^{3} \mathbf{J}^a_{j\ell}\mathbf{J}^a_{ik} - \sum_{a=1}^{3} \mathbf{J}^a_{jk}\mathbf{J}^a_{i\ell} + 2\sum_{a=1}^{3} \mathbf{J}^a_{ij}\mathbf{J}^a_{k\ell}\right) \\ &= {}^{(g)}R_{ijk\ell} - (g_{j\ell}\delta_{ik} - g_{jk}\delta_{i\ell}) + R_{\mathbb{HP}}. \end{aligned} \tag{9.2}$$

We now state the uniformization theorem.

Theorem 9.3. (1) *Let M be a $(4n+3)$-dimensional p-c qCR manifold $(n \geq 1)$. If the curvature tensor T vanishes, then M is locally modelled on $S^{4p+3,4q}$ with respect to the group* $\mathrm{PSp}(p+1, q+1)$.

(2) *If M is a 3-dimensional p-c qCR manifold whose curvature tensor TW vanishes, then M is conformally flat (locally modelled on S^3 with respect to the group* $\mathrm{PSp}(1,1)$ *).*

Proof. Using (5.2) and (9.1), the formula of Theorem 9.1 becomes

$$T^i_{jk\ell} = \pi^* \hat{R}^i_{jk\ell} - R_{\mathbb{HP}}. \tag{9.3}$$

Compared this with (9.2), we obtain that

$$T^i_{jk\ell} = {}^{(g)}R^i_{jk\ell} - (g_{j\ell}\delta^i_k - g_{jk}\delta^i_\ell). \tag{9.4}$$

The equality (9.4) is also true for $n=1$. In fact, when $n=1$, $R_{\mathbb{HP}} = 4(g_{j\ell}\delta^i_k - g_{jk}\delta^i_\ell)$ (cf. (8.12), (8.13)) and from (9.2), ${}^{(g)}R^i_{jk\ell} - (g_{j\ell}\delta^i_k - g_{jk}\delta^i_\ell) = \pi^*\hat{R}^i_{jk\ell} - R_{\mathbb{HP}} = T^i_{jk\ell}$ by (8.14).

Suppose that T (respectively TW) vanishes identically on M. First we show that M is locally isomorphic to $S^{4p+3,4q}$ (respectively M is locally isomorphic to S^3.) As $T|U_\alpha = ({}^{(\alpha)}T^i_{jk\ell}) = 0$ on U_α, for brevity, we omit α so that $T = (T^i_{jk\ell})$ vanishes identically on U for $n \geq 2$. As a consequence,

$${}^{(g)}R^i_{jk\ell} = g_{j\ell}\delta^i_k - g_{jk}\delta^i_\ell \ \text{ on } \mathcal{D}|U. \tag{9.5}$$

Since (U, g) is a pseudo-Sasakian 3-structure with Killing fields $\{\xi_1, \xi_2, \xi_3\}$, the normality of (4.18) can be stated as ${}^{(g)}R(X, \xi_a)Y = g(X,Y)\xi_a - g(\xi_a, Y)X$ (cf. [33]). It turns out that

$${}^{(g)}R(\xi_a, X, Y, Z) = g(X,Z)g(\xi_a, Y) - g(X,Y)g(\xi_a, Z) \tag{9.6}$$

$(\forall X, Y, Z \in TU)$. Then (9.5) and (9.6) imply that (U, g) is the space of positive constant curvature. As $\hat{R}^i_{jk\ell} = R_{\mathbb{HP}}$ by (9.3), the quotient space $(U/\mathcal{E}, \hat{g})$ is locally isometric to the quaternionic pseudo-Kähler projective space $(\mathbb{HP}^{p,q}, \hat{g}_0)$. (Note that if $T^i_{jk\ell} = 0$ for $n=1$, then $\pi^*\hat{R}^i_{jk\ell} = R^i_{jk\ell} = 4(\delta_{j\ell}\delta^i_k - \delta_{jk}\delta^i_\ell)$ from (8.14). When $p=1, q=0$, the base space $(U/\mathcal{E}, \hat{g})$ is locally isometric to the standard sphere S^4 which is identified with the 1-dimensional quaternionic projective space $\mathbb{HP}^1$. If $p=0, q=1$, then $(U/\mathcal{E}, \hat{g})$ is locally isometric to the quaternionic hyperbolic space $\mathbb{H}^1_{\mathbb{H}} = \mathbb{HP}^{0,1}$ in which we remark that the metric $\hat{g}$ is negative definite.) Hence, the bundle: $\mathcal{E} \to (U, g) \xrightarrow{\pi} (U/\mathcal{E}, \hat{g})$ is locally isometric to the Hopf bundle as the Riemannian submersion $(n \geq 1)$ (cf. Theorem 3.4):

$$\mathrm{Sp}(1) \to (\Sigma^{4p+3,4q}_{\mathbb{H}}, g_0) \longrightarrow (\mathbb{HP}^{p,q}, \hat{g}_0).$$

This is obviously true for $n=0$.

Let $\varphi : (U, g) \to (\Sigma^{4p+3,4q}_{\mathbb{H}}, g_0)$ be an isometric immersion preserving the above principal bundle. If $V_0 = \{\xi^0_1, \xi^0_2, \xi^0_3\}$ is the distribution of Killing vector fields which generates $\mathrm{Sp}(1)$ of the above Hopf bundle, then we can assume that $\varphi_*\xi_a = \xi^0_a$ $(a = 1,2,3)$ (by a composite of some element of $\mathrm{Sp}(1)$ if necessary). As $\omega_a(X) = g(\xi_a, X)$ $(X \in TU)$ and $\omega^0_a(X) = g_0(\xi_a, X)$ $(X \in T\Sigma^{4p+3,4q}_{\mathbb{H}})$ respectively, the equality $g = \varphi^* g_0$ implies that

$$\omega_a = \varphi^*\omega^0_a \ \ (a = 1,2,3), \ \ \omega = \varphi^*\omega_0. \tag{9.7}$$

If we represent $\varphi^*\theta^i = \theta^k T^i_k + \sum_a \omega_a v^i_a$ for some matrix T^i_j and $v^i_a \in \mathbb{R}$, then the equality $\varphi_*\xi_a = \xi^0_a$ shows that $v^i_a = 0$ for $i = 1, \cdots, 4n$. Thus,

$$\varphi^*\theta^i = \theta^k T^i_k. \tag{9.8}$$

For each $\alpha \in \Lambda$, we have an immersion $\varphi_\alpha : U_\alpha \to \Sigma^{4p+3,4q}_{\mathbb{H}}$ as above so that there is a collection of charts $\{U_\alpha, \varphi_\alpha\}_{\alpha\in\Lambda}$ on M. Put $g_{\alpha\beta} = \varphi_\beta \circ {\varphi_\alpha}^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ when $U_\alpha \cap U_\beta \neq \emptyset$. It suffices to prove that $g_{\alpha\beta}$ extends uniquely to an element of $\mathrm{PSp}(p+1, q+1) = \mathrm{Aut}_{qCR}(S^{4p+3,4q})$. Suppose that

$$\omega^{(\beta)} = \lambda \cdot \omega^{(\alpha)} \cdot \overline{\lambda} = u^2 \cdot a \cdot \omega^{(\alpha)} \cdot \bar{a} \quad \text{on } U_\alpha \cap U_\beta \neq \emptyset \tag{9.9}$$

where $\lambda = u \cdot a$. The immersions $\varphi_\alpha : U_\alpha \to \Sigma^{4p+3,4q}_{\mathbb{H}}$, $\varphi_\beta : U_\beta \to \Sigma^{4p+3,4q}_{\mathbb{H}}$ satisfy $\omega^{(\alpha)} = \varphi^*_\alpha\omega_0$, $\omega^{(\beta)} = \varphi^*_\beta\omega_0$ as in (9.7). If we put $\mu = \lambda \circ {\varphi_\alpha}^{-1}$ on $\varphi_\alpha(U_\alpha \cap U_\beta)$, then the above relation shows that

$$g^*_{\alpha\beta}\omega_0 = \mu \cdot \omega_0 \cdot \bar{\mu}. \tag{9.10}$$

Using the fact that $d\omega^{(\alpha)}_a(J^{(\alpha)}_a X, Y) = g^{(\alpha)}(X, Y)$ ($\forall\ X, Y \in \mathcal{D}, a = 1, 2, 3$) from (1.1) and $g^{(\alpha)} = \varphi^*_{\alpha*}g_0$, calculate that

$$\omega^0_a(\varphi_{\alpha*}J^{(\alpha)}_a X, \varphi_{\alpha*}Y) = d\omega_a(J^{(\alpha)}_a X, Y) = g_0(\varphi_{\alpha*}X, \varphi_{\alpha*}Y) = d\omega^0_a(J^0_a\varphi_{\alpha*}X, \varphi_{\alpha*}Y).$$

As $d\omega^0_a$ is nondegenerate on $\mathcal{D}$, for each $\alpha \in \Lambda$ we have

$$\varphi_{\alpha*} \circ J^{(\alpha)}_a = J^0_a \circ \varphi_{\alpha*} \quad \text{on } \mathcal{D} \ \ (a = 1, 2, 3). \tag{9.11}$$

Let $\varphi^*_\alpha\theta^i = \theta^k_{(\alpha)} \cdot {}^{(\alpha)}T^i_k$ for some matrix ${}^{(\alpha)}T^i_k$ as in (9.8). Then (9.11) means that ${}^{(\alpha)}T^k_i \cdot (J^a)^j_k = (J^a)^k_i \cdot {}^{(\alpha)}T^j_k$, which implies that ${}^{(\alpha)}T^i_k \in \mathrm{GL}(n, \mathbb{H})$. Noting that $g^{(\alpha)}(X, Y) = g_0(\varphi_{\alpha*}X, \varphi_{\alpha*}Y)$, this reduces to

$${}^{(\alpha)}T^i_k \in \mathrm{Sp}(p, q). \tag{9.12}$$

Let $\{\omega_{(\alpha)}, \omega^i_{(\alpha)}\}_{i=1,\cdots,n}$, $\{\omega_{(\beta)}, \omega^i_{(\beta)}\}_{i=1,\cdots,n}$ be two coframes on the intersection $U_\alpha \cap U_\beta$ where $\omega_{(\alpha)}$ is a $\mathrm{Im}\mathbb{H}$-valued 1-form and each $\omega^i_{(\alpha)}$ is a $\mathbb{H}$-valued 1-form, simlarly for β. Noting (6.3) and (9.9), the coordinate change of the fiber $\mathbb{H}^n$ satisfies that

$$\begin{pmatrix} \omega_{(\beta)} \\ \omega^1_{(\beta)} \\ \vdots \\ \omega^n_{(\beta)} \end{pmatrix} = \left(\begin{array}{c|c} \lambda & 0 \\ \hline \tilde{v}^i & U' \end{array}\right) \begin{pmatrix} \omega_{(\alpha)} \\ \omega^1_{(\alpha)} \\ \vdots \\ \omega^n_{(\alpha)} \end{pmatrix} \cdot \bar{\lambda}. \tag{9.13}$$

In order to transform them into the real forms, recall that $\mathrm{GL}(n, \mathbb{H}) \cdot \mathrm{GL}(1, \mathbb{H})$ is the maximal closed subgroup of $\mathrm{GL}(4n, \mathbb{R})$ acting on $\mathbb{R}^{4n}$ preserving the standard quaternionic structure $\{I, J, K\}$. For each fiber of $\mathcal{D}_\alpha$ $(= \mathcal{D}_\beta)$ on the intersection, there exists a matrix $\tilde{U} = (\tilde{U}^i_j) = U' \cdot \lambda \in \mathrm{GL}(n, \mathbb{H}) \cdot \mathrm{GL}(1, \mathbb{H})$ such that:

$$e^{(\alpha)}_j = \tilde{U}^i_j e^{(\beta)}_i \tag{9.14}$$

with respect to the basis $\{e_i^{(\alpha)}\}_x \in (\mathcal{D}_\alpha)_x$, $\{e_i^{(\beta)}\}_x \in (\mathcal{D}_\beta)_x$. From Corollary 1.4,

$$\pm u^2\delta_{k\ell} = u^2 g_{(\alpha)}(e_k^{(\alpha)}, e_\ell^{(\alpha)}) = g_{(\beta)}(\tilde{U}_k^i e_i^{(\beta)}, \tilde{U}_\ell^j e_j^{(\beta)}) = \pm\delta_{ij}\tilde{U}_k^i\tilde{U}_\ell^j,$$

so $(u^{-1}\tilde{U}_k^i) \in \mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1) = \mathrm{GL}(n,\mathbb{H})\cdot\mathrm{GL}(1,\mathbb{H})\cap\mathrm{SO}(4p,4q)$ up to conjugacy $(n \geq 1)$. Put $U = (U_k^i) = (u^{-1}\tilde{U}_k^i) \in \mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$, then

$$\tilde{U} = uU = (uU_k^i) \in \mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)\times\mathbb{R}^+. \tag{9.15}$$

Using coframes $\{\theta^i_{(\alpha)}\}, \{\theta^i_{(\beta)}\}$ (induced from $\{\omega^i_{(\alpha)}, \omega^i_{(\beta)}\}_{i=1,\cdots,n}$), the equation (9.14) translates into $\theta^i_{(\beta)} = \theta^k_{(\alpha)}\tilde{U}_k^i$ on $\mathcal{D}$. Using (9.13), it follows that

$$\theta^i_{(\beta)} = \theta^k_{(\alpha)}\tilde{U}_k^i + \sum_{a=1}^3 \omega_a^{(\alpha)}\cdot v_a^i \ \text{ on } U_\alpha\cap U_\beta.$$

Here v_a^i are determined by $\tilde{v}^i$, see (6.12). Then,

$$\begin{aligned} g_{\alpha\beta}{}^*(\theta^i) &= (\varphi_\alpha{}^{-1})^*\varphi_\beta{}^*(\theta^i) = (\varphi_\alpha{}^{-1})^*(\theta^j_{(\beta)}\cdot{}^{(\beta)}T_j^i) \\ &= (\varphi_\alpha{}^{-1})^*\left((\theta^k_{(\alpha)}\tilde{U}_k^j + \sum_{a=1}^3\omega_a^{(\alpha)}\cdot v_a^j)\cdot{}^{(\beta)}T_j^i\right) \\ &= \theta^\ell\cdot({}^{(\alpha)}T^{-1})_\ell^k\tilde{U}_k^j\cdot{}^{(\beta)}T_j^i + \sum_{a=1}^3\omega_a^0\cdot(v_a^j\cdot{}^{(\beta)}T_j^i). \end{aligned} \tag{9.16}$$

If we put $S = (S_\ell^i) = (({}^{(\alpha)}T^{-1})_\ell^k\cdot\tilde{U}_k^j\cdot{}^{(\beta)}T_j^i)$, then (9.15) and (9.12) imply $S \in \mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)\times\mathbb{R}^+$. By (9.10), (9.16), $g_{\alpha\beta}$ satisfies the conditions of (6.12). Therefore the diffeomorphism $g_{\alpha\beta} : \varphi_\alpha(U_\alpha\cap U_\beta)\rightarrow\varphi_\beta(U_\alpha\cap U_\beta)$ is viewed locally as an element of $\mathrm{Aut}_{qCR}(S^{4p+3,4q}) = \mathrm{PSp}(p+1,q+1)$ because $\Sigma_{\mathbb{H}}^{4p+3,4q} \subset S^{4p+3,4q}$. As $\mathrm{PSp}(p+1,q+1)$ acts real analytically on $S^{4p+3,4q}$, $g_{\alpha\beta}$ extends uniquely to an element of $\mathrm{PSp}(p+1,q+1)$. Therefore, the collection of charts $\{U_\alpha,\varphi_\alpha\}_{\alpha\in\Lambda}$ gives rise to a uniformization of a p-c qCR manifold M with respect to $(\mathrm{PSp}(p+1,q+1), S^{4p+3,4q})$.

Recall that the orthogonal Lorentz group $\mathrm{PO}(4,1)^0$ is isomorphic to $\mathrm{PSp}(1,1)$ as a Lie group. The same is true for the 3-dimensional conformal geometry $(\mathrm{PSp}(1,1), S^3) = (\mathrm{PO}(4,1)^0, S^3)$ $(n = 0)$. □

10. Quaternionic bundle

It is known that the first Stiefel-Whitney class is the obstruction to the existence of a global 1-form of the contact structure (cf. [13], [32]) and the first Chern class is the obstruction to the existence of a global 1-form of the complex contact structure (cf. [22],[7],[37],[25]) respectively. It is natural to ask whether the first Pontrjagin class $p_1(M)$ is the obstruction to the existence of global 1-form of p-c q structure (respectively p-c qCR structure) on a $(4n+3)$- manifold M $(n \geq 1)$. In order to consider this, we need the elementary properties of the quaternionic bundle theory whose structure group is $\mathrm{GL}(n,\mathbb{H})\cdot\mathrm{GL}(1,\mathbb{H})$ but not $\mathrm{GL}(n,\mathbb{H})$. To our knowledge, the fundamental properties of the quaternionic bundle with $\mathrm{GL}(n,\mathbb{H})\cdot\mathrm{GL}(1,\mathbb{H})$ as the structure group are not provided explicitly. So we prepare the necessary facts here. Let $\mathcal{D}$ be the $4n$-dimensional bundle defined by $\mathcal{D} = \bigcup_{\alpha} \mathcal{D}_\alpha$ where $\mathcal{D}_\alpha = \mathcal{D}|U_\alpha = \text{Null}\ \omega^{(\alpha)}$ in which there is the relation on the intersection $U_\alpha \cap U_\beta$:

$$\omega^{(\beta)} = \bar{\lambda} \cdot \omega^{(\alpha)} \cdot \lambda = u^2 \cdot \bar{a}\omega^{(\beta)} \cdot a \quad \text{where } \lambda = u \cdot a \in \mathbb{H}^*. \tag{10.1}$$

We have already discussed the transition functions on $\mathcal{D}$ in (9.13). In fact, the gluing condition of $\mathcal{D}$ in $U_\alpha \cap U_\beta$ is given by

$$\begin{pmatrix} v_1^{(\alpha)} \\ \vdots \\ v_n^{(\alpha)} \end{pmatrix} = uT \begin{pmatrix} v_1^{(\beta)} \\ \vdots \\ v_n^{(\beta)} \end{pmatrix} \cdot a, \tag{10.2}$$

in which $u(T \cdot \bar{a}) \in \mathrm{Sp}(p,q) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+$ $(p+q=n)$.

Definition 10.1. *A quaternionic n-dimensional bundle is a vector bundle over a paracompact Hausdorff space M with fiber isomorphic to the n-dimensional quaternionic vector space $\mathbb{H}^n$. For an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of M, if $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n,\mathbb{H}) \cdot \mathrm{GL}(1,\mathbb{H})$.*

As a consequence, $\mathcal{D}$ is a quaternionic n-dimensional bundle on M. Note that as $\mathrm{GL}(1,\mathbb{H}) \cdot \mathrm{GL}(1,\mathbb{H}) \approx \mathrm{SO}(4) \times \mathbb{R}^+$, the quaternionic line bundle is isomorphic to an oriented real 4-dimensional bundle. Define the inner product $\langle\ ,\rangle$ of type (p,q) on $\mathbb{H}^n$ $(p+q=n)$ by

$$\langle z, w \rangle = \bar{z}_1 w_1 + \cdots + \bar{z}_p w_p - \bar{z}_{p+1} w_{p+1} - \cdots - \bar{z}_n w_n.$$

Then $\langle\ ,\rangle$ satisfies that $\langle z, w \cdot \lambda \rangle = \langle z, w \rangle \cdot \lambda$, $\langle z \cdot \lambda, w \rangle = \bar{\lambda}\langle z, w \rangle$, $\langle z, w \rangle = \overline{\langle w, z \rangle}$ for $\lambda \in \mathbb{H}$, and so on. By a subspace W in $\mathbb{H}^n$ we mean a right $\mathbb{H}$-module. Choosing $v_0 \in \mathbb{H}^n$ with $\langle v_0, v_0 \rangle > 0$, let $V = \{v_0 \cdot \lambda \mid \lambda \in \mathbb{H}\}$ be a 1-dimensional subspace of $\mathbb{H}^n$. Denote $V^\perp = \{v \in \mathbb{H}^n | \langle v, x \rangle = 0, \forall\, x \in V\}$. Then it is easy to check that $V^\perp$ is a right $\mathbb{H}$-module for which there is a decomposition: $\mathbb{H}^n = V \oplus V^\perp$ as a right $\mathbb{H}$-module. The following is a quaternionic analogue of the splitting theorem.

Proposition 10.2. *Given a quaternionic n-dimensional bundle ξ with an (indefinite) inner product $\langle\,\rangle$ on each fiber, there exists a quaternionic line bundle ξ_i $(i = 1, \cdots, n)$ over a paracompact Hausdorff space N and a (splitting) map $f : N \to M$ for which:*

(1) $f^*\xi = \xi_1 \oplus \cdots \oplus \xi_n$.

(2) $f^* : H^*(M) \to H^*(N)$ *is injective. Moreover,*

(3) *The bundle isomorphism $b : \xi_1 \oplus \cdots \oplus \xi_n \to \xi$ compatible with f can be chosen to preserve the (indefinite) inner product.*

Proof. Let $\mathbb{H}^n - \{0\} \to \xi_0 \xrightarrow{\pi} M$ be the subbundle of ξ consisting of nonzero sections. Noting that $\mathbb{H}^n$ is a right $\mathbb{H}$-module, it induces a fiber bundle with fiber $\mathbb{HP}^{n-1}$: $\mathbb{HP}^{n-1} \to Q \xrightarrow{q} M$. Since the cohomology group $H^*(\mathbb{HP}^{n-1};\mathbb{Z})$ is a free abelian group, $q^* : H^*(M) \to H^*(Q)$ is injective by the Leray-Hirsch's theorem (cf. [28].) Put

$$q^*\xi = \{(\ell, v) \in Q \times \xi \mid q(\ell) = \pi(v)\}.$$

Then, $(q^*\xi, \mathrm{pr}, Q)$ is a quaternionic bundle. Choose $\ell = v_1\mathbb{H}$ with $\langle v_1, v_1\rangle > 0$. Let $\xi_1 = \{(\ell, v) \in q^*\xi \mid v \in \ell\}$ which is the quaternionic 1-dimensional subbundle of $q^*\xi$. The (right) $\mathbb{H}$-inner product $\langle\, ,\ \rangle$ on ξ induces a (right) $\mathbb{H}$-inner product on $q^*\xi$ such that the bundle projection $\mathrm{Pr} : q^*\xi \to \xi$ preserves the inner product obviously. Moreover, we obtain that

$$q^*\xi = \xi_1 \oplus {\xi_1}^{\perp}.$$

Since ${\xi_1}^{\perp}$ is a quaternionic $(n-1)$-dimensional bundle over Q, an induction hypothesis for $n-1$ implies that there exist a paracompact Hausdorff space N and a splitting map $f_1 : N \to Q$ such that $f_1^*\xi_1^{\perp} = \xi_2 \oplus \cdots \oplus \xi_n$ and $f_1^* : H^*(Q) \to H^*(N)$ is injective. Moreover if $b_1 : \xi_2 \oplus \cdots \oplus \xi_n \to {\xi_1}^{\perp}$ is the bundle map compatible with f_1, then b_1 preserves the inner product on the fiber between $\xi_2 \oplus \cdots \oplus \xi_n$ and ${\xi_1}^{\perp}$ by induction. Putting $f = q \circ f_1 : N \to M$, we see that $f^* : H^*(M) \to H^*(N)$ is injective and $f^*\xi = f_1^*\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$. If $\mathrm{Pr}_1 : f_1^*\xi_1 \to \xi_1$ is the bundle map, then $\mathrm{Pr}_1 \oplus b_1 : f_1^*\xi_1 \oplus (\xi_2 \oplus \cdots \oplus \xi_n) \to \xi_1 \oplus \xi_1^{\perp}$ is the bundle map. Then the map $\mathrm{Pr} \circ (\mathrm{Pr}_1 \oplus b_1) : f_1^*\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n \longrightarrow \xi$ is compatible with f and preserves the inner product $\langle\, ,\ \rangle$. This proves the induction step for n. □

Let ξ be a quaternionic line bundle over M with gluing condition on $U_\alpha \cap U_\beta$:

$$z_\alpha = \bar{\lambda}(x) z_\beta \mu(x) = u(x) \cdot \bar{b}(x) z_\beta a(x) \quad (u > 0, a, b \in \mathrm{Sp}(1)). \tag{10.3}$$

Consider the tensor $\bar{\xi} \underset{\mathbb{H}}{\otimes} \xi$ so that the gluing condition on $U_\alpha \cap U_\beta$ is given by

$$\begin{aligned}(\bar{z}_\alpha \underset{\mathbb{H}}{\otimes} z_\alpha) &= u^2(x)\bar{a}(x)(\bar{z}_\beta b(x) \underset{\mathbb{H}}{\otimes} \bar{b}(x) z_\beta) a(x) \\ &= u^2(x)\bar{a}(x)(\bar{z}_\beta \underset{\mathbb{H}}{\otimes} z_\beta) a(x).\end{aligned}$$

Then $\bar{\xi} \underset{\mathbb{H}}{\otimes} \xi$ is a quaternionic line bundle over M whenever ξ is a quaternionic line bundle.

Lemma 10.3. *If $\bar{\xi} \underset{\mathbb{H}}{\otimes} \xi$ is viewed as a real 4-dimensional vector bundle, then $p_1(\bar{\xi} \underset{\mathbb{H}}{\otimes} \xi) = p_1(\bar{\xi}) + p_1(\xi)$. Moreover, $p_1(\bar{\xi}) = p_1(\xi)$ so that $p_1(\bar{\xi} \underset{\mathbb{H}}{\otimes} \xi) = 2p_1(\xi)$.*

Proof. Let γ be the canonical real 4-dimensional vector bundle over $B\mathrm{SO}(4)$ (cf. [28]). Then, ξ is determined by a classifying map $f : M \to B\mathrm{SO}(4)$ such that $f^*\gamma = \xi$. Let $\mathrm{pr}_i : B\mathrm{SO}(4) \times B\mathrm{SO}(4) \to B\mathrm{SO}(4)$ be the projection $(i = 1, 2)$. As γ inherits a quaternionic structure from ξ through the bundle map, there is a quaternionic line bundle $\mathrm{pr}_1^*\bar{\gamma} \underset{\mathbb{H}}{\otimes} \mathrm{pr}_2^*\gamma$ over $B\mathrm{SO}(4) \times B\mathrm{SO}(4)$. Now, let $h : B\mathrm{SO}(4) \times B\mathrm{SO}(4) \to B\mathrm{SO}(4)$ be a classifying map of this bundle so that $h^*\gamma = \mathrm{pr}_1^*\bar{\gamma} \underset{\mathbb{H}}{\otimes} \mathrm{pr}_2^*\gamma$. When $\iota_i : B\mathrm{SO}(4) \to B\mathrm{SO}(4) \times B\mathrm{SO}(4)$ is the inclusion map on each factor, $\iota_1^*\mathrm{pr}_2^*\gamma$ is the trivial quaternionic line bundle (we simply put $\theta^1_{\mathbf{h}}$) and so $\iota_1^*h^*p_1(\gamma) = \iota_1^*p_1(\mathrm{pr}_1^*\bar{\gamma} \underset{\mathbb{H}}{\otimes} \mathrm{pr}_2^*\gamma) = p_1(\bar{\gamma} \underset{\mathbb{H}}{\otimes} \theta^1_{\mathbf{h}}) = p_1(\bar{\gamma})$. Similarly, $\iota_2^*h^*p_1(\gamma) = p_1(\gamma)$. Hence we obtain that

$$h^*p_1(\gamma) = p_1(\bar{\gamma}) \times 1 + 1 \times p_1(\gamma).$$

Let $f' : M \to B\mathrm{SO}(4)$ be a classifying map for $\bar{\xi}$ such that $f'^*\gamma = \bar{\xi}$. Then the map $h(f' \times f)d$ composed of the diagonal map $d : M \to M \times M$ satisfies that

$$(h(f' \times f)d)^*\gamma = f'^*\bar{\gamma} \underset{\mathbb{H}}{\otimes} f^*\gamma = \bar{\xi} \underset{\mathbb{H}}{\otimes} \xi.$$

Therefore, $p_1(\bar{\xi} \underset{\mathbb{H}}{\otimes} \xi) = d^*(f' \times f)^*(p_1(\bar{\gamma}) \times 1 + 1 \times p_1(\gamma)) = p_1(f'^*\bar{\gamma}) + p_1(f^*\gamma) = p_1(\bar{\xi}) + p_1(\xi)$.

Next, the conjuagte $\bar{\xi}$ is isomorphic to ξ as real 4-dimensional vector bundle without orientation. But the correspondence $(1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}) \mapsto (1, -\boldsymbol{i}, -\boldsymbol{j}, -\boldsymbol{k})$ gives an isomorphism of $\bar{\xi}$ onto $(-1)^3\xi$. And so, the complexification $\bar{\xi}_{\mathbb{C}}$ of $\bar{\xi}$ (viewed as a real vector bundle) is isomorphic to $(-1)^6\xi_{\mathbb{C}} = \xi_{\mathbb{C}}$. By definition, $p_1(\bar{\xi}) = p_1(\xi)$. □

10.1. **Relation between the first Pontrjagin classes.**
Suppose that $\{\omega^{(\alpha)}, (I^{(\alpha)}, J^{(\alpha)}, K^{(\alpha)}), g_{(\alpha)}, U_\alpha\}_{\alpha \in \Lambda}$ represents a p-c q structure $\mathcal{D}$ on a $(4n+3)$-manifold $M = \underset{\alpha \in \Lambda}{\cup} U_\alpha$. Let L be the quotient bundle $TM/\mathcal{D}$. Choose the local vector fields $\{\xi_1^{(\alpha)}, \xi_2^{(\alpha)}, \xi_3^{(\alpha)}\}$ on each neighborhood U_α such that $\omega_a^{(\alpha)}(\xi_b^{(\alpha)}) = \delta_{ab}$. Then, $L|U_\alpha$ is spanned by $\{\xi_1^{(\alpha)}\}_{i=1,2,3}$ for each $\alpha \in \Lambda$. Moreover, the gluing condition between $L|U_\alpha$ and $L|U_\beta$ is exactly given by

$$\begin{pmatrix} \xi_1^{(\alpha)} \\ \xi_2^{(\alpha)} \\ \xi_3^{(\alpha)} \end{pmatrix} = u^2 A \begin{pmatrix} \xi_1^{(\beta)} \\ \xi_2^{(\beta)} \\ \xi_3^{(\beta)} \end{pmatrix}. \tag{10.4}$$

(Compare Definition 1.6.) It is easy to see that $\sum_{a=1}^{3} \omega_a^{(\alpha)} \cdot \xi_a^{(\alpha)} = \sum_{a=1}^{3} \omega_a^{(\beta)} \cdot \xi_a^{(\beta)}$ on $L|U_\alpha \cap U_\beta$. We can define a section $\theta : TM \rightarrow L$ which is an L-valued 1-form by setting

$$\theta|U_\alpha = \omega_1^{(\alpha)} \cdot \xi_1^{(\alpha)} + \omega_2^{(\alpha)} \cdot \xi_2^{(\alpha)} + \omega_3^{(\alpha)} \cdot \xi_3^{(\alpha)} \tag{10.5}$$

which induces the exact sequence of bundles: $1 \rightarrow \mathcal{D} \rightarrow TM \overset{\theta}{\longrightarrow} L \rightarrow 1$.

Let E be the quaternionic line bundle obtained from the union $\bigcup_{\alpha \in \Lambda} U_\alpha \times \mathbb{H}$ by identifying

$$(p, z_\alpha) \sim (q, z_\beta) \text{ if and only if } \begin{cases} p = q \in U_\alpha \cap U_\beta, \\ z_\alpha = \lambda \cdot z_\beta \cdot \bar{\lambda} = u^2 a \cdot z_\beta \cdot \bar{a} \text{ for a fnction } \lambda \in \mathbb{H} \end{cases} \tag{10.6}$$

If $L \oplus \theta$ is the Whitney sum composed of the trivial (real) line bundle θ on M, then it is easy to see that $L \oplus \theta$ is isomorphic to the quaternionic line bundle E. In particular, $p_1(E) = p_1(L \oplus \theta)$. We prove that

Theorem 10.4. *The first Pontrjagin classes of M and the bundle L has the relation:*

$$2p_1(M) = (n+2)p_1(L \oplus \theta).$$

Proof. As $\mathcal{D}$ is a quaternionic bundle in our sense, there is a splitting map $f : N \rightarrow M$ such that $f^*\mathcal{D} = \xi_1 \oplus \cdots \oplus \xi_n$ from Proposition 10.2. Let $\Psi : \xi_1 \oplus \cdots \oplus \xi_n \rightarrow \mathcal{D}$ be a bundle map which is compatible with f. Since Ψ is a right $\mathbb{H}$-linear map on the fiber at each point $x \in N$, we can describe

$$\Psi \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_x = P(x) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{f(x)}$$

for some function $P : N \to \mathrm{GL}(n, \mathbb{H})$. By (3) of Theorem 10.2, choosing an appropriate inner product $\langle , \rangle$ of typw (p, q) on $\mathcal{D}$ and the direct inner product on $\xi_1 \oplus \cdots \oplus \xi_n$, Ψ preserves the inner product between them. We may assume that

$$P(x) \in \mathrm{Sp}(p, q) \quad (p + q = n). \tag{10.7}$$

We examine the gluing condition of each ξ_i on $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \neq \emptyset$. For $x \in f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)$, let $v_i^{(\alpha)} \in \xi_i | f^{-1}(U_\alpha)$. Suppose that there is an element $v_i^{(\beta)} \in \xi_i | f^{-1}(U_\beta)$ such that $v_i^{(\alpha)} \sim v_i^{(\beta)}$, i.e. $v_i^{(\alpha)} = \bar{\lambda}_i v_i^{(\beta)} \mu_i$ ($\lambda_i, \mu_i \in \mathbb{H}^*; i = 1, \cdots, n$). Since $\Psi(v_i^{(\alpha)}) \sim \Psi(v_i^{(\beta)})$ at $f(x)$, it follows from (10.2) that $\Psi \begin{pmatrix} v_1^{(\alpha)} \\ \vdots \\ v_n^{(\alpha)} \end{pmatrix} = uT \cdot \Psi \begin{pmatrix} v_1^{(\beta)} \\ \vdots \\ v_n^{(\beta)} \end{pmatrix} \cdot a$ at $f(x) \in U_a \cap U_\beta$. As

$$P \begin{pmatrix} v_1^{(\alpha)} \\ \vdots \\ v_n^{(\alpha)} \end{pmatrix} = \Psi \begin{pmatrix} v_1^{(\alpha)} \\ \vdots \\ v_n^{(\alpha)} \end{pmatrix} = uT \cdot P \begin{pmatrix} v_1^{(\beta)} \\ \vdots \\ v_n^{(\beta)} \end{pmatrix} \cdot a = P \cdot uP^{-1}TP \begin{pmatrix} v_1^{(\beta)} \\ \vdots \\ v_n^{(\beta)} \end{pmatrix} \cdot a,$$

it follows that

$$\begin{pmatrix} v_1^{(\alpha)} \\ \vdots \\ v_n^{(\alpha)} \end{pmatrix} = u \cdot P^{-1}TP \begin{pmatrix} v_1^{(\beta)} \\ \vdots \\ v_n^{(\beta)} \end{pmatrix} \cdot a.$$

Since $v_i^{(\alpha)} = \bar{\lambda}_i v_i^{(\beta)} \mu_i$ as above, we have that ($x \in f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)$):

$$(\mathbf{1}) \quad u(x)P(x)^{-1}T(f(x))P(x) = \begin{pmatrix} \bar{\lambda}_1(x) & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \bar{\lambda}_n(x) \end{pmatrix}.$$

$$(\mathbf{2}) \quad \begin{pmatrix} \mu_1(x) & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mu_n(x) \end{pmatrix} = \begin{pmatrix} a(x) & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a(x) \end{pmatrix}.$$

Recall that $\mathrm{Sp}(p, q) = \{A | A^* \cdot \mathrm{I}_{p,q} \cdot A = \mathrm{I}_{p,q}\}$ where $\mathrm{I}_{p,q} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$.

From the fact that $T, P \in \mathrm{Sp}(p, q)$ (cf. (10.2),(10.7)), the equality $(\mathbf{1})$ shows that

$$\begin{pmatrix} |\lambda_1|^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -|\lambda_n|^2 \end{pmatrix} = u^2 (P^{-1}TP)^* \cdot \mathrm{I}_{p,q} \cdot (P^{-1}TP) = u^2 \mathrm{I}_{p,q}.$$

Hence, $\lambda_i = u \cdot \lambda_i/|\lambda_i| = u \cdot \nu_i$ where $\nu_i = \lambda_i/|\lambda_i| \in \mathrm{Sp}(1)$. It follows from (**2**) that $\mu_i = a$ for each i. We obtain that

$$v_i^{(\alpha)} = u(x) \cdot \bar{\nu}_i(x) v_i^{(\beta)} a(x) \quad (i = 1, \cdots, n). \tag{10.8}$$

Each ξ_i is a quaternionic line bundle over N equipped with (10.8) on $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)$. If we consider $\bar{\xi}_i \underset{\mathbb{H}}{\otimes} \xi_i$, then the gluing condition on $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)$ is given by

$$(\bar{v}_i^{(\alpha)} \underset{\mathbb{H}}{\otimes} v_i^{(\alpha)}) = u^2(x)\bar{a}(x)(\bar{v}_i^{(\beta)} \underset{\mathbb{H}}{\otimes} v_i^{(\beta)})a(x).$$

Since $\lambda = u \cdot a$ is the same as that of E from (10.6), each $\bar{\xi}_i \underset{\mathbb{H}}{\otimes} \xi_i$ is isomorphic to $f^*(E)$. As $E \cong L \oplus \theta^1$, we see that $f^*(L \oplus \theta^1) = \bar{\xi}_i \underset{\mathbb{H}}{\otimes} \xi_i \quad (i = 1, \cdots, n)$. By Lemma 10.3, $f^* p_1(L \oplus \theta^1) = 2p_1(\xi_i)$ for each i. Since $f^* p_1(\mathcal{D}) \equiv p_1(\xi_1) + \cdots + p_1(\xi_n)$ mod 2-torsion in $H^4(N; \mathbb{Z})$, $f^*(2p_1(\mathcal{D})) = 2p_1(\xi_1) + \cdots + 2p_1(\xi_n) = nf^* p_1(L \oplus \theta^1) = nf^* p_1(L)$. Noting that the splitting map f^* is injective, $2p_1(\mathcal{D}) = np_1(L)$ in $H^4(M; \mathbb{Z})$. As $TM \cong \mathcal{D} \oplus L$, we have $2p_1(M) = (n + 2)p_1(L)$. □

Corollary 10.5. *Let $(M, \mathcal{D})$ be a $(4n + 3)$-dimensional simply connected p-c q manifold associated with the local forms $\{\omega^{(\alpha)}, (I^{(\alpha)}, J^{(\alpha)}, K^{(\alpha)}), g_{(\alpha)}, U_\alpha\}_{\alpha \in \Lambda}$. Then the following are equivalent.*

(1) $2p_1(M) = 0$. *In particular, the rational Pontrjagin class vanishes.*
(2) L *is the trivial bundle so that* $\{\xi_\alpha\}_{\alpha=1,2,3}$ *exists globally on* M.
(3) *There exists a* $\mathrm{Im}\mathbb{H}$*-valued* 1*-form* ω *on* M *which represents a p-c q structure* $\mathcal{D}$. *In particular, there exists a hypercomplex structure* $\{I, J, K\}$ *on* $\mathcal{D}$.

Proof. First note that the Whitney sum $L \oplus \theta^1$ is the quaternionic line bundle E with structure group lying in $\mathrm{SO}(3) \times \mathbb{R}^+ \subset \mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+$. As above we have the quaternionic line bundle of ℓ-times tensor $\overset{\ell}{\underset{\mathbb{H}}{\otimes}} E$ with structure group $\mathrm{SO}(3) \times \mathbb{R}^+$. Viewed as the 4-dimensional real vector bundle, it determines a classifying map $g : M \to B(\mathrm{SO}(3) \times \mathbb{R}^+) = B\mathrm{SO}(3)$. Note that $p : B(\mathrm{Sp}(1) \times \mathbb{R}^+) \to B(\mathrm{SO}(3) \times \mathbb{R}^+)$ is the two-fold covering map. As M is simply connected by the hypothesis, the map g lifts to a classifying map $\tilde{g} : M \to B\mathrm{Sp}(1)$ such that $g = p \circ \tilde{g}$. Let γ be the 4-dimensional universal bundle over $B\mathrm{SO}(3)$. (Compare [28].) Then the pull back $p^*\gamma$ is the 4-dimensional canonical bundle over $B\mathrm{Sp}(1) = \mathbb{HP}^\infty$ whose first Pontrjagin class $p_1(p^*\gamma)$ generates the cohomology ring $H^*(\mathbb{HP}^\infty; \mathbb{Z})$. So the bundle $\overset{\ell}{\underset{\mathbb{H}}{\otimes}} E$ is classified by the map $\tilde{g}$ where $[\tilde{g}] = \tilde{g}^* p_1(p^*\gamma) \in H^4(M; \mathbb{Z})$, which coincides with $p_1(\overset{\ell}{\underset{\mathbb{H}}{\otimes}} E)$.

(1) ⇒ (2). If $2p_1(M) = 0$, then Theorem 10.4 shows $(n+2)p_1(L) = 0$, i.e. $p_1((\overset{n+2}{\underset{\mathbb{H}}{\otimes}} E)) = 0$. (See Lemma 10.3.) Hence, the classifying map $\tilde{g} : M \to B\mathrm{Sp}(1)$ for $\overset{n+2}{\underset{\mathbb{H}}{\otimes}} E$ is null homotopic so that $\tilde{g}^* p^* \gamma = \overset{n+2}{\underset{\mathbb{H}}{\otimes}} E$ is trivial. There exists a family of functions $\{h_\alpha\} \in \mathrm{Sp}(1) \times \mathbb{R}^+$ such that the transition function $g_{\alpha\beta}(x) = \delta^1 h(\alpha, \beta)(x) \quad (x \in U_\alpha \cap U_\beta)$. As the gluing relation

for $\overset{n+2}{\underset{\mathbb{H}}{\otimes}} E$ is given by $z \mapsto u_{\alpha\beta}^{2(n+2)} \bar{a}_{\alpha\beta} \cdot z \cdot a_{\alpha\beta}$, letting $h_\alpha = a_\alpha \cdot u_\alpha \in \mathrm{Sp}(1) \times \mathbb{R}^+$, it follows that

$$u_{\alpha\beta}^{2(n+2)} \cdot \bar{a}_{\alpha\beta} \cdot z \cdot a_{\alpha\beta} = (h_\alpha^{-1} h_\beta) z = u_\alpha^{-1} u_\beta a_\alpha \bar{a}_\beta \cdot z \cdot a_\beta \bar{a}_\alpha \quad (z \in \mathbb{H}).$$

Then, $u_{\alpha\beta}^{2(n+2)} = u_\alpha^{-1} u_\beta \in \mathbb{R}^+$ and $a_{\alpha\beta} = \pm a_\beta \bar{a}_\alpha$. As $u_{\alpha\beta} > 0$, $u_{\alpha\beta} = (u_\alpha^{-1})^{\frac{1}{2(n+2)}} \cdot u_\beta^{\frac{1}{2(n+2)}}$. Since the gluing relation of $E = L \oplus \theta$ is given by $z_\alpha = u_{\alpha\beta}^2 \cdot \bar{a}_{\alpha\beta} \cdot z_\beta \cdot a_{\alpha\beta}$, putting $u'_\alpha = (u_\alpha)^{\frac{1}{(n+2)}}, u'_\beta = (u_\beta)^{\frac{1}{(n+2)}}$, a calculation shows $z_\alpha = {u'_\alpha}^{-1} u'_\beta \cdot a_\alpha \bar{a}_\beta \cdot z_\beta \cdot a_\beta \bar{a}_\alpha$. Moreover if $C(\alpha) \in \mathrm{SO}(3)$ is the matrix defined by $\bar{a}_\alpha \cdot \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \cdot a_\alpha = C(\alpha) \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix}$ (similarly for $C(\beta)$), then

$$u_{\alpha\beta}^2 \cdot A^{\alpha\beta} = {u'}_\alpha^{-1} u'_\beta \cdot C(\alpha)^{-1} \circ C(\beta). \tag{10.9}$$

Substitute this into (10.4), it follows that

$$u'_\alpha \cdot C(\alpha) \begin{pmatrix} \xi_1^{(\alpha)} \\ \xi_2^{(\alpha)} \\ \xi_3^{(\alpha)} \end{pmatrix} = u'_\beta \cdot C(\beta) \begin{pmatrix} \xi_1^{(\beta)} \\ \xi_2^{(\beta)} \\ \xi_3^{(\beta)} \end{pmatrix} \quad \text{on } U_\alpha \cap U_\beta.$$

We can define the vector fields $\{\xi_1, \xi_2, \xi_3\}$ on M to be

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \Big| U_\alpha = u'_\alpha \cdot C(\alpha) \begin{pmatrix} \xi_1^{(\alpha)} \\ \xi_2^{(\alpha)} \\ \xi_3^{(\alpha)} \end{pmatrix}. \tag{10.10}$$

Then $\{\xi_1, \xi_2, \xi_3\}$ spans L, therefore, L is trivial.

(2) $\Rightarrow$ (3). Since $(\omega_1^{(\beta)}, \omega_2^{(\beta)}, \omega_3^{(\beta)}) = (\omega_1^{(\alpha)}, \omega_2^{(\alpha)}, \omega_3^{(\alpha)}) u_{\alpha\beta}^2 \cdot A^{\alpha\beta}$, (10.9) implies that

$$(\omega_1^{(\beta)}, \omega_2^{(\beta)}, \omega_3^{(\beta)}) {u'}_\beta^{-1} \cdot C(\beta)^{-1} = (\omega_1^{(\alpha)}, \omega_2^{(\alpha)}, \omega_3^{(\alpha)}) {u'}_\alpha^{-1} \cdot C(\alpha)^{-1} \text{ on } U_\alpha \cap U_\beta.$$

Then, a Im$\mathbb{H}$-valued 1-form ω on M can be defined by

$$\omega | U_\alpha = (\omega_1^{(\alpha)}, \omega_2^{(\alpha)}, \omega_3^{(\alpha)}) {u'}_\alpha^{-1} \cdot C(\alpha)^{-1} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix}. \tag{10.11}$$

Note that ω satisfies that $\omega | U_\alpha = \bar{\lambda}_\alpha \cdot \omega^{(\alpha)} \cdot \lambda_\alpha$ for some function $\lambda_\alpha : U_\alpha \to \mathbb{H}^*$ $(\alpha \in \Lambda)$. Recall that two quaternionic structures on $U_\alpha \cap U_\beta$ are related:

$$\begin{pmatrix} I^{(\alpha)} \\ J^{(\alpha)} \\ K^{(\alpha)} \end{pmatrix} = A^{\alpha\beta} \begin{pmatrix} I^{(\beta)} \\ J^{(\beta)} \\ K^{(\beta)} \end{pmatrix}.$$

As $A^{\alpha\beta} = C(\alpha)^{-1} \circ C(\beta)$, it follows that

$$C(\alpha) \cdot \begin{pmatrix} I^{(\alpha)} \\ J^{(\alpha)} \\ K^{(\alpha)} \end{pmatrix} = C(\beta) \cdot \begin{pmatrix} I^{(\beta)} \\ J^{(\beta)} \\ K^{(\beta)} \end{pmatrix}. \tag{10.12}$$

Letting $\begin{pmatrix} I \\ J \\ K \end{pmatrix} |U_\alpha = C(\alpha)\cdot \begin{pmatrix} I^{(\alpha)} \\ J^{(\alpha)} \\ K^{(\alpha)} \end{pmatrix}$, there exists a hypercomplex structure $\{I, J, K\}$ on $\mathcal{D}$.

(3) $\Rightarrow$ (1). If the global Im$\mathbb{H}$-valued 1-form ω exists, then ω defines a three independent vector fields isomorphic to L, i.e. $p_1(L) = 0$. Hence apply Theorem 10.4. $\square$

References

[1] D. A. Alekseevsky, "Riemannian spaces with exceptional holonomy groups," *Functional Anal. Appl.* Vol. 2 (1968) 97 – 105.

[2] D. A. Alekseevsky and Y. Kamishima, "Quaternionic CR structure on (4n+3)-dimensional manifolds", *Preprint.*

[3] D. A. Alekseevsky and Y. Kamishima, "Quaternionic pseudo-conformal structures," *preprint.*

[4] D. A. Alekseevsky and Y. Kamishima, "Quaternionic and para-quaternionic CR structure on (4n+3)-dimensional manifolds", *Central European J. of Mathematics (electronic),* vol. 2(5) (2004), 732–753.

[5] D.E. Blair, 'Riemannian geometry of contact and symplectic manifolds Contact manifolds ,' *Progress in Math., 203 Birkhäuser*, 2002.

[6] O. Biquard, "Quaternionic structures in mathematics and physics,"Rome, 1999, World Sci. Publishing, River Edge, NJ, 2001; 23–30.

[7] W.M. Boothby, "A note on homogeneous complex contact manifolds," *Proc. Amer. Math. Soc.,* Vol. 13 (1962) 276 – 280.

[8] C.P. Boyer, K.Galicki and B.M. Mann, "The geometry and topology of 3-Sasakian manifolds," *Jour. reine ange. Math.,* Vol. 455 (1994) 183 – 220.

[9] G. Bredon, 'Introduction to compact transformation groups,' *Academic Press, New York*, 1972.

[10] D. M. Burns and S. Shnider, "Spherical hypersurfaces in complex manifolds," *Invent. Math.*, Vol. 33 (1976) 223 – 246.

[11] S.S. Chen and L. Greenberg, 'Hyperbolic Spaces,' *Contribution to Analysis* (A Collection of Papers Dedicated to Lipman Bers, eds. L. Ahlfors and others), Academic Press, New York and London, (1974) 49 – 87.

[12] S.S. Chern and J. Moser, "Real hypersurfaces in complex manifolds," *Acta Math.*, Vol. 133 (1974) 48 – 69.

[13] J. W. Gray, "Some global properties of contact structures," *Ann. of Math.,* Vol. 69, (1959) 421 – 450.

[14] N. J. Hitchin, "The self-duality equations on a Riemannian surface," *Proc. London Math. Soc.,* Vol. 55, (1987) 59 – 126.

[15] S. Ishihara, "Quaternion Kählerian manifolds," *J. Diff. Geom.,* Vol. 9 (1974) 483 – 500.

[16] Y. Kamishima, "Geometric rigidity of spherical hypersurfaces in quaternionic manifolds," *Asian J. Math.*, Vol. 3, No.3 (1999) 519 – 556

[17] Y. Kamishima, "Geometric flows on compact manifolds and Global rigidity," *Topology,* Vol. 35 (1996) 439 – 450.

[18] Y.Kamishima and T. Udono, "Three dimensional Lie group actions on compact ($4n$+3)-dimensional geometric manifolds" *Differential Geometrty and its Applications*, Vol. 21 (2004) 1 – 26.

[19] T. Kashiwada, "On a contact metric structure," *Math. Z.*, Vol. 238, (2001) 829 – 832.

[20] S. Kobayashi, 'Transformation groups in differential geometry', Ergebnisse der Math. 70, Springer Verlag, 1972.

[21] S. Kobayashi and K. Nomizu, 'Foundations of differential geometry I,II,' *Interscience John Wiley & Sons, New York*, 1969.

[22] S. Kobayashi, "Remarks on complex contact manifolds," *Proc. Amer. Math.,* Vol. 10 (1959) 164 – 167.

[23] R. Kulkarni, "On the principle of uniformization," *J. Diff. Geom.,* Vol. 13 (1978) 109 –138.

[24] R. Kulkarni, "Proper actions and pseudo-Riemannian space forms," *Advances in Math.,* Vol. 40 (1981) 10 – 51.

[25] C. Lebrun, "Fano manifolds, contact structures, and quaternionic geometry," *Int. J. Math.* Vol. 3 (1995) 419-437.

[26] J. Lelong-Ferrand, *Transformations conformes et quasi conformes des variétés riemanniennes compactes*, Acad. Roy. Belgique Sci. Mem. Coll., Vol. 8 (1971), 1 – 44.

[27] P. Libermann, "Sur les structures presque complexes et autres structures infinitésimales régulières," *Bull. Soc. Math., France* Vol. 83 (1955) 195 – 224.

[28] J. Milnor and J. Stasheff, 'Characteristic Classes,' *Ann. of Math. Studies,* Vol. 76 (1974) Princeton Univ. Press.

[29] M. Obata, "The conjectures on conformal transformations of Riemannian manifolds," *J. Diff. Geom.,* Vol. 6 (1971), 247-258.

[30] R. S. Palais, 'A global formulation of the Lie theory of transformation groups,' *Mem. Amer. Math. Soc.*, Vol 22 (1957).

[31] S. Sasaki, "Spherical space forms with normal contact metric 3-structure," *J. Diff. Geom.*, Vol. 6 (1972) 307–315.

[32] R. Stong, "Contact manifolds," *J. Diff. Geom.*, Vol. 9 (1974) 219–238.

[33] S. Tanno, "Killing vector fields on contact Riemannian manifolds and fibering related to the Hopf fibrations," *Tôhoku Math. Jour.*, Vol. 23 (1971) 313 – 333.

[34] S. Tanno, "Remarks on a triple of K-contact structures," *Tôhoku Math. Jour.*, Vol. 48 (1996) 519 – 531.

[35] S. Webster, "Pseudo-Hermitian structures on a real hypersurface," *J. Diff. Geom.*, vol. 13 (1978) 25–41.

[36] S. Webster, "On the pseudo conformal geometry of a Kähler manifold," *Math. Z.*, vol. 157 (1977) 265 – 270.

[37] J. Wolf, "Complex homogeneous contact structures and quaternionic symmetric spaces," *Jour. Math. Mech.,* Vol. 14 (1965) 1033 – 1047.

[38] J. Wolf, 'Spaces of constant curvature,' *McGraw-Hill, Inc.*, 1967.

Department of Mathematics, Hull University, GEN. Cottingham Road HU6 7RX, Hull, England

E-mail address: `d.v.alekseevsky@maths.hull.ac.uk`

Department of Mathematics, Tokyo Metropolitan University,Minami-Ohsawa 1-1,Hachioji, Tokyo 192-0397, Japan

E-mail address: `kami@comp.metro-u.ac.jp`