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Pseudo-Differential Calculus in Anisotropic Gelfand–Shilov Setting

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Abstract. We study some classes of pseudo-differential operators with symbols a admitting anisotropic exponential type growth at infinity. We deduce mapping properties for these operators on Gelfand–Shilov spaces. Moreover, we deduce algebraic and certain invariance properties of these classes.

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0. Introduction

In the paper we deduce algebraic and continuity properties for a family of anisotropic pseudo-differential operators of infinite orders when acting on Gelfand–Shilov spaces. We permit superexponential growth on corresponding symbols and deduce continuity properties in the full range of (classical) Gelfand–Shilov spaces. We also deduce that the operator classes are closed under compositions.

Pseudo-differential operators (as well as Fourier integral operators) with ultra-differentiable symbols $a(x, \xi)$ which are permitted to grow faster than polynomials at infinity, are commonly known as operators of infinite order. Operators of infinite order appear naturally when dealing with various kinds of partial differential equations, usually emerging in science and engineering. Such operators have been studied in different ways, e. g. in [2–9, 11, 12, 15–17, 29, 35]. Keyparts of these investigations consist of deducing fundamental algebraic and continuity properties.

The assumptions on the symbols for pseudo-differential operators with infinite order are more extreme compared to classical pseudo-differential operators. For the symbols to operators of infinite order, stronger regularity are imposed while growth conditions are relaxed compared to symbols of classical operators (Gevrey regularity and exponential type bound conditions instead of smoothness and polynomial bound conditions).

In order to meet the more extreme conditions on symbols to operators of infinite order, the spaces of Schwartz functions and their distribution spaces, feasible when dealing with classical pseudo-differential operators, are replaced by Gelfand–Shilov spaces and their distribution spaces. For fixed $s,\sigma>0$, the Gelfand–Shilov space $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ ($\Sigma_s^\sigma(\mathbf{R}^d)$) consists of all $f\in C^\infty(\mathbf{R}^d)$ such that

$$|\partial^\beta f(x)| \lesssim h^{|\beta|}\beta!^\sigma e^{-r|x|^{\frac{1}{s}}} \tag{0.1}$$

for some (for every) $h,r>0$. (See [22] and Sect. 1 for notations.) For $\sigma>1$, $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ represents a natural global counterpart of the Gevrey class $G^\sigma(\mathbf{R}^d)$ but, in addition, the condition (0.1) encodes a precise description of the behavior at infinity of f.

Continuity properties for operators of infinite order are important when investigating well-posedness for partial differential equations in the framework of Gelfand–Shilov spaces. Some studies are performed in [2,11,25,35] where the symbols have exponential growth with respect to the momentum variable. In [11,12,14,24,25] such operators are applied to Cauchy problems for hyperbolic and Schrödinger equations in Gevrey classes. Parallel results have also been obtained in Gelfand–Shilov spaces (see [3,4,7,8,10,29]). In the latter case, the symbols of the involved operators of infinite order admit exponential growth both in configuration and momentum variables, i. e. in the phase space variables.

For pseudo-differential operators of infinite order, their symbols should obey conditions of the form

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \lesssim h^{|\alpha+\beta|}\alpha!^{\sigma_2}\beta!^{s_2}e^{r(|x|^{\frac{1}{s_1}}+|\xi|^{\frac{1}{\sigma_1}})} \tag{0.2}$$

or, what seems to be more general,

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \lesssim h^{|\alpha+\beta|}\alpha!^{\sigma}\beta!^{s}\omega(x,\xi) \tag{0.3}$$

for some positive constants s,s_j,σ,σ_j and some positive function $\omega(x,\xi)$ defined on the phase space $\mathbf{R}^{2d}$, $j=1,2$. A common condition is that ω should be moderate, meaning that it exists a positive function v on $\mathbf{R}^{2d}$ such that

$$\omega(x+y,\xi+\eta)\lesssim \omega(x,\xi)v(y,\eta).$$

This implies that ω must satisfy conditions of the form

$$\omega(x,\xi)\lesssim e^{r(|x|+|\xi|)}$$

(cf. [20] or Proposition 1.6). Hence, for such ω, (0.3) does not need to be fulfilled when (0.2) holds for some $s_1<1$ or $\sigma_1<1$. If instead ω fails to be moderate, then in reality it is always assumed that

$$\omega(x,\xi)=e^{r(|x|^{\frac{1}{s_1}}+|\xi|^{\frac{1}{\sigma_1}})} \tag{0.4}$$

for some positive constants s_1 and σ_1. Consequently, if ω is not moderate, then in reality, the cases (0.2) and (0.3) agree.

In most of the contributions [2–9,11,12,15–17,29,35] mentioned above, it is assumed that $s,s_j,\sigma,\sigma_j>1$ and that $\omega(x,\xi)$ is allowed to grow at most subexponentially.

An exception concerns [16] by Cordero, Nicola and Rodino, where it is merely assumed that $s = \sigma > 0$ and it is evident that in their analysis, ω must be moderate, admitting exponential growth of the symbols.

In [16] it also seems to be the first time where characterizations of symbols satisfying (0.3) in terms of estimates with corresponding short-time Fourier transforms are performed (cf. [16, Theorem 3.1]) and where continuity of operators with infinite order is obtained for Gelfand–Shilov spaces of the forms $\mathcal{S}_s^s(\mathbf{R}^d)$ with *s less than one* (cf. [16, Proposition 4.7]). Here we remark that some implicit steps in such directions are given in [31]. (Cf. Theorems 3.9 and 6.15 in [31].) These continuity properties are established by using methods based on modulation space theory and short-time estimates on the symbols of the operators, instead of the usual micro-local techniques. We also remark that the extension of the complete calculus developed in [3,4] in this case is out of reach due to the lack of compactly supported functions in $\mathcal{S}_s^s(\mathbf{R}^d)$ and $\Sigma_s^s(\mathbf{R}^d)$ when $s \le 1$.

In [9], pseudo-differential operators with symbols satisfying (0.2) with

$$s_1 = s_2 = \sigma_1 = \sigma_2 = s \ge \frac{1}{2} \tag{0.5}$$

are considered, which for example is interesting in connection with Shubin-type pseudo-differential operators. In particular, superexponential growth on the symbols is permitted, giving that the growth conditions on the symbols are even more relaxed compared to [16].

In [9] it is deduced that such operators of infinite order are continuous on the Gelfand–Shilov spaces $\mathcal{S}_s^s$ or Σ_s^s, depending on the precise conditions on h and r in (0.2), and their distribution spaces. Here it is also proved that such operator classes are algebras under compositions.

In Sect. 3 we extend the results in [9] to the anisotropic case, where the conditions in (0.5) for (0.2) are relaxed into

$$s_1 = s_2 = s, \quad \sigma_1 = \sigma_2 = \sigma, \quad s + \sigma \ge 1. \tag{0.5$'$}$$

We prove that operators with such symbols are continuous on the (anisotropic) Gelfand–Shilov spaces $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$ (again depending on the precise conditions on h and r in (0.2)), and their distribution spaces. We also prove that our operator classes are algebras under compositions, thereby receiving full extension of the results in [9] to the anisotropic case.

In a similar way as in [9,16], our analysis is based on characterizations of our symbols in terms of suitable estimates of their short-time Fourier transforms. On the other hand, an essential part of the analysis in [16] is based on suitable applications of almost diagonalization property for the operators. Such technique works well when ω in (0.2) is moderate. Since this is not the case in our situation when $s < 1$ or $\sigma < 1$, we can not use such approach. Instead we accept certain types of gaps between symbol estimates and estimates on corresponding short-time Fourier transforms, which neither harm our analysis nor threat our conclusions.

Finally we remark that rates of growth and Gevrey regularity are usually different and not so related to each others, leading to differences between the choice of s and the choice of σ. Hence, the restriction $s = \sigma$ in [9] and in several other contributions or problems, is not natural. We therefore believe that it is relevant to consider, as it is done in Sect. 3, the anisotropic case where s and σ are allowed to be different.

An example where anisotropic operators of infinite order appear concerns certain initial value problems for Schrödinger type equations with data in Gelfand–Shilov spaces (see [1]). On the other hand, in the present paper we do not give specific applications which require a long treatise.

The paper is organized as follows. In Sect. 1, after recalling some basic properties of the spaces $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$, we introduce several general symbol classes. In Sect. 2 we characterize these symbols in terms of the behavior of their short time Fourier transform. In Sect. 3 we deduce continuity on $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$ and their distribution spaces, composition and invariance properties for pseudo-differential operators in our classes. Finally, in order to make it easy for the reader and the community we have, in Appendix A, collected some essential properties and included some short proofs for moderate weights. These properties can essentially be found in the literature, but at different places (see e. g. [20,31]). For example, here we show that moderate weights are bounded by exponential functions.

1. Preliminaries

In this section we recall some basic facts, especially concerning Gelfand-Shilov spaces, the short-time Fourier transform and pseudo-differential operators.

We let $\mathscr{S}(\mathbf{R}^d)$ be the Schwartz space of rapidly decreasing functions on $\mathbf{R}^d$ together with their derivatives, and by $\mathscr{S}'(\mathbf{R}^d)$ the corresponding dual space of tempered distributions.

1.1. Gelfand–Shilov Spaces

We start by recalling some facts about Gelfand–Shilov spaces. Let $0 < h, s, \sigma \in \mathbf{R}$ be fixed. Then $\mathcal{S}_{s;h}^\sigma(\mathbf{R}^d)$ is the Banach space of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s;h}^\sigma} \equiv \sup_{\alpha,\beta \in \mathbf{N}^d} \sup_{x \in \mathbf{R}^d} \frac{|x^\alpha \partial^\beta f(x)|}{h^{|\alpha+\beta|}\alpha!^s\,\beta!^\sigma} < \infty, \tag{1.1}$$

endowed with the norm (1.1).

The *Gelfand–Shilov spaces* $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$ are defined as the inductive and projective limits respectively of $\mathcal{S}_{s;h}^\sigma(\mathbf{R}^d)$. This implies that

$$\mathcal{S}_s^\sigma(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s;h}^\sigma(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s^\sigma(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s;h}^\sigma(\mathbf{R}^d), \tag{1.2}$$

and that the topology for $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ is the strongest possible one such that the inclusion map from $\mathcal{S}_{s;h}^\sigma(\mathbf{R}^d)$ to $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ is continuous, for every choice of $h > 0$. The space $\Sigma_s^\sigma(\mathbf{R}^d)$ is a Fréchet space with seminorms $\|\cdot\|_{\mathcal{S}_{s;h}^\sigma}$, $h > 0$. Moreover, $\Sigma_s^\sigma(\mathbf{R}^d) \neq \{0\}$, if and only if $s + \sigma \geq 1$ and $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, and $\mathcal{S}_s^\sigma(\mathbf{R}^d) \neq \{0\}$, if and only if $s + \sigma \geq 1$.

The spaces $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$ can be characterized also in terms of the exponential decay of their elements, namely $f \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$ (respectively $f \in \Sigma_s^\sigma(\mathbf{R}^d)$), if and only if

$$|\partial^\alpha f(x)| \lesssim h^{|\alpha|}(\alpha!)^\sigma e^{-r|x|^{\frac{1}{s}}}$$

for some $h, r > 0$ (respectively for every $h, r > 0$). Moreover we recall that for $s < 1$ the elements of $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ admit entire extensions to $\mathbf{C}^d$ satisfying suitable exponential bounds, cf. [18] for details.

The *Gelfand–Shilov distribution spaces* $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ and $(\Sigma_s^\sigma)'(\mathbf{R}^d)$ are the projective and inductive limit respectively of $(\mathcal{S}_{s;h}^\sigma)'(\mathbf{R}^d)$. This means that

$$(\mathcal{S}_s^\sigma)'(\mathbf{R}^d) = \bigcap_{h>0} (\mathcal{S}_{s;h}^\sigma)'(\mathbf{R}^d) \quad \text{and} \quad (\Sigma_s^\sigma)'(\mathbf{R}^d) = \bigcup_{h>0} (\mathcal{S}_{s;h}^\sigma)'(\mathbf{R}^d). \qquad (1.2)'$$

We remark that in [28] it is proved that $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ is the dual of $\mathcal{S}_{s,\sigma}(\mathbf{R}^d)$, and $(\Sigma_s^\sigma)'(\mathbf{R}^d)$ is the dual of $\Sigma_s^\sigma(\mathbf{R}^d)$ (also in topological sense).

For every $s, \sigma > 0$ we have

$$\Sigma_s^\sigma(\mathbf{R}^d) \hookrightarrow \mathcal{S}_s^\sigma(\mathbf{R}^d) \hookrightarrow \Sigma_{s+\varepsilon}^{\sigma+\varepsilon}(\mathbf{R}^d) \hookrightarrow \mathscr{S}(\mathbf{R}^d) \qquad (1.3)$$

for every $\varepsilon > 0$. If $s + \sigma \geq 1$, then the last two inclusions in (1.3) are dense, and if in addition $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, then the first inclusion in (1.3) is dense.

From these properties it follows that $\mathscr{S}'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ when $s+\sigma \geq 1$, and if in addition $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, then $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d) \hookrightarrow (\Sigma_s^\sigma)'(\mathbf{R}^d)$.

The Gelfand–Shilov spaces possess several convenient mapping properties. For example they are nuclear and invariant under translations, dilations, and to some extent tensor products and (partial) Fourier transformations, cf. [18, 26, 27]).

The Fourier transform $\mathscr{F}$ is the linear and continuous map on $\mathscr{S}(\mathbf{R}^d)$, given by the formula

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in \mathscr{S}(\mathbf{R}^d)$. Here $\langle \,\cdot\, , \,\cdot\, \rangle$ denotes the usual scalar product on $\mathbf{R}^d$. The Fourier transform extends uniquely to homeomorphisms from $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ to $(\mathcal{S}_\sigma^s)'(\mathbf{R}^d)$, and from $(\Sigma_s^\sigma)'(\mathbf{R}^d)$ to $(\Sigma_\sigma^s)'(\mathbf{R}^d)$. Furthermore, it restricts to homeomorphisms from $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ to $\mathcal{S}_\sigma^s(\mathbf{R}^d)$, and from $\Sigma_s^\sigma(\mathbf{R}^d)$ to $\Sigma_\sigma^s(\mathbf{R}^d)$.

Some considerations later on involve a broader family of Gelfand–Shilov spaces. More precisely, for $s_j, \sigma_j \in \mathbf{R}_+$, $j = 1, 2$, the Gelfand–Shilov spaces $\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})$ and $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})$ consist of all functions $F \in C^\infty(\mathbf{R}^{d_1+d_2})$ such that

$$|x_1^{\alpha_1} x_2^{\alpha_2} \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} F(x_1, x_2)| \lesssim h^{|\alpha_1+\beta_1|+|\alpha_2+\beta_2|} \alpha_1!^{s_1} \alpha_2!^{s_2} \beta_1!^{\sigma_1} \beta_2!^{\sigma_2} \qquad (1.4)$$

for some $h > 0$ respectively for every $h > 0$. The topologies, and the duals

$$(\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbf{R}^{d_1+d_2}) \quad \text{and} \quad (\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbf{R}^{d_1+d_2})$$

of

$$\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2}) \quad \text{and} \quad \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2}),$$

respectively, and their topologies are defined in analogous ways as for the spaces $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$ above.

The following proposition explains mapping properties of partial Fourier transforms on Gelfand–Shilov spaces, and follows by similar arguments as in analogous situations in [18]. The proof is therefore omitted. Here, $\mathscr{F}_1 F$ and $\mathscr{F}_2 F$ are the partial Fourier transforms of $F(x_1, x_2)$ with respect to $x_1 \in \mathbf{R}^{d_1}$ and $x_2 \in \mathbf{R}^{d_2}$, respectively.

Proposition 1.1. *Let $s_j, \sigma_j > 0$, $j = 1, 2$. Then the following is true:*

(1) *the mappings $\mathscr{F}_1$ and $\mathscr{F}_2$ on $\mathscr{S}(\mathbf{R}^{d_1+d_2})$ restrict to homeomorphisms*

$$\mathscr{F}_1 \, : \, \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2}) \to \mathcal{S}_{\sigma_1,s_2}^{s_1,\sigma_2}(\mathbf{R}^{d_1+d_2})$$

and

$$\mathscr{F}_2 \, : \, \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2}) \to \mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbf{R}^{d_1+d_2});$$

(2) *the mappings $\mathscr{F}_1$ and $\mathscr{F}_2$ on $\mathscr{S}(\mathbf{R}^{d_1+d_2})$ are uniquely extendable to homeomorphisms*

$$\mathscr{F}_1 \, : \, (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbf{R}^{d_1+d_2}) \to (\mathcal{S}_{\sigma_1,s_2}^{s_1,\sigma_2})'(\mathbf{R}^{d_1+d_2})$$

and

$$\mathscr{F}_2 \, : \, (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbf{R}^{d_1+d_2}) \to (\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbf{R}^{d_1+d_2}).$$

The same holds true if the $\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}$-spaces and their duals are replaced by corresponding $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}$-spaces and their duals.

The next two results follow from [13]. The proofs are therefore omitted.

Proposition 1.2. *Let $s_j, \sigma_j > 0$, $j = 1, 2$. Then the following conditions are equivalent:*

(1) $F \in \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2}) \quad (F \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2}))$*;*

(2) *for some $r > 0$ (for every $r > 0$) it holds*

$$|F(x_1, x_2)| \lesssim e^{-r(|x_1|^{\frac{1}{s_1}} + |x_2|^{\frac{1}{s_2}})} \quad \text{and} \quad |\widehat{F}(\xi_1, \xi_2)| \lesssim e^{-r(|\xi_1|^{\frac{1}{\sigma_1}} + |\xi_2|^{\frac{1}{\sigma_2}})}.$$

We notice that if $s_j + \sigma_j < 1$ for some $j = 1, 2$, then $\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})$ and $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})$ are equal to the trivial space $\{0\}$. Likewise, if $s_j = \sigma_j = \frac{1}{2}$ for some $j = 1, 2$, then $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2}) = \{0\}$.

1.2. The Short Time Fourier Transform and Gelfand–Shilov Spaces

We recall here some basic facts about the short-time Fourier transform and weights.

Let $\phi \in \mathcal{S}_s^\sigma(\mathbf{R}^d) \backslash 0$ be fixed. Then the short-time Fourier transform of $f \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ is given by

$$(V_\phi f)(x, \xi) = (2\pi)^{-\frac{d}{2}} (f, \phi(\,\cdot\, - x) e^{i\langle\,\cdot\,,\xi\rangle})_{L^2}.$$

Here $(\,\cdot\,,\,\cdot\,)_{L^2}$ is the unique extension of the L^2-form on $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ to a continuous sesqui-linear form on $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d) \times \mathcal{S}_s^\sigma(\mathbf{R}^d)$. In the case $f \in L^p(\mathbf{R}^d)$, for some $p \in [1,\infty]$, then $V_\phi f$ is given by

$$V_\phi f(x,\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)}e^{-i\langle y,\xi\rangle}\, dy.$$

The following characterizations of the $\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})$, $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})$ and their duals follow by similar arguments as in the proofs of Propositions 2.1 and 2.2 in [32]. The details are left for the reader.

Proposition 1.3. *Let $s_j,\sigma_j > 0$ be such that $s_j+\sigma_j \ge 1$, $j=1,2$. Also let $\phi \in \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})\backslash 0$ and let f be a Gelfand–Shilov distribution on $\mathbf{R}^{d_1+d_2}$. Then the following is true:*

(1) *$f \in \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})$, if and only if*

$$|V_\phi f(x_1,x_2,\xi_1,\xi_2)| \lesssim e^{-r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}}+|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})}, \tag{1.5}$$

holds for some $r>0$;

(2) *if in addition $\phi \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})\backslash 0$, then $f \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})$ if and only if*

$$|V_\phi f(x_1,x_2,\xi_1,\xi_2)| \lesssim e^{-r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}}+|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})} \tag{1.6}$$

holds for every $r>0$.

A proof of Proposition 1.3 can be found in e. g. [21] (cf. [21, Theorem 2.7]). The corresponding result for Gelfand–Shilov distributions is the following improvement of [31, Theorem 2.5]. See also [32].

Proposition 1.4. *Let $s_j,\sigma_j > 0$ be such that $s_j+\sigma_j \ge 1$, $j=1,2$. Also let $\phi \in \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})\backslash 0$ and let f be a Gelfand–Shilov distribution on $\mathbf{R}^{d_1+d_2}$. Then the following is true:*

(1) *$f \in (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbf{R}^{d_1+d_2})$, if and only if*

$$|V_\phi f(x_1,x_2,\xi_1,\xi_2)| \lesssim e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}}+|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})} \tag{1.7}$$

holds for every $r>0$;

(2) *if in addition $\phi \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})\backslash 0$, then $f \in (\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbf{R}^{d_1+d_2})$, if and only if*

$$|V_\phi f(x_1,x_2,\xi_1,\xi_2)| \lesssim e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}}+|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})} \tag{1.8}$$

holds for some $r>0$.

Remark 1.5. We notice that any short-time Fourier transform of a Gelfand–Shilov distribution with window function as Gelfand–Shilov function or even a Schwartz function makes sense as a Gelfand–Shilov distribution.

In fact, let

$$T_1 : (\mathcal{S}_s^\sigma)'(\mathbf{R}^d) \times (\mathcal{S}_s^\sigma)'(\mathbf{R}^d) \to (\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d}),$$

and

$$T_2 : (\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d}) \to (\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d})$$

be the continuous mappings

$$T_1(f,\phi) = f \otimes \overline{\phi}, \quad f,\phi \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d),$$

and

$$(T_2F)(x,y) = F(y, y-x), \quad F \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d}).$$

Also let $(\mathscr{F}_2F)(x,\,\cdot\,)$ be the partial Fourier transform of $F(x,y)$ with respect to $y \in \mathbf{R}^d$, which is continuous from $(\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d})$ to $(\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$. Then

$$V_\phi f = (\mathscr{F}_2 \circ T_2 \circ T_1)(f,\phi). \tag{1.9}$$

By defining $V_\phi f$ as the right-hand side of (1.9) when $f,\phi \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$, it follows that the map

$$(f,\phi) \mapsto V_\phi f \tag{1.10}$$

is continuous from $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d) \times (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ to $(\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$.

In the same way (1.10) extends uniquely to a continuous map from $(\Sigma_s^\sigma)'(\mathbf{R}^d) \times (\Sigma_s^\sigma)'(\mathbf{R}^d)$ to $(\Sigma_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$.

By similar arguments it follows that if $f,\phi \in (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbf{R}^{d_1+d_2})$ ($f,\phi \in (\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbf{R}^{d_1+d_2})$), then $V_\phi f$ is still defined as some sort of Gelfand–Shilov distibution, given as the dual of a Gelfand–Shilov space, defined in terms of Komatsu functions (see e. g. [13]).

1.3. Weight Functions

A function ω on $\mathbf{R}^d$ is called a *weight* or *weight function* if $\omega, 1/\omega \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$ are positive everywhere. It is often assumed that ω is *moderate*. This means that

$$\omega(x+y) \lesssim \omega(x)v(y), \quad x,y \in \mathbf{R}^d. \tag{1.11}$$

for some positive function v on $\mathbf{R}^d$. If (1.11) is fulfilled for some positive function v on $\mathbf{R}^d$, then ω is also called v-moderate. The positive function $v \in L^\infty_{loc}(\mathbf{R}^d)$ is called submultiplicative if it is even and (1.11) holds with $\omega = v$. We let $\mathscr{P}_E(\mathbf{R}^d)$ be the set of all moderate functions on $\mathbf{R}^d$.

For any $s > 0$, let $\mathscr{P}_s(\mathbf{R}^d)$ ($\mathscr{P}_s^0(\mathbf{R}^d)$) be the set of all weights ω on $\mathbf{R}^d$ such that

$$\omega(x+y) \lesssim \omega(x)e^{r|y|^{\frac{1}{s}}}$$

for some $r > 0$ (for every $r > 0$).

More generally, if $d = d_1 + d_2$ with $d_2 \geq 0$ and $(s_1, s_2) \in \mathbf{R}_+^2$, then we let

$$\mathscr{P}_s(\mathbf{R}^d) = \mathscr{P}_{s_1,s_2}(\mathbf{R}^{d_1+d_2}) \quad (\mathscr{P}_s^0(\mathbf{R}^d) = \mathscr{P}^0_{s_1,s_2}(\mathbf{R}^{d_1+d_2}))$$

be the set of all weight functions ω on $\mathbf{R}^{d_1+d_2}$ such that

$$\omega(x_1+y_1, x_2+y_2) \lesssim \omega(x_1,x_2)e^{r(|y_1|^{\frac{1}{s_1}} + |y_2|^{\frac{1}{s_2}})}, \quad x_j, y_j \in \mathbf{R}^{d_j},\ j = 1,2, \tag{1.12}$$

for some $r > 0$ (for every $r > 0$). In particular, if $\omega \in \mathscr{P}_{s_1,s_2}(\mathbf{R}^{d_1+d_2})$ ($\omega \in \mathscr{P}^0_{s_1,s_2}(\mathbf{R}^{d_1+d_2})$), then

$$e^{-r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}})} \lesssim \omega(x_1,x_2) \lesssim e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}})} \tag{1.13}$$

for some $r > 0$ (for every $r > 0$).

The following proposition shows among others limitations concerning growths and decays for moderate weights.

Proposition 1.6. *Let $d = d_1 + d_2$ and $s, s_j, t_j \in \mathbf{R}^2_+$ be such that $d_j \ge 0$ are integers and $t_j = \max(1, s_j)$, $j = 1, 2$. Then the set $\mathscr{P}_s(\mathbf{R}^d)$ is non-increasing with respect to s,*

$$\mathscr{P}_{s_1,s_2}(\mathbf{R}^{d_1+d_2}) = \mathscr{P}_{t_1,t_2}(\mathbf{R}^{d_1+d_2}), \tag{1.14}$$

and

$$\mathscr{P}_{1,1}(\mathbf{R}^{d_1+d_2}) = \mathscr{P}_E(\mathbf{R}^{d_1+d_2}). \tag{1.15}$$

The statements in Proposition 1.6 are essentially presented at different places in the literature (cf. [20,30,31]). For conveniency we present a proof in Appendix A, and refer to [20,31] for more facts about weights.

1.4. Pseudo-Differential Operators

Let $\mathbf{M}(d,\mathbf{R})$ be the set of all $d \times d$-matrices with entries in $\mathbf{R}$, $A \in \mathbf{M}(d,\mathbf{R})$ and $s \ge \frac{1}{2}$ be fixed, and let $a \in \mathcal{S}_s(\mathbf{R}^{2d})$. Then the *pseudo-differential operator* $\operatorname{Op}_A(a)$ with symbol a is the continuous operator on $\mathcal{S}_s(\mathbf{R}^d)$ is defined by the formula

$$(\operatorname{Op}_A(a)f)(x) = (2\pi)^{-d} \iint a(x - A(x-y), \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi. \tag{1.16}$$

We set $\operatorname{Op}_t(a) = \operatorname{Op}_A(a)$ when $t \in \mathbf{R}$, $A = t \cdot I$ and I is the identity matrix, and notice that this definition agrees with the Shubin type pseudo-differential operators (cf. e. g. [30]).

If instead $s, \sigma > 0$ are such that $s + \sigma \ge 1$, $a \in (\mathcal{S}^{\sigma,s}_{s,\sigma})'(\mathbf{R}^{2d})$, then $\operatorname{Op}_A(a)$ is defined to be the linear and continuous operator from $\mathcal{S}^\sigma_s(\mathbf{R}^d)$ to $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$ with the kernel in $(\mathcal{S}^\sigma_s)'(\mathbf{R}^{2d})$, given by

$$K_{a,A}(x,y) \equiv (\mathscr{F}_2^{-1}a)(x - A(x-y), x-y). \tag{1.17}$$

It is easily seen that the latter definition agrees with (1.16) when $a \in L^1(\mathbf{R}^{2d})$.

If $t = \frac{1}{2}$, then $\operatorname{Op}_t(a)$ is equal to the Weyl operator $\operatorname{Op}^w(a)$ for a. If instead $t = 0$, then the standard (Kohn–Nirenberg) representation $a(x,D)$ is obtained.

1.5. Symbol Classes

Next we introduce function spaces related to symbol classes of the pseudo-differential operators. These functions should obey various conditions of the form

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x,\xi), \tag{1.18}$$

on the phase space $\mathbf{R}^{2d}$. For this reason we consider norms of the form

$$\|a\|_{\Gamma^{\sigma,s;h}_{(\omega)}} \equiv \sup_{\alpha,\beta\in\mathbf{N}^d} \left(\sup_{x,\xi\in\mathbf{R}^d} \left(\frac{|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|}{h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x,\xi)} \right) \right), \tag{1.19}$$

indexed by $h > 0$.

Definition 1.7. Let s, σ and h be positive constants, let ω be a weight on $\mathbf{R}^{2d}$, and let

$$\omega_r(x,\xi) \equiv e^{r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}.$$

(1) The set $\Gamma^{\sigma,s;h}_{(\omega)}(\mathbf{R}^{2d})$ consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that $\|a\|_{\Gamma^{\sigma,s;h}_{(\omega)}}$ in (1.19) is finite. The set $\Gamma^{\sigma,s;h}_{0}(\mathbf{R}^{2d})$ consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that $\|a\|_{\Gamma^{\sigma,s;h}_{(\omega_r)}}$ is finite for every $r > 0$, and the topology is the projective limit topology of $\Gamma^{\sigma,s;h}_{(\omega_r)}(\mathbf{R}^{2d})$ with respect to $r > 0$;

(2) The sets $\Gamma^{\sigma,s}_{(\omega)}(\mathbf{R}^{2d})$ and $\Gamma^{\sigma,s;0}_{(\omega)}(\mathbf{R}^{2d})$ are given by

$$\Gamma^{\sigma,s}_{(\omega)}(\mathbf{R}^{2d}) \equiv \bigcup_{h>0} \Gamma^{\sigma,s;h}_{(\omega)}(\mathbf{R}^{2d}) \quad \text{and} \quad \Gamma^{\sigma,s;0}_{(\omega)}(\mathbf{R}^{2d}) \equiv \bigcap_{h>0} \Gamma^{\sigma,s;h}_{(\omega)}(\mathbf{R}^{2d}),$$

and their topologies are the inductive and the projective topologies of $\Gamma^{\sigma,s;h}_{(\omega)}(\mathbf{R}^{2d})$ respectively, with respect to $h > 0$.

Furthermore we have the following classes.

Definition 1.8. For $s_j, \sigma_j > 0$, $j = 1, 2$, $h, r > 0$ and $f \in C^\infty(\mathbf{R}^{d_1+d_2})$, let

$$\|f\|_{(h,r)} \equiv \sup \left(\frac{|\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} f(x_1,x_2)|}{h^{|\alpha_1+\alpha_2|}\alpha_1!^{\sigma_1}\alpha_2!^{\sigma_2} e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}})}} \right), \tag{1.20}$$

where the supremum is taken over all $\alpha_1 \in \mathbf{N}^{d_1}, \alpha_2 \in \mathbf{N}^{d_2}, x_1 \in \mathbf{R}^{d_1}$ and $x_2 \in \mathbf{R}^{d_2}$.

(1) $\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbf{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbf{R}^{d_1+d_2})$ such that $\|f\|_{(h,r)}$ is finite for some $h, r > 0$;

(2) $\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2;0}(\mathbf{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbf{R}^{d_1+d_2})$ such that for some $h > 0$, $\|f\|_{(h,r)}$ is finite for every $r > 0$;

(3) $\Gamma^{\sigma_1,\sigma_2;0}_{s_1,s_2}(\mathbf{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbf{R}^{d_1+d_2})$ such that for some $r > 0$, $\|f\|_{(h,r)}$ is finite for every $h > 0$;

(4) $\Gamma^{\sigma_1,\sigma_2;0}_{s_1,s_2;0}(\mathbf{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbf{R}^{d_1+d_2})$ such that $\|f\|_{(h,r)}$ is finite for every $h, r > 0$.

In order to define suitable topologies of the spaces in Definition 1.8, let $(\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbf{R}^{d_1+d_2})$ be the set of $f \in C^\infty(\mathbf{R}^{d_1+d_2})$ such that $\|f\|_{(h,r)}$ is finite. Then $(\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbf{R}^{d_1+d_2})$ is a Banach space, and the sets in Definition 1.8 are given by

$$\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbf{R}^{d_1+d_2}) = \bigcup_{h,r>0} (\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbf{R}^{d_1+d_2}),$$

$$\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2;0}(\mathbf{R}^{d_1+d_2}) = \bigcup_{h>0}\left(\bigcap_{r>0}(\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbf{R}^{d_1+d_2})\right),$$

$$\Gamma^{\sigma_1,\sigma_2;0}_{s_1,s_2}(\mathbf{R}^{d_1+d_2}) = \bigcup_{r>0}\left(\bigcap_{h>0}(\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbf{R}^{d_1+d_2})\right)$$

and

$$\Gamma^{\sigma_1,\sigma_2;0}_{s_1,s_2;0}(\mathbf{R}^{d_1+d_2}) = \bigcap_{h,r>0}(\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbf{R}^{d_1+d_2}),$$

and we equip these spaces by suitable mixed inductive and projective limit topologies of $(\Gamma^{\sigma_1,\sigma_2}_{s_1,s_2})_{(h,r)}(\mathbf{R}^{d_1+d_2})$.

2. Characterizations of Symbols via the Short-Time Fourier Transform

In this section we characterize the symbol class from the previous section in term of estimates of their short-time Fourier transform.

In what follows we let κ be defined as

$$\kappa(r) = \begin{cases} 1 & \text{when } r \le 1, \\ 2^{r-1} & \text{when } r > 1. \end{cases} \tag{2.1}$$

In the sequel we shall frequently use the inequality

$$|x+y|^{\frac{1}{s}} \le \kappa(s^{-1})\left(|x|^{\frac{1}{s}} + |y|^{\frac{1}{s}}\right), \qquad s>0, \quad x,y \in \mathbf{R}^d,$$

which follows by straight-forward computations.

Proposition 2.1. *Let $s,\sigma > 0$ be such that $s+\sigma \ge 1$ and $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, $\phi \in \Sigma_s^\sigma(\mathbf{R}^d)\backslash 0$, $r>0$ and let f be a Gelfand–Shilov distribution on $\mathbf{R}^d$. Then the following is true:*

(1) *If $f \in C^\infty(\mathbf{R}^d)$ and satisfies*

$$|\partial^\alpha f(x)| \lesssim h^{|\alpha|}\alpha!^\sigma e^{r|x|^{\frac{1}{s}}} \tag{2.2}$$

for every $h>0$ (for some $h>0$), then

$$|V_\phi f(x,\xi)| \lesssim e^{\kappa(s^{-1})r|x|^{\frac{1}{s}} - h|\xi|^{\frac{1}{\sigma}}} \tag{2.3}$$

for every $h>0$ (for some new $h>0$);

(2) *If*

$$|V_\phi f(x,\xi)| \lesssim e^{r|x|^{\frac{1}{s}} - h|\xi|^{\frac{1}{\sigma}}} \tag{2.4}$$

for every $h>0$ (for some $h>0$), then $f \in C^\infty(\mathbf{R}^d)$ and satisfies

$$|\partial^\alpha f(x)| \lesssim h^{|\alpha|}\alpha!^\sigma e^{\kappa(s^{-1})r|x|^{\frac{1}{s}}}$$

for every $h>0$ (for some new $h>0$).

Remark 2.2. If $s \ge 1$, then Proposition 2.1 is a special case of [16, Theorem 3.1]. In fact, the latter result asserts that Proposition 2.1 holds true for a larger class of window functions and with a general moderate function $\omega(x)$ in place of the moderate functions

$$x \mapsto e^{r|x|^{\frac{1}{s}}} \quad \text{and} \quad x \mapsto e^{\kappa(s^{-1})r|x|^{\frac{1}{s}}}.$$

On the other hand, if instead $s < 1$, then the latter functions fail to be moderate. Consequently, [16, Theorem 3.1] does not cover Proposition 2.1 in the case when $s < 1$.

Proof of Proposition 2.1. We only prove the assertion when (2.2) or (2.4) are true for every $h > 0$, leaving the straight-forward modifications of the other cases to the reader.

Assume that (2.2) holds. Then for every $x \in \mathbf{R}^d$ the function

$$y \mapsto F_x(y) \equiv f(y+x)\overline{\phi(y)}$$

belongs to $\Sigma_s^\sigma(\mathbf{R}^d)$, and

$$|\partial_y^\alpha F_x(y)| \lesssim h^{|\alpha|}\alpha!^\sigma e^{\kappa(s^{-1})r|x|^{\frac{1}{s}}} e^{-r_0|y|^{\frac{1}{s}}},$$

for every $h, r_0 > 0$. In particular,

$$|F_x(y)| \lesssim e^{\kappa(s^{-1})r|x|^{\frac{1}{s}}} e^{-r_0|y|^{\frac{1}{s}}} \quad \text{and} \quad |\widehat{F}_x(\xi)| \lesssim e^{\kappa(s^{-1})r|x|^{\frac{1}{s}}} e^{-r_0|\xi|^{\frac{1}{\sigma}}}, \tag{2.5}$$

for every $r_0 > 0$. Since $|V_\phi f(x,\xi)| = |\widehat{F}_x(\xi)|$, the estimate (2.3) follows from the second inequality in (2.5), and (1) follows.

Next we prove (2). By the inversion formula we get

$$f(x) = (2\pi)^{-\frac{d}{2}}\|\phi\|_{L^2}^{-2} \iint_{\mathbf{R}^{2d}} V_\phi f(y,\eta)\phi(x-y)e^{i\langle x,\eta\rangle}\, dy d\eta. \tag{2.6}$$

Here we notice that

$$(x,y,\eta) \mapsto V_\phi f(y,\eta)\phi(x-y)e^{i\langle x,\eta\rangle}$$

is smooth and

$$(y,\eta) \mapsto \eta^\alpha V_\phi f(y,\eta)\partial^\beta\phi(x-y)e^{i\langle x,\eta\rangle}$$

is an integrable function for every x, α and β, giving that f in (2.6) is smooth.

By differentiation and the fact that $\phi \in \Sigma_s^\sigma$ we get

$$\begin{aligned}
|\partial^\alpha f(x)| &\asymp \left|\sum_{\beta\le\alpha}\binom{\alpha}{\beta} i^{|\beta|}\iint_{\mathbf{R}^{2d}} \eta^\beta V_\phi f(y,\eta)(\partial^{\alpha-\beta}\phi)(x-y)e^{i\langle x,\eta\rangle}\, dy d\eta\right| \\
&\lesssim \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\iint_{\mathbf{R}^{2d}} |\eta^\beta e^{r|y|^{\frac{1}{s}}} e^{-h|\eta|^{\frac{1}{\sigma}}}(\partial^{\alpha-\beta}\phi)(x-y)|\, dy d\eta \\
&\lesssim \sum_{\beta\le\alpha}\binom{\alpha}{\beta} h_2^{|\alpha-\beta|}(\alpha-\beta)!^\sigma \iint_{\mathbf{R}^{2d}} |\eta^\beta| e^{-h|\eta|^{\frac{1}{\sigma}}} e^{r|y|^{\frac{1}{s}}} e^{-h_1|x-y|^{\frac{1}{s}}}\, dy d\eta,
\end{aligned}$$

for every $h_1, h_2 > 0$. Since

$$|\eta^\beta e^{-h|\eta|^{\frac{1}{\sigma}}}| \lesssim h_2^{|\beta|}(\beta!)^\sigma e^{-\frac{h}{2}\cdot|\eta|^{\frac{1}{\sigma}}}, \tag{2.7}$$

we get

$$\begin{aligned}&|\partial^\alpha f(x)|\\ &\lesssim h_2^{|\alpha|}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(\beta!(\alpha-\beta)!)^\sigma \iint_{\mathbf{R}^{2d}} e^{-\frac{h}{2}\cdot|\eta|^{\frac{1}{\sigma}}} e^{r|y|^{\frac{1}{s}}} e^{-h_1|x-y|^{\frac{1}{s}}}\, dy d\eta\\ &\lesssim (4h_2)^{|\alpha|}\alpha!^\sigma \int_{\mathbf{R}^d} e^{r|y|^{\frac{1}{s}}} e^{-h_1|x-y|^{\frac{1}{s}}}\, dy. \end{aligned}\tag{2.8}$$

Since $|y|^{\frac{1}{s}} \le \kappa(s^{-1})(|x|^{\frac{1}{s}} + |y-x|^{\frac{1}{s}})$ and h_1 can be chosen arbitrarily large, it follows from the last estimate that

$$|\partial^\alpha f(x)| \lesssim (4h_2)^{|\alpha|}\alpha!^\sigma e^{r\kappa(s^{-1})|x|^{\frac{1}{s}}},$$

for every $h_2 > 0$. □

By similar arguments we get the following result. The details are left for the reader.

Proposition 2.1′. *Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \ge 1$ and $(s_j, \sigma_j) \ne (\frac{1}{2}, \frac{1}{2})$, $j = 1, 2$, $\phi \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})\backslash 0$, $r > 0$ and let f be a Gelfand–Shilov distribution on $\mathbf{R}^{d_1+d_2}$. Then the following is true:*

(1) *if $f \in C^\infty(\mathbf{R}^{d_1+d_2})$ and satisfies*

$$|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} f(x_1, x_2)| \lesssim h^{|\alpha_1+\alpha_2|}\alpha_1!^{\sigma_1}\alpha_2!^{\sigma_2} e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}})} \tag{2.2$'$}$$

for every $h > 0$ (resp. for some $h > 0$), then

$$|V_\phi f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^{\kappa(s_1^{-1})r|x_1|^{\frac{1}{s_1}}+\kappa(s_2^{-1})r|x_2|^{\frac{1}{s_2}}-h(|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})} \tag{2.3$'$}$$

for every $h > 0$ (resp. for some new $h > 0$);

(2) *if*

$$|V_\phi f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^{r(|x_1|^{\frac{1}{s_1}}+|x_2|^{\frac{1}{s_2}})-h(|\xi_1|^{\frac{1}{\sigma_1}}+|\xi_2|^{\frac{1}{\sigma_2}})} \tag{2.4$'$}$$

for every $h > 0$ (resp. for some $h > 0$), then $f \in C^\infty(\mathbf{R}^{d_1+d_2})$ and satisfies

$$|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} f(x_1, x_2)| \lesssim h^{|\alpha_1+\alpha_2|}\alpha_1!^{\sigma_1}\alpha_2!^{\sigma_2} e^{\kappa(s_1^{-1})r|x_1|^{\frac{1}{s_1}}+\kappa(s_2^{-1})r|x_2|^{\frac{1}{s_2}}},$$

for every $h > 0$ (resp. for some new $h > 0$).

As a consequence of the previous result we get the following.

Proposition 2.3. *Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \ge 1$ and $(s_j, \sigma_j) \ne (\frac{1}{2}, \frac{1}{2})$, $j = 1, 2$, $\phi \in \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})\backslash 0$ and let f be a Gelfand–Shilov distribution on $\mathbf{R}^{d_1+d_2}$. Then the following is true:*

(1) *there exist $h > 0$ and $r > 0$ such that (2.4)′ holds if and only if $f \in \Gamma_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbf{R}^{d_1+d_2})$;*

(2) *there exists* $r > 0$ *such that* (2.4)′ *holds for every* $h > 0$ *if and only if* $f \in \Gamma^{\sigma_1,\sigma_2;0}_{s_1,s_2}(\mathbf{R}^{d_1+d_2})$;

(3) (2.4)′ *holds for every* $h > 0$ and $r > 0$ *if and only if* $f \in \Gamma^{\sigma_1,\sigma_2;0}_{s_1,s_2;0}(\mathbf{R}^{d_1+d_2})$.

By similar arguments that led to Proposition 2.3 we also get the following. The details are left for the reader.

Proposition 2.4. *Let* $s_j, \sigma_j > 0$ *be such that* $s_j + \sigma_j \ge 1$, $j = 1, 2$, $\phi \in \mathcal{S}^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbf{R}^{d_1+d_2})\setminus 0$ *and let* f *be a Gelfand–Shilov distribution on* $\mathbf{R}^{d_1+d_2}$. *Then there exists* $h > 0$ *such that* (2.4)′ *holds for every* $r > 0$, *if and only if* $f \in \Gamma^{\sigma_1,\sigma_2}_{s_1,s_2;0}(\mathbf{R}^{d_1+d_2})$.

We also have the following version of Proposition 2.1′, involving certain types of moderate weights.

Proposition 2.5. *Let* $s, \sigma > 0$ *be such that* $s + \sigma \ge 1$, $\phi \in \mathcal{S}^{\sigma,s}_{s,\sigma}(\mathbf{R}^{2d})\setminus 0$ ($\phi \in \Sigma^{\sigma,s}_{s,\sigma}(\mathbf{R}^{2d})\setminus 0$), $\omega \in \mathscr{P}^0_{s,\sigma}(\mathbf{R}^{2d})$ ($\omega \in \mathscr{P}_{s,\sigma}(\mathbf{R}^{2d})$) *and let* a *be a Gelfand–Shilov distribution on* $\mathbf{R}^{2d}$. *Then the following is true:*

(1) *if* $a \in C^\infty(\mathbf{R}^{2d})$ *and satisfies*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x,\xi), \tag{2.9}$$

for some $h > 0$ (*for every* $h > 0$), *then*

$$|V_\phi a(x,\xi,\eta,y)| \lesssim \omega(x,\xi) e^{-r(|\eta|^{\frac{1}{\sigma}} + |y|^{\frac{1}{s}})}, \tag{2.10}$$

for some $r > 0$ (*for every* $r > 0$);

(2) *if* (2.10) *holds for some* $r > 0$ (*for every* $r > 0$), *then* $a \in C^\infty(\mathbf{R}^{2d})$ *and* (2.9) *holds for some* $h > 0$ (*for every* $h > 0$).

We note that [16, Theorem 3.1] is more general than Proposition 2.5 when $s = \sigma$, since the former result is valid for a strictly larger class of window functions. It is also evident that Proposition 2.5 follows from [16, Theorem 3.1] and its proof, also when s and σ are allowed to be different. In order to be self-contained we present a short proof of Proposition 2.5 in Appendix B, where the first part is slightly different compared to the proof of [16, Theorem 3.1].

3. Invariance, Continuity and Algebraic Properties for Pseudo-Differential Operators

In this section we deduce invariance, continuity and composition properties for pseudo-differential operators with symbols in the classes considered in the previous sections. In the first part we show that for any such class S, the set $\operatorname{Op}_A(S)$ of pseudo-differential operators is independent of the matrix A. Thereafter we show that such operators are continuous on Gelfand–Shilov spaces and their duals. In the last part we deduce that these operator classes are closed under compositions.

3.1. Invariance Properties

An essential part of the study of invariance properties concerns the operator $e^{i\langle AD_\xi,D_x\rangle}$ when acting on the symbol classes in the previous sections.

Theorem 3.1. *Let $s,s_1,s_2,\sigma,\sigma_1,\sigma_2>0$ be such that*

$$s+\sigma\ge 1,\quad s_1+\sigma_1\ge 1,\quad s_2+\sigma_2\ge 1,\quad s_2\le s_1\quad \text{and}\quad \sigma_1\le\sigma_2,$$

and let $A\in \mathbf{M}(d,\mathbf{R})$. Then the following is true:

1. *$e^{i\langle AD_\xi,D_x\rangle}$ on $\mathscr{S}(\mathbf{R}^{2d})$ restricts to a homeomorphism on $\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbf{R}^{2d})$, and extends uniquely to a homeomorphism on $(\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbf{R}^{2d})$;*
2. *if in addition $(s_1,\sigma_1)\neq(\frac12,\frac12)$ and $(s_2,\sigma_2)\neq(\frac12,\frac12)$, then $e^{i\langle AD_\xi,D_x\rangle}$ restricts to a homeomorphism on $\Sigma_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbf{R}^{2d})$, and extends uniquely to a homeomorphism on $(\Sigma_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbf{R}^{2d})$;*
3. *$e^{i\langle AD_\xi,D_x\rangle}$ is a homeomorphism on $\Gamma^{\sigma,s}_{s,\sigma;0}(\mathbf{R}^{2d})$;*
4. *if in addition $(s,\sigma)\neq(\frac12,\frac12)$, then $e^{i\langle AD_\xi,D_x\rangle}$ is a homeomorphism on $\Gamma^{\sigma,s;0}_{s,\sigma}(\mathbf{R}^{2d})$ and on $\Gamma^{\sigma,s;0}_{s,\sigma;0}(\mathbf{R}^{2d})$.*

The assertion (1) in the previous theorem is proved in [9] and is essentially a special case of Theorem 32 in [34], whereas (2) can be found in [9,10]. Thus we only need to prove (3) and (4) in the previous theorem, which are extensions of [9, Theorem 4.6 (3)].

Proof. Let $\phi\in\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$, $\phi_A=e^{i\langle AD_\xi,D_x\rangle}\phi$ and let A^* be the transpose of A. Then $\phi_A\in\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ in view of (1) and

$$|(V_{\phi_A}(e^{i\langle AD_\xi,D_x\rangle}a))(x,\xi,\eta,y)|=|(V_\phi a)(x-Ay,\xi-A^*\eta,\eta,y)| \tag{3.1}$$

by straight-forward computations. Then $a\in\Gamma^{\sigma,s}_{s,\sigma;0}(\mathbf{R}^{2d})$ is equivalent to that for some $h>0$,

$$|V_\phi a(x,\xi,\eta,y)|\lesssim e^{r(|x|^{\frac1s}+|\xi|^{\frac1\sigma})-h(|\eta|^{\frac1\sigma}+|y|^{\frac1s})},$$

holds for every $r>0$, in view of Proposition 2.4. By (3.1) and (1) it follows by straight-forward computation, that the latter condition is invariant under the mapping $e^{i\langle AD_\xi,D_x\rangle}$, and (3) follows from these invariance properties. By similar arguments, taking $\phi\in\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ and using (2) instead of (1), we deduce (4). □

Corollary 3.2. *Let $s,\sigma>0$ be such that $s+\sigma\ge1$ and $\sigma\le s$. Then $e^{i\langle AD_\xi,D_x\rangle}$ is a homeomorphism on $\mathcal{S}_s^\sigma(\mathbf{R}^{2d})$, $\Sigma_s^\sigma(\mathbf{R}^{2d})$, $(\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d})$ and on $(\Sigma_s^\sigma)'(\mathbf{R}^{2d})$.*

We also have the following extension of (4) in [9, Theorem 4.1].

Theorem 3.3. *Let $\omega\in\mathscr{P}_{s,\sigma}(\mathbf{R}^{2d})$, $s,\sigma>0$ be such that $s+\sigma\ge1$. Then $a\in\Gamma^{\sigma,s;0}_{(\omega)}(\mathbf{R}^{2d})$ if and only if $e^{i\langle AD_\xi,D_x\rangle}a\in\Gamma^{\sigma,s;0}_{(\omega)}(\mathbf{R}^{2d})$.*

We need some preparation for the proof and start with the following proposition.

Proposition 3.4. *Let $s,\sigma > 0$ be such that $s+\sigma \ge 1$ and $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, $\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})\backslash 0$, $\omega \in \mathscr{P}_{s,\sigma}(\mathbf{R}^{2d})$ and let a be a Gelfand–Shilov distribution on $\mathbf{R}^{2d}$. Then the following conditions are equivalent:*

(1) $a \in \Gamma_{(\omega)}^{\sigma,s;0}(\mathbf{R}^{2d})$;

(2) *for every $\alpha,\beta \in \mathbf{N}^d$, $h>0$, $r>0$ and x,y,ξ,η in $\mathbf{R}^d$ it holds*

$$\left|\partial_x^\alpha \partial_\xi^\beta \left(e^{i(\langle x,\eta\rangle + \langle y,\xi\rangle)} V_\phi a(x,\xi,\eta,y)\right)\right| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x,\xi) e^{-r(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}; \tag{3.2}$$

(3) *for $\alpha=\beta=0$, (3.2) holds for every $h>0$, $r>0$ and $x,y,\xi,\eta \in \mathbf{R}^d$.*

Proof. Obviously, (2) implies (3). Assume now that (1) holds. Let

$$F_a(x,\xi,y,\eta) = a(x+y,\xi+\eta)\phi(y,\eta).$$

By straight-forward application of Leibniz rule in combination with (1.12) we obtain

$$|\partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma \partial_\eta^\delta F_a(x,\xi,y,\eta)| \lesssim h^{|\alpha+\beta|}(\alpha!\gamma!)^\sigma (\beta!\delta!)^s \omega(x,\xi) e^{-r(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}$$

for every $h>0$ and $R>0$. Hence, if

$$G_{a,h,x,\xi}(y,\eta) = \frac{\partial_x^\alpha \partial_\xi^\beta F_a(x,\xi,y,\eta)}{h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x,\xi)},$$

then $\{\, G_{a,h,x,\xi}\, ;\, x,\xi \in \mathbf{R}^d \,\}$ is a bounded set in $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ for every fixed $h>0$. Let $\mathscr{F}_2 F_a$ be the partial Fourier transform of $F_a(x,\xi,y,\eta)$ with respect to the (y,η)-variable. Then

$$|\partial_x^\alpha \partial_\xi^\beta (\mathscr{F}_2 F_a)(x,\xi,\zeta,z)| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s \omega(x,\xi) e^{-r(|z|^{\frac{1}{s}}+|\zeta|^{\frac{1}{\sigma}})},$$

for every $h>0$ and $r>0$. This is the same as (2).

It remains to prove that (3) implies (1), but this follows by similar arguments as in the proof of Proposition 2.1. The details are left for the reader. □

Proposition 3.5. *Let $r>0$, $q \in [1,\infty]$, $s,\sigma>0$ be such that $s+\sigma \ge 1$ and $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, $\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})\backslash 0$, $\omega \in \mathscr{P}_{s,\sigma}(\mathbf{R}^{2d})$, and let*

$$\omega_r(x,\xi,\eta,y) = \omega(x,\xi) e^{-r(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}.$$

Then

$$\Gamma_{(\omega)}^{\sigma,s;0}(\mathbf{R}^{2d}) = \bigcap_{r>0} \{\, a \in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})\, ;\, \|\omega_r^{-1} V_\phi a\|_{L^{\infty,q}} < \infty \,\}. \tag{3.3}$$

Proof. Let $\phi_0 \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})\backslash 0$, $a \in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$, and set

$$F_{0,a}(X,Y) = |(V_{\phi_0} a)(x,\xi,\eta,y)|, \quad F_a(X,Y) = |(V_\phi a)(x,\xi,\eta,y)|$$
$$\text{and} \quad G(x,\xi,\eta,y) = |(V_\phi \phi_0)(x,\xi,\eta,y)|,$$

where $X=(x,\xi)$ and $Y=(y,\eta)$. Since $V_\phi \phi_0 \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{4d})$, we have

$$0 \le G(x,\xi,\eta,y) \lesssim e^{-r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}}+|\eta|^{\frac{1}{\sigma}}+|y|^{\frac{1}{s}})} \quad \text{for every} \quad r>0. \tag{3.4}$$

By [19, Lemma 11.3.3], we have $F_a \lesssim F_{0,a} * G$. We obtain

$$\begin{aligned}(\omega_r^{-1} \cdot F_a)(X,Y) \\ &\lesssim \omega(X)^{-1} e^{r(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})} \iint_{\mathbf{R}^{4d}} F_{0,a}(X-X_1,Y-Y_1)G(X_1,Y_1)\,dX_1dY_1 \\ &\lesssim \iint_{\mathbf{R}^{4d}} (\omega_{cr}^{-1} \cdot F_{0,a})(X-X_1,Y-Y_1)G_1(X_1,Y_1)\,dX_1dY_1 \end{aligned} \tag{3.5}$$

for some G_1 satisfying (3.4) in place of G and some $c > 0$ independent of R. By applying the L^∞-norm on the last inequality we get

$$\begin{aligned}\|\omega_r^{-1} F_a\|_{L^\infty(\mathbf{R}^{4d})} \\ &\lesssim \sup_Y \left(\iint_{\mathbf{R}^{4d}} \big(\sup(\omega_{cr}^{-1} \cdot F_{0,a})(\cdot, Y-Y_1)\big) G_1(X_1,Y_1)\,dX_1dY_1 \right) \\ &\le \sup_Y \big(\|(\omega_{cr}^{-1} \cdot F_{0,a})(\cdot - (0,Y)\|_{L^{\infty,q}} \big) \|G_1\|_{L^{1,q'}} \asymp \|\omega_{cr}^{-1} \cdot F_{0,a}\|_{L^{\infty,q}}.\end{aligned}$$

We only consider the case $q < \infty$ when proving the opposite inequality. The case $q = \infty$ follows by similar arguments and is left for the reader.

By (3.5) we have

$$\|\omega_r^{-1} \cdot F_a\|_{L^{\infty,q}}^q \lesssim \int_{\mathbf{R}^{2d}} \left(\sup H(\,\cdot\,,Y)^q\right)\,dY,$$

where $H = K_1 * G$ and $K_j = \omega_{jcr}^{-1} \cdot F_{0,a}$, $j \ge 1$. Let $Y_1 = (y_1,\eta_1)$ be new variables of integration. Then Minkowski's inequality gives

$$\begin{aligned}\sup_X H(X,Y) \\ &\lesssim \iint_{\mathbf{R}^{4d}} (\sup K_2(\,\cdot\,,Y-Y_1)) e^{-cr(|y-y_1|^{\frac{1}{s}}+|\eta-\eta_1|^{\frac{1}{\sigma}})} G(X_1,Y_1)\,dX_1dY_1 \\ &\lesssim \|K_2\|_{L^\infty} \iint_{\mathbf{R}^{4d}} e^{-cr(|y-y_1|^{\frac{1}{s}}+|\eta-\eta_1|^{\frac{1}{\sigma}})} G(X_1,Y_1)\,dX_1dY_1.\end{aligned}$$

By combining these estimates we get

$$\begin{aligned}\|\omega_r^{-1} \cdot F_a\|_{L^{\infty,q}}^q \\ &\lesssim \|K_2\|_{L^\infty}^q \int_{\mathbf{R}^{2d}} \left(\iint_{\mathbf{R}^{4d}} e^{-cr(|y-y_1|^{\frac{1}{s}}+|\eta-\eta_1|^{\frac{1}{\sigma}})} G(X_1,Y_1)\,dX_1dY_1 \right) dY \\ &\asymp \|K_2\|_{L^\infty}^q.\end{aligned}$$

That is,

$$\|\omega_r^{-1} \cdot F_a\|_{L^{\infty,q}} \lesssim \|\omega_{2cr}^{-1} \cdot F_{0,a}\|_{L^\infty},$$

and the result follows. □

Proof of Theorem 3.3. The case $s = \sigma = \frac{1}{2}$ follows from [9, Theorem 4.1]. We may therefore assume that $(s,\sigma) \ne (\frac{1}{2},\frac{1}{2})$. Let $\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ and $\phi_A = e^{i\langle AD_\xi, D_x\rangle}\phi$. Then $\phi_A \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$, in view of (2) in Theorem 3.1.

Also let

$$\omega_{A,r}(x,\xi,\eta,y) = \omega(x-Ay,\xi-A^*\eta)e^{-r(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}.$$

By straight-forward applications of Parseval's formula, we get

$$|(V_{\phi_A}(e^{i\langle AD_\xi,D_x\rangle}a))(x,\xi,\eta,y)| = |(V_\phi a)(x-Ay,\xi-A^*\eta,\eta,y)|$$

(cf. Proposition 1.7 in [30] and its proof). This gives

$$\|\omega_{0,r}^{-1}V_\phi a\|_{L^{p,q}} = \|\omega_{A,r}^{-1}V_{\phi_A}(e^{i\langle AD_\xi,D_x\rangle}a)\|_{L^{p,q}}.$$

Hence Proposition 3.5 gives

$$\begin{aligned}
a\in \Gamma^{\sigma,s;0}_{(\omega)}(\mathbf{R}^{2d}) \quad &\Leftrightarrow \quad \|\omega_{0,r}^{-1}V_\phi a\|_{L^\infty}<\infty \quad \text{for every } R>0\\
&\Leftrightarrow \quad \|\omega_{A,r}^{-1}V_{\phi_A}(e^{i\langle AD_\xi,D_x\rangle}a)\|_{L^\infty}<\infty \quad \text{for every } r>0\\
&\Leftrightarrow \quad \|\omega_{0,r}^{-1}V_{\phi_A}(e^{i\langle AD_\xi,D_x\rangle}a)\|_{L^\infty}<\infty \quad \text{for every } r>0\\
&\Leftrightarrow \quad e^{i\langle AD_\xi,D_x\rangle}a\in \Gamma^{\sigma,s;0}_{(\omega)}(\mathbf{R}^{2d}),
\end{aligned}$$

and the result follows in this case. Here the third equivalence follows from the fact that

$$\omega_{0,r+c}\lesssim \omega_{t,r}\lesssim \omega_{0,r-c},$$

for some $c>0$. □

We note that if $A,B\in \mathbf{M}(d,\mathbf{R})$ and $a,b\in (\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$ or $a,b\in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$, then Proposition 1.1 and its proof in [33] give

$$\operatorname{Op}_A(a)=\operatorname{Op}_B(b) \quad \Leftrightarrow \quad e^{i\langle AD_\xi,D_x\rangle}a=e^{i\langle BD_\xi,D_x\rangle}b. \tag{3.6}$$

The following result follows from Theorems 3.1 and 3.3. The details are left for the reader.

Theorem 3.6. *Let $s,s_1,s_2,\sigma,\sigma_1,\sigma_2>0$ be such that*

$$s+\sigma\ge 1,\quad s_1+\sigma_1\ge 1,\quad s_2+\sigma_2\ge 1,\quad s_2\le s_1 \quad \text{and} \quad \sigma_1\le\sigma_2,$$

$A,B\in \mathbf{M}(d,\mathbf{R})$, $\omega\in \mathscr{P}_{s,\sigma}(\mathbf{R}^{2d})$, and let a and b be Gelfand–Shilov distributions such that $\operatorname{Op}_A(a)=\operatorname{Op}_B(b)$. Then the following is true:

1. *$a\in \mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbf{R}^{2d})$ (resp. $a\in (\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbf{R}^{2d})$) if and only if $b\in \mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbf{R}^{2d})$ (resp. $b\in (\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbf{R}^{2d})$);*
2. *$a\in \Sigma_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbf{R}^{2d})$ (resp. $a\in (\Sigma_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbf{R}^{2d})$) if and only if $b\in \Sigma_{s_1,\sigma_2}^{\sigma_1,s_2}(\mathbf{R}^{2d})$ (resp. $b\in (\Sigma_{s_1,\sigma_2}^{\sigma_1,s_2})'(\mathbf{R}^{2d})$);*
3. *$a\in \Gamma^{\sigma,s}_{s,\sigma;0}(\mathbf{R}^{2d})$ if and only if $b\in \Gamma^{\sigma,s}_{s,\sigma;0}(\mathbf{R}^{2d})$. If in addition $(s,\sigma)\neq(\frac{1}{2},\frac{1}{2})$, then $a\in \Gamma^{\sigma,s;0}_{s,\sigma}(\mathbf{R}^{2d})$ if and only if $b\in \Gamma^{\sigma,s;0}_{s,\sigma}(\mathbf{R}^{2d})$, and $a\in \Gamma^{\sigma,s;0}_{s,\sigma;0}(\mathbf{R}^{2d})$ if and only if $b\in \Gamma^{\sigma,s;0}_{s,\sigma;0}(\mathbf{R}^{2d})$;*
4. *$a\in \Gamma^{\sigma,s;0}_{(\omega)}(\mathbf{R}^{2d})$ if and only if $b\in \Gamma^{\sigma,s;0}_{(\omega)}(\mathbf{R}^{2d})$.*

3.2. Continuity for Pseudo-Differential Operators with Symbols of Infinite Order on Gelfand–Shilov Spaces of Functions and Distributions

Next we deduce continuity for pseudo-differential operators with symbols in the classes given in Definitions 1.7 and 1.8. We begin with the case when the symbols belong to $\Gamma^{\sigma,s}_{(\omega)}(\mathbf{R}^{2d})$ or $\Gamma^{\sigma,s}_{s,\sigma;0}(\mathbf{R}^{2d})$.

Theorem 3.7. *Let* $A \in \mathbf{M}(d, \mathbf{R})$*,* $s, \sigma > 0$ *be such that* $s + \sigma \ge 1$*,* $\omega \in \mathscr{P}^0_{s,\sigma}(\mathbf{R}^{2d})$ *and let* $a \in \Gamma^{\sigma,s}_{(\omega)}(\mathbf{R}^{2d})$*. Then* $\operatorname{Op}_A(a)$ *is continuous on* $\mathcal{S}^\sigma_s(\mathbf{R}^d)$ *and on* $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$*.*

Since $\Gamma^{\sigma,s}_{(\omega)}(\mathbf{R}^{2d}) \subseteq \Gamma^{\sigma,s}_{s,\sigma;0}(\mathbf{R}^{2d})$ when $\omega \in \mathscr{P}^0_{s,\sigma}(\mathbf{R}^{2d})$, the preceding result is an immediate consequence of the following.

Theorem 3.8. *Let* $A \in \mathbf{M}(d, \mathbf{R})$*,* $s, \sigma > 0$ *be such that* $s + \sigma \ge 1$ *and let* $a \in \Gamma^{\sigma,s}_{s,\sigma;0}(\mathbf{R}^{2d})$*. Then* $\operatorname{Op}_A(a)$ *is continuous on* $\mathcal{S}^\sigma_s(\mathbf{R}^d)$ *and on* $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$*.*

Remark 3.9. Let s, σ and A be the same as in Theorem 3.8, $\omega \in \mathscr{P}^0_{s,\sigma}(\mathbf{R}^{2d})$ and let $a \in \Gamma^{\sigma,s}_{(\omega)}(\mathbf{R}^{2d})$. Then the following is true:

(1) Theorems 3.7 and 3.8 agree in the case when $s, \sigma \ge 1$;

(2) Theorem 3.7 is a strict subcase of Theorem 3.8 when $s < 1$ or $\sigma < 1$, because $\Gamma^{\sigma,s}_{(\omega)}(\mathbf{R}^{2d})$ is strictly contained in $\Gamma^{\sigma,s}_{s,\sigma;0}(\mathbf{R}^{2d})$ for such choices s and σ. For example, in this case, there are symbols a which satisfy the hypothesis in Theorem 3.8 and which grow superexponentially in some directions, while the symbols in Theorem 3.7 are allowed to grow at most exponentially, in view of Proposition 1.6;

(3) Proposition 4.7 in [16] is a consequence of Theorem 3.7. More precisely, if $s = \sigma \ge \frac{1}{2}$ and $\omega = 1$, then Theorem 3.7 agrees with Proposition 4.7 in [16], and asserts that $\operatorname{Op}^w(a)$ is continuous on $\mathcal{S}_s(\mathbf{R}^d)$;

(4) the analysis in [16] which lead to Proposition [16, Proposition 4.7], involving a technique on almost diagonalization for pseudo-differential operators can be performed to deduce Theorem 3.7 in the case $s = \sigma \ge \frac{1}{2}$. We note that as a corner stone in the analysis in [16], the weight ω needs to be moderate, giving that the symbols in [16] need to be bounded by exponential functions. Consequently, it seems impossible to include symbols with superexponential growth in both x and ξ in the analysis in [16]. In particular, Theorem 3.8 in the case $s = \sigma < 1$ seems not possible to reach with the methods in [16].

For the proof of Theorem 3.8 we need the following result.

Lemma 3.10. *Let* $s, \sigma > 0$ *be such that* $s + \sigma \ge 1$*,* $h_1 \ge 1$*,* Ω_1 *be a bounded set in* $\mathcal{S}^\sigma_{s;h_1}(\mathbf{R}^d)$*, and let*

$$h_2 \ge 2^{2+s} h_1 \quad \text{and} \quad h_3 \ge 2^{4+s+\sigma} h_1.$$

Then

$$\Omega_2 = \left\{ x \mapsto \frac{x^\gamma f(x)}{(2^{1+s}h_1)^{|\gamma|} \gamma!^s} \, ; \, f \in \Omega_1, \ \gamma \in \mathbf{N}^d \right\}$$

is a bounded set in $\mathcal{S}^\sigma_{s;h_2}(\mathbf{R}^d)$*, and*

$$\Omega_3 = \left\{ x \mapsto \frac{D^\delta x^\gamma f(x)}{(2^{3+s+\sigma}h_1)^{|\gamma+\delta|} \gamma!^s \delta!^\sigma} \, ; \, f \in \Omega_1, \ \gamma, \delta \in \mathbf{N}^d \right\}$$

is a bounded set in $\mathcal{S}^\sigma_{s;h_3}(\mathbf{R}^d)$*.*

Proof. Since Ω_1 is a bounded set in $\mathcal{S}^\sigma_{s;h_1}(\mathbf{R}^d)$, there is a constant $C>0$ such that

$$|x^\alpha D^\beta f(x)| \le C h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma, \quad \alpha,\beta\in\mathbf{N}^d, \tag{3.7}$$

for every $f\in\Omega_1$. We shall prove that (3.7) is true for all $f\in\Omega_2$ for a new choice of $C>0$, and h_2 in place of h_1.

Let $f\in\Omega_2$. Then

$$f(x)=\frac{x^\gamma f_0(x)}{(2^{1+s}h_1)^{|\gamma|}\gamma!^s}$$

for some $f_0\in\Omega_1$ and $\gamma\in\mathbf{N}^d$. Then

$$\begin{aligned}
|x^\alpha D^\beta f(x)| &= \left|\frac{x^\alpha D^\beta(x^\gamma f_0)(x)}{(2^{1+s}h_1)^{|\gamma|}\gamma!^s}\right|\\
&\le \sum_{\gamma_0\le\gamma,\beta}\binom{\beta}{\gamma_0}\frac{\gamma!}{(\gamma-\gamma_0)!}\cdot\frac{|x^{\alpha+\gamma-\gamma_0}\partial^{\beta-\gamma_0}f_0(x)|}{(2^{1+s}h_1)^{|\gamma|}\gamma!^s}\\
&\lesssim \sum_{\gamma_0\le\gamma,\beta}\binom{\beta}{\gamma_0}\binom{\gamma}{\gamma_0}\gamma_0!\cdot\frac{h_1^{|\alpha+\beta+\gamma-2\gamma_0|}(\alpha+\gamma-\gamma_0)!^s(\beta-\gamma_0)!^\sigma}{(2^{1+s}h_1)^{|\gamma|}\gamma!^s}\\
&\lesssim h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma\sum_{\gamma_0\le\gamma,\beta}\binom{\beta}{\gamma_0}\binom{\gamma}{\gamma_0}2^{-(1+s)|\gamma|}\left(\frac{(\alpha+\gamma-\gamma_0)!\gamma_0!}{\alpha!\gamma!}\right)^s\left(\frac{(\beta-\gamma_0)!\gamma_0!}{\beta!}\right)^\sigma\\
&\lesssim h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma\sum_{\gamma_0\le\gamma,\beta}\binom{\beta}{\gamma_0}\binom{\gamma}{\gamma_0}2^{-(1+s)|\gamma|}\left(\frac{(\alpha+\gamma-\gamma_0)!\gamma_0!}{(\alpha+\gamma)!}\right)^s\binom{\alpha+\gamma}{\gamma}^s\\
&\lesssim h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma\sum_{\gamma_0\le\gamma,\beta}2^{|\beta|}2^{|\gamma|}2^{-(1+s)|\gamma|}2^{s|\alpha+\gamma|}\\
&\lesssim 2^{s|\alpha|}2^{|\beta|}h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma\sum_{\gamma_0\le\beta}1.
\end{aligned}$$

Since

$$\sum_{\gamma_0\le\beta}1\lesssim 2^{|\beta|},$$

we get

$$|x^\alpha D^\beta f(x)|\le C2^{s|\alpha|}2^{2|\beta|}h_1^{|\alpha+\beta|}\alpha!^s\beta!^\sigma\le Ch_2^{|\alpha+\beta|}\alpha!^s\beta!^\sigma$$

for some constant C which is independent of f, and the assertion on Ω_2 follows.

The same type of arguments shows that

$$\left\{x\mapsto\frac{D^\delta f(x)}{(2^{1+s}h_1)^{|\delta|}\delta!^\sigma}\ ;\ f\in\Omega_1,\ \delta\in\mathbf{N}^d\right\} \tag{3.8}$$

is a bounded set in $\mathcal{S}^\sigma_{s;2^{2+\sigma}h_1}(\mathbf{R}^d)$, and the boundedness of Ω_3 in $\mathcal{S}^\sigma_{s;h_3}(\mathbf{R}^d)$ follows by combining the boundedness of Ω_2 and the boundedness of (3.8) in $\mathcal{S}^\sigma_{s;h_2}(\mathbf{R}^d)$. □

Lemma 3.11. *Let $s,\tau>0$, and set*

$$m_s(t)=\sum_{j=0}^\infty\frac{t^j}{(j!)^{2s}}\quad \textit{and}\quad m_{s,\tau}(x)=m_s(\tau\langle x\rangle^2)\quad t\ge 0,\ x\in\mathbf{R}^d.$$

Then

$$C^{-1}e^{(2s-\varepsilon)\tau^{\frac{1}{2s}}\langle x\rangle^{\frac{1}{s}}} \le m_{s,\tau}(x) \le Ce^{(2s+\varepsilon)\tau^{\frac{1}{2s}}\langle x\rangle^{\frac{1}{s}}}, \tag{3.9}$$

for every $\varepsilon > 0$, *and*

$$\frac{x^\alpha}{m_{s,\tau}(x)} \lesssim h_0^{|\alpha|}\alpha!^s e^{-r|x|^{\frac{1}{s}}}, \tag{3.10}$$

for some positive constant r *which depends on* d, s *and* τ *only.*

The estimate (3.9) follows from [23], and (3.10) also follows from computations given in e. g. [9,23]. For sake of completeness we present a proof of (3.10).

Proof. We have

$$\frac{x^\alpha}{m_{s,\tau}(x)} \lesssim \prod_{j=1}^{d} g_{\alpha_j}(x_j),$$

where

$$g_k(t) = t^k e^{-2r_0 t^{\frac{1}{s}}}, \qquad t \ge 0,$$

for some $r_0 > 0$ depending only on d, s and τ. Let

$$g_{0,k}(t) = C_k e^{-r_0 t}, \qquad t \ge 0,$$

where

$$C_k = \sup_{t\ge 0}(t^{sk}e^{-r_0 t}).$$

Then $g_k(t) \le g_{0,k}(t^{\frac{1}{s}})$, and the result follows if we prove $C_k \lesssim h_0^k k!^s$.

By straight-forward computations it follows that the maximum of $t^{sk}e^{-r_0 t}$ is attained at $t = sk/r_0$, giving that

$$C_k = \left(\frac{s}{r_0 e}\right)^{sk}(k^k)^s \lesssim h_0^k k!^s, \qquad h_0 = \left(\frac{s}{r_0}\right)^s,$$

where the last inequality follows from Stirling's formula. □

Proof of Theorem 3.8. By Theorem 3.1 it suffices to consider the case $A = 0$, that is for the operator

$$\mathrm{Op}_0(a)f(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbf{R}^d} a(x,\xi)\widehat{f}(\xi)e^{i\langle x,\xi\rangle}\, d\xi.$$

Observe that

$$\frac{1}{m_{s,\tau}(x)}\sum_{j=0}^{\infty}\frac{\tau^j}{(j!)^{2s}}(1-\Delta_\xi)^j e^{i\langle x,\xi\rangle} = e^{i\langle x,\xi\rangle}.$$

Let $h_1 > 0$ and $f \in \Omega$, where Ω is a bounded subset of $\mathcal{S}_{s,h_1}^{\sigma}(\mathbf{R}^d)$. For fixed $\alpha, \beta \in \mathbf{N}^d$ we get

$$\begin{aligned}
&(2\pi)^{\frac{d}{2}}x^\alpha D_x^\beta(\mathrm{Op}_0(a)f)(x)\\
&\quad = x^\alpha\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\int_{\mathbf{R}^d}\xi^\gamma D_x^{\beta-\gamma}a(x,\xi)\widehat{f}(\xi)e^{i\langle x,\xi\rangle}\, d\xi
\end{aligned}$$

$$= \frac{x^\alpha}{m_{s,\tau}(x)} \sum_{\gamma \le \beta} \binom{\beta}{\gamma} g_{\tau,\beta,\gamma}(x), \tag{3.11}$$

where

$$g_{\tau,\beta,\gamma}(x) = \sum_{j=0}^\infty \frac{\tau^j}{(j!)^{2s}} \int_{\mathbf{R}^d} (1-\Delta_\xi)^j \left(\xi^\gamma D_x^{\beta-\gamma} a(x,\xi) \widehat{f}(\xi) \right) e^{i\langle x,\xi \rangle}\, d\xi.$$

By Lemma 3.10 and the fact that $(2j)! \le 4^j j!^2$, it follows that for some $h>0$,

$$\Omega = \left\{ \xi \mapsto \frac{(1-\Delta_\xi)^j (\xi^\gamma D_x^\beta a(x,\xi) \widehat{f}(\xi))}{h^{|\beta+\gamma|+j} j!^{2s} \gamma!^\sigma \beta!^\sigma e^{r|x|^{\frac{1}{s}}}}\, ;\, j \ge 0,\ \beta,\gamma \in \mathbf{N}^d \right\}$$

is bounded in $\mathcal{S}_\sigma^s(\mathbf{R}^d)$ for every $r>0$. This implies that for some positive constants h and r_0 we get

$$|(1-\Delta_\xi)^j (\xi^\gamma D_x^\beta a(x,\xi) \widehat{f}(\xi))| \lesssim h^{|\beta+\gamma|+j} j!^{2s} \gamma!^\sigma \beta!^\sigma e^{r|x|^{\frac{1}{s}} - r_0 |\xi|^{\frac{1}{\sigma}}},$$

for every $r>0$. Hence,

$$\begin{aligned} |g_{\tau,\beta,\gamma}(x)| &\lesssim \sum_{j=0}^\infty \frac{\tau^j}{(j!)^{2s}} h^{|\beta|+j} j!^{2s} \gamma!^\sigma (\beta-\gamma)!^\sigma e^{r|x|^{\frac{1}{s}}} \int_{\mathbf{R}^d} e^{-r_0|\xi|^{\frac{1}{\sigma}}}\, d\xi \\ &\lesssim h^{|\beta|} \beta!^\sigma e^{r|x|^{\frac{1}{s}}} \sum_{j=0}^\infty (\tau h)^j \asymp h^{|\beta|} \beta!^\sigma e^{r|x|^{\frac{1}{s}}} \end{aligned}$$

for every $r>0$, provided τ is chosen such that $\tau h < 1$.

By inserting this into (3.11) and using Lemma 3.11 we get for some $h>0$ and some $r_0>0$ that

$$\begin{aligned} |x^\alpha D_x^\beta (\mathrm{Op}_0(a) f)(x)| &\lesssim h^{|\alpha|} \alpha!^s e^{-r_0|x|^{\frac{1}{s}}} \sum_{\gamma \le \beta} \binom{\beta}{\gamma} |g_{\tau,\beta,\gamma}(x)| \\ &\lesssim h^{|\alpha+\beta|} \alpha!^s \beta!^\sigma e^{-(r_0-r)|x|^{\frac{1}{s}}} \left(\sum_{\gamma \le \beta} 1 \right) \lesssim (2h)^{|\alpha+\beta|} \alpha!^s \beta!^\sigma, \end{aligned}$$

provided that r above is chosen to be smaller than r_0. Then the continuity of $\mathrm{Op}_A(a)$ on $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ follows. The continuity of $\mathrm{Op}_A(a)$ on $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ now follows from the preceding continuity and duality. □

Next we shall discuss corresponding continuity in the Beurling case. The main idea is to deduce such properties by suitable estimates on short-time Fourier transforms of involved functions and distributions. First we have the following relation between the short-time Fourier transforms of the symbols and kernels of a pseudo-differential operator.

Lemma 3.12. *Let $A \in \mathbf{M}(d,\mathbf{R})$, $s,\sigma > 0$ be such that $s+\sigma \ge 1$, $a \in (\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$ ($a \in (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$), $\phi \in \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ ($\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$), and let*

$$K_{a,A}(x,y) = (2\pi)^{-\frac{d}{2}} (\mathscr{F}_2^{-1} a)(x - A(x-y), x-y)$$

and

$$\psi(x,y) = K_{\phi,A}(x,y) = (2\pi)^{-\frac{d}{2}}(\mathscr{F}_2^{-1}\phi)(x-A(x-y),x-y)$$

be the kernels of $\operatorname{Op}_A(a)$ *and* $\operatorname{Op}_A(\phi)$*, respectively. Then*

$$\begin{aligned}(2\pi)^d e^{-i\langle x-y,\eta-A^*(\xi+\eta)\rangle}&(V_\psi K_{a,A})(x,y,\xi,\eta)\\ &= (V_\phi a)(x-A(x-y),-\eta+A^*(\xi+\eta),\xi+\eta,y-x). \qquad (3.12)\end{aligned}$$

The essential parts of (3.12) is presented in the proof of [33, Proposition 2.5]. In order to be self-contained we here present a short proof.

Proof. Let

$$T_A(x,y) = x - A(x-y)$$

and

$$Q = Q(x,x_1,y,\xi,\xi_1,\eta) = \langle x-y,\xi_1 - T_{A^*}(-\eta,\xi)\rangle - \langle x_1,\xi+\eta\rangle.$$

By formal computations, using Fourier's inversion formula we get

$$\begin{aligned}&(V_\psi K_{a,A})(x,y,\xi,\eta)\\ &= (2\pi)^{-3d}\iint K_{a,A}(x_1,y_1)\overline{\psi(x_1-x,y_1-y)}e^{-i(\langle x_1,\xi\rangle + \langle y_1,\eta\rangle)}\,dx_1dy_1\\ &= (2\pi)^{-2d}\iint a(x_1,\xi_1)\overline{\phi(x_1-T_A(x,y),\xi_1-T_{A^*}(-\eta,\xi))}e^{iQ(x,x_1,y,\xi,\xi_1,\eta)}\,dx_1d\xi_1\\ &= (2\pi)^{-d}e^{i\langle x-y,\eta-A^*(\xi+\eta)\rangle}(V_\phi a)(T_A(x,y),T_{A^*}(-\eta,\xi),\xi+\eta,y-x),\end{aligned}$$

where all integrals should be interpreted as suitable Fourier transforms. □

Before continuing discussing continuity of pseudo-differential operators, we observe that the previous lemma in combination with Propositions 2.3 and 2.4 give the following.

Proposition 3.13. *Let* $A \in \mathbf{M}(d,\mathbf{R})$*,* $s,\sigma > 0$ *be such that* $s+\sigma \ge 1$ *and* $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$*,* $\phi \in \Sigma_s^\sigma(\mathbf{R}^{2d})\backslash 0$*, a be a Gelfand–Shilov distribution on* $\mathbf{R}^{2d}$ *and let* $K_{a,A}$ *be the kernel of* $\operatorname{Op}_A(a)$*. Then the following conditions are equivalent:*

(1) $a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ *(resp.* $a \in \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d})$*);*

(2) *for some* $r > 0$*,*

$$|V_\phi K_{a,A}(x,y,\xi,\eta)| \lesssim e^{r(|x-A(x-y)|^{\frac{1}{s}}+|\eta-A^*(\xi+\eta)|^{\frac{1}{\sigma}})-h(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})}$$

holds for some $h > 0$ *(for every* $h > 0$*).*

By similar arguments we also get the following result. The details are left for the reader.

Proposition 3.14. *Let* $A \in \mathbf{M}(d,\mathbf{R})$*,* $s,\sigma > 0$ *be such that* $s+\sigma \ge 1$*,* $\phi \in \mathcal{S}_s^\sigma(\mathbf{R}^{2d})\backslash 0$*, a be a Gelfand–Shilov distribution on* $\mathbf{R}^{2d}$ *and let* $K_{a,A}$ *be the kernel of* $\operatorname{Op}_A(a)$*. Then the following conditions are equivalent:*

(1) $a \in \Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d})$ *(resp.* $a \in \Gamma_{s,\sigma;0}^{\sigma,s;0}(\mathbf{R}^{2d})$*);*

(2) *for some* $h>0$ *(for every* $h>0$*),*

$$|V_\phi K_{a,A}(x,y,\xi,\eta)| \lesssim e^{r(|x-A(x-y)|^{\frac{1}{s}}+|\eta-A^*(\xi+\eta)|^{\frac{1}{\sigma}})-h(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})}$$

holds for every $r>0$.

Theorem 3.15. *Let* $A\in \mathbf{M}(d,\mathbf{R})$*,* $s,\sigma>0$ *be such that* $s+\sigma\ge 1$ *and* $(s,\sigma)\neq(\frac{1}{2},\frac{1}{2})$*, and let* $a\in \Gamma^{\sigma,s;0}_{s,\sigma}(\mathbf{R}^{2d})$*. Then* $\operatorname{Op}_A(a)$ *is continuous on* $\Sigma_s^\sigma(\mathbf{R}^d)$*, and is uniquely extendable to a continuous map on* $(\Sigma_s^\sigma)'(\mathbf{R}^d)$*.*

Proof. By Theorem 3.1 we may assume that $A=0$. Let

$$g(x)=\operatorname{Op}_0(a)f(x)=(K_{a,0}(x,\,\cdot\,),\overline{f})=(h_{a,x},\overline{f}),$$

where $h_{a,x}=K_{a,0}(x,\,\cdot\,)$, and let $\phi_j\in\Sigma_s^\sigma(\mathbf{R}^d)$ be such that $\|\phi_j\|_{L^2}=1$, $j=1,2$. By Moyal's identity we get

$$g(x)=(h_{a,x},\overline{f})_{L^2(\mathbf{R}^d)}=(V_{\phi_1}h_{a,x},V_{\phi_1}\overline{f})_{L^2(\mathbf{R}^{2d})},$$

and applying the short-time Fourier transform on g and using Fubini's theorem on distributions we get

$$V_{\phi_2}g(x,\xi)=\langle J(x,\xi,\,\cdot\,),F\rangle,$$

where

$$F(y,\eta)=V_{\phi_1}f(y,-\eta),\quad J(x,\xi,y,\eta)=V_\phi K_{a,0}(x,y,\xi,\eta)$$

and $\phi=\phi_2\otimes\phi_1$.

Now suppose that $r>0$ is arbitrarily chosen. By Proposition 2.3 we get for some $c\in(0,1)$ which depends on s and σ only, that for some $r_0>0$ and with $r_1=(r+r_0)/c$ that

$$\begin{aligned}|J(x,\xi,y,\eta)| &\lesssim e^{r_0(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}e^{-r_1(|y-x|^{\frac{1}{s}}+|\xi+\eta|^{\frac{1}{\sigma}})}\\ &\lesssim e^{-((cr_1-r_0)|x|^{\frac{1}{s}}+cr_1|\xi|^{\frac{1}{\sigma}})}e^{r_1|y|^{\frac{1}{s}}+(r_1+r_0)|\eta|^{\frac{1}{\sigma}}}\\ &\lesssim e^{-r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}e^{r_2(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})},\end{aligned}$$

where r_2 only depends on r and r_0.

Since $f\in\Sigma_s^\sigma(\mathbf{R}^d)$ we have

$$|F(x,\xi)|\lesssim \|f\|_{S^\sigma_{s;h}}e^{-(1+r_2)(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})},$$

where $h>0$ only depends on r_2, and thereby depends only on r and r_0. This implies

$$\begin{aligned}|V_{\phi_2}g(x,\xi)| &= |\langle J(x,\xi,\,\cdot\,),F\rangle|\\ &\lesssim \|f\|_{S^\sigma_{s;h}}\left(\iint e^{r_2(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}e^{-(1+r_2)(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}\,dyd\eta\right)e^{-r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}\\ &\asymp \|f\|_{S^\sigma_{s;h}}e^{-r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{\sigma}})}\end{aligned}\tag{3.13}$$

which shows that $g\in\Sigma_s^\sigma(\mathbf{R}^d)$ in view of [32, Proposition 2.1].

Since the topology of $\Sigma_s^\sigma(\mathbf{R}^d)$ is given by the semi-norms

$$g \mapsto \sup_{x,\xi \in \mathbf{R}^d} \left| V_{\phi_2} g(x,\xi) e^{r(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{\sigma}})} \right|$$

it follows from (3.13) that $\operatorname{Op}(a)$ is continuous on $\Sigma_s^\sigma(\mathbf{R}^d)$.

By duality it follows that $\operatorname{Op}(a)$ is uniquely extendable to a continuous map on $(\Sigma_s^\sigma)'(\mathbf{R}^d)$. □

The following result follows by similar arguments as in the previous proof. The verifications are left for the reader.

Theorem 3.16. *Let $A \in \mathbf{M}(d,\mathbf{R})$, $s,\sigma > 0$ be such that $s+\sigma \ge 1$ and $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, and let $a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$. Then $\operatorname{Op}_A(a)$ is continuous from $\Sigma_s^\sigma(\mathbf{R}^d)$ to $\mathcal{S}_s^\sigma(\mathbf{R}^d)$, and from $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ to $(\Sigma_s^\sigma)'(\mathbf{R}^d)$.*

3.3. Compositions of Pseudo-Differential Operators

Next we deduce algebraic properties of pseudo-differential operators considered in Theorems 3.8, 3.15 and 3.16. We recall that for pseudo-differential operators with symbols in e. g. Hörmander classes, we have

$$\operatorname{Op}_0(a_1 \#_0 a_2) = \operatorname{Op}_0(a_1) \circ \operatorname{Op}_0(a_2),$$

when

$$a_1 \#_0 a_2(x,\xi) \equiv \big(e^{i\langle D_\xi, D_y\rangle}(a_1(x,\xi)a_2(y,\eta))\big)\Big|_{(y,\eta)=(x,\xi)}.$$

More generally, if $A \in \mathbf{M}(d,\mathbf{R})$ and $a_1 \#_A a_2$ is defined by

$$a_1 \#_A a_2 \equiv e^{i\langle AD_\xi, D_x\rangle}\Big(\big(e^{-i\langle AD_\xi, D_x\rangle}a_1\big)\#_0\big(e^{-i\langle AD_\xi, D_x\rangle}a_2\big)\Big), \tag{3.14}$$

for a_1 and a_2 belonging to certain Hörmander symbol classes, then it follows from the analysis in [22] that

$$\operatorname{Op}_A(a_1 \#_A a_2) = \operatorname{Op}_A(a_1) \circ \operatorname{Op}_A(a_2) \tag{3.15}$$

for suitable a_1 and a_2.

We recall that the map $a \mapsto K_{a,A}$ is a homeomorphism from $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\mathcal{S}_s^\sigma(\mathbf{R}^{2d})$ and from $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Sigma_s^\sigma(\mathbf{R}^{2d})$. It is also obvious that the map

$$(K_1,K_2) \mapsto \left((x,y) \mapsto (K_1 \circ K_2)(x,y) = \int_{\mathbf{R}^d} K_1(x,z)K_2(z,y)\,dz\right)$$

is sequentially continuous from $\mathcal{S}_s^\sigma(\mathbf{R}^{2d}) \times \mathcal{S}_s^\sigma(\mathbf{R}^{2d})$ to $\mathcal{S}_s^\sigma(\mathbf{R}^{2d})$, and from $\Sigma_s^\sigma(\mathbf{R}^{2d}) \times \Sigma_s^\sigma(\mathbf{R}^{2d})$ to $\Sigma_s^\sigma(\mathbf{R}^{2d})$. Here we have identified operators with their kernels. For compositions with three operator kernels we have

$$\begin{aligned} (K_1 \circ K_2 \circ K_3)(x,y) &= \langle K_2, T_{K_1,K_3}(x,y,\,\cdot\,)\rangle \\ \text{with}\quad T_{K_1,K_3}(x,y,z_1,z_2) &= K_1(x,z_1)K_3(z_2,y) \end{aligned} \tag{3.16}$$

when $K_j \in L^2(\mathbf{R}^{2d})$, $j = 1,2,3$. Notice that

$$(K_1,K_2,K_3) \mapsto ((x,y) \mapsto \langle K_2, T_{K_1,K_3}(x,y,\,\cdot\,)\rangle)$$

is sequentially continuous from $\mathcal{S}_s^\sigma(\mathbf{R}^{2d}) \times (\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d}) \times \mathcal{S}_s^\sigma(\mathbf{R}^{2d})$ to $\mathcal{S}_s^\sigma(\mathbf{R}^{2d})$, and from $\Sigma_s^\sigma(\mathbf{R}^{2d}) \times (\Sigma_s^\sigma)'(\mathbf{R}^{2d}) \times \Sigma_s^\sigma(\mathbf{R}^{2d})$ to $\Sigma_s^\sigma(\mathbf{R}^{2d})$. The following result follows from these continuity properties and (3.15).

Proposition 3.17. *Let $A \in \mathbf{M}(d,\mathbf{R})$, and let $s,\sigma > 0$ be such that $s+\sigma \geq 1$. Then the following is true:*

1. *the map $(a_1,a_2) \mapsto a_1 \#_A a_2$ is continuous from $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$;*
2. *the map $(a_1,a_2) \mapsto a_1 \#_A a_2$ is continuous from $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$;*
3. *the map $(a_1,a_2,a_3) \mapsto a_1 \#_A a_2 \#_A a_3$ from $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ extends uniquely to a sequentially continuous and associative map from $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times (\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d}) \times \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$;*
4. *the map $(a_1,a_2,a_3) \mapsto a_1 \#_A a_2 \#_A a_3$ from $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ extends uniquely to a sequentially continuous and associative map from $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d}) \times \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$.*

We have the following corresponding algebra result for $\Gamma_{s,\sigma}^{\sigma,s;0}$ and related symbol classes.

Theorem 3.18. *Let $A \in \mathbf{M}(d,\mathbf{R})$, and let $s,\sigma > 0$ be such that $s+\sigma \geq 1$. Then the following is true:*

1. *the map (1) in Proposition 3.17 extends uniquely to a continuous map from $\Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d}) \times \Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d})$, and from $\Gamma_{s,\sigma;0}^{\sigma,s;0}(\mathbf{R}^{2d}) \times \Gamma_{s,\sigma;0}^{\sigma,s;0}(\mathbf{R}^{2d})$ to $\Gamma_{s,\sigma;0}^{\sigma,s;0}(\mathbf{R}^{2d})$;*
2. *if in addition $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, the map (2) in Proposition 3.17 extends uniquely to a continuous map from $\Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d}) \times \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d})$ to $\Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d})$, and from $\Gamma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d})$ or from $\Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d}) \times \Gamma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Gamma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$.*

Proof. We prove only the first assertion in (2). The other statements follow by similar arguments and are left for the reader.

By Theorem 3.6 it suffices to consider the case when $A = 0$. Let $\phi_1,\phi_2,\phi_3 \in \Sigma_s^\sigma(\mathbf{R}^d)\setminus 0$, $a_j \in \Gamma_{s,\sigma}^{\sigma,s;0}(\mathbf{R}^{2d})$, $j = 1,2$, and let K be the kernel of $\operatorname{Op}_0(a_1) \circ \operatorname{Op}_0(a_2)$. By Proposition 3.13 we need to prove that for some $r > 0$,

$$|V_{\phi_1\otimes\phi_3}K(x,y,\xi,\eta)| \lesssim e^{r(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})-h(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})} \tag{3.17}$$

for every $h > 0$.

Therefore, let $h > 0$ be arbitrarily chosen but fixed, and let K_j be the kernel of $\operatorname{Op}_0(a_j)$, $j = 1,2$,

$$F_1(x,y,\xi,\eta) = V_{\phi_1\otimes\phi_2}K_1(x,y,\xi,\eta),$$
$$F_2(x,y,\xi,\eta) = V_{\phi_2\otimes\phi_3}K_2(x,y,-\xi,\eta)$$

and

$$G(x,y,\xi,\eta) = V_{\phi_1\otimes\phi_3}K(x,y,\xi,\eta).$$

Then

$$G(x,y,\xi,\eta) = \iint_{\mathbf{R}^{2d}} F_1(x,z,\xi,\zeta)F_2(z,y,\zeta,\eta)\,dzd\zeta \tag{3.18}$$

by Moyal's identity (cf. proof of Theorem 3.15). Since $a_j \in \Gamma^{\sigma,s;0}_{s,\sigma}(\mathbf{R}^{2d})$ we have for some $r_0 > 0$ that

$$|F_1(x,y,\xi,\eta)| \lesssim e^{r_0(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})-r(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})}$$

and

$$|F_2(x,y,\xi,\eta)| \lesssim e^{r_0(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})-r(|\xi-\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})}$$

for every $r > 0$. By combining this with (3.18) we get for some $r_0 > 0$,

$$|G(x,y,\xi,\eta)| \lesssim \iint_{\mathbf{R}^{2d}} e^{\varphi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta)+\psi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta)}\,dzd\zeta, \tag{3.19}$$

where $r_1 \ge 2cr + cr_0$,

$$\varphi_{r_0,r}(x,y,z,\xi,\eta,\zeta) = r_0\left(|x|^{\frac{1}{s}} + |\zeta|^{\frac{1}{\sigma}}\right) - r\left(|\zeta-\eta|^{\frac{1}{\sigma}} + |y-z|^{\frac{1}{s}}\right),$$

$$\psi_{r_0,r}(x,y,z,\xi,\eta,\zeta) = r_0\left(|z|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}\right) - r\left(|\xi+\zeta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}}\right)$$

and $c \ge 1$ is chosen such that

$$|x+y|^{\frac{1}{s}} \le c\left(|x|^{\frac{1}{s}} + |y|^{\frac{1}{s}}\right) \text{ and } \quad |\xi+\eta|^{\frac{1}{\sigma}} \le c\left(|\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}}\right),\ x,y,\xi,\eta \in \mathbf{R}^d.$$

Then

$$\begin{aligned}\varphi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta) &\le cr_0\left(|x|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}\right) - (r_1 - cr_0)\left(|\zeta-\eta|^{\frac{1}{\sigma}} + |y-z|^{\frac{1}{s}}\right)\\ &\le cr_0\left(|x|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}\right) - 2cr\left(|\zeta-\eta|^{\frac{1}{\sigma}} + |y-z|^{\frac{1}{s}}\right)\end{aligned}$$

and

$$\psi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta) \le cr_0\left(|x|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}\right) - 2cr\left(|\xi+\zeta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}}\right).$$

This gives

$$\begin{aligned}&\varphi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta) + \psi_{r_0,r_1}(x,y,z,\xi,\eta,\zeta)\\ &\quad\le 2cr_0\left(|x|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}}\right) - 2cr\left(|\xi+\zeta|^{\frac{1}{\sigma}} + |\zeta-\eta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}} + |y-z|^{\frac{1}{s}}\right).\end{aligned}$$

Since

$$\begin{aligned}&-2cr\left(|\xi+\zeta|^{\frac{1}{\sigma}} + |\zeta-\eta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}} + |y-z|^{\frac{1}{s}}\right)\\ &\quad\le -r\left(|\xi+\eta|^{\frac{1}{\sigma}} + |x-y|^{\frac{1}{s}}\right) - cr\left(|\xi+\zeta|^{\frac{1}{\sigma}} + |\zeta-\eta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}} + |y-z|^{\frac{1}{s}}\right)\\ &\quad\le -r\left(|\xi+\eta|^{\frac{1}{\sigma}} + |x-y|^{\frac{1}{s}}\right) - cr\left(|\xi+\zeta|^{\frac{1}{\sigma}} + |x-z|^{\frac{1}{s}}\right)\end{aligned}$$

we get by combining these estimates with (3.19) that

$$\begin{aligned}|G(x,y,\xi,\eta)| &\lesssim \iint_{\mathbf{R}^{2d}} e^{2cr_0(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})-r(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})-cr(|\xi+\zeta|^{\frac{1}{\sigma}}+|x-z|^{\frac{1}{s}})}\,dzd\zeta,\\ &\asymp e^{2cr_0(|x|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})-r(|\xi+\eta|^{\frac{1}{\sigma}}+|x-y|^{\frac{1}{s}})}.\end{aligned}$$

Since $r_0 > 0$ is fixed and $r > 0$ can be chosen arbitrarily, the result follows. $\square$

Theorem 3.19. *Let $A \in \mathbf{M}(d, \mathbf{R})$, $s, \sigma > 0$ be such that $s + \sigma \geq 1$, and let $\omega_j \in \mathscr{P}_{s,\sigma}(\mathbf{R}^{2d})$, $j = 1, 2$. Then the following is true:*

(1) *the map $(a_1, a_2) \mapsto a_1 \#_A a_2$ from $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ is uniquely extendable to a continuous map from $\Gamma_{(\omega_1)}^{\sigma,s;0}(\mathbf{R}^{2d}) \times \Gamma_{(\omega_2)}^{\sigma,s;0}(\mathbf{R}^{2d})$ to $\Gamma_{(\omega_1\omega_2)}^{\sigma,s;0}(\mathbf{R}^{2d})$;*

(2) *if in addition $\omega_j \in \mathscr{P}_{s,\sigma}^0(\mathbf{R}^{2d})$, $j = 1, 2$, then the map $(a_1, a_2) \mapsto a_1 \#_A a_2$ from $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \times \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ is uniquely extendable to a continuous map from $\Gamma_{(\omega_1)}^{\sigma,s}(\mathbf{R}^{2d}) \times \Gamma_{(\omega_2)}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Gamma_{(\omega_1\omega_2)}^{\sigma,s}(\mathbf{R}^{2d})$.*

For the proof we need the following lemma.

Lemma 3.20. *Let ω be a weight on $\mathbf{R}^{4d}$, $\omega_0(x, \xi) = \omega(x, x, \xi, \xi)$ when $x, \xi \in \mathbf{R}^d$, $s, \sigma > 0$ be such that $s + \sigma \geq 1$. Then the trace map which takes*

$$\mathbf{R}^{4d} \ni (x, y, \xi, \eta) \mapsto F(x, y, \xi, \eta)$$

to

$$\mathbf{R}^{2d} \ni (x, \xi) \mapsto F(x, x, \xi, \xi)$$

is linear and continuous from $\Gamma_{(\omega)}^{\sigma,s}(\mathbf{R}^{4d})$ into $\Gamma_{(\omega_0)}^{\sigma,s}(\mathbf{R}^{2d})$. The same holds true with $\Gamma_{(\omega)}^{\sigma,s;0}$ and $\Gamma_{(\omega_0)}^{\sigma,s;0}$ in place of $\Gamma_{(\omega)}^{\sigma,s}$ and $\Gamma_{(\omega_0)}^{\sigma,s}$, respectively, at each occurrence.

Lemma 3.20 follows by similar arguments as in the proof of Lemma 3.10, using the Leibniz type rule

$$\partial_x^\alpha \partial_\xi^\beta (F(x, x, \xi, \xi)) = \sum_{\gamma \leq \alpha} \sum_{\delta \leq \beta} \binom{\alpha}{\gamma} \binom{\beta}{\delta} (\partial_1^{\alpha-\gamma} \partial_2^{\beta-\delta} \partial_3^\gamma \partial_4^\delta F)(x, x, \xi, \xi).$$

The details are left for the reader.

Proof of Theorem 3.19. We may assume that $A = 0$ by Theorem 3.1. We only prove (2). The assertion (1) follows by similar arguments and is left for the reader.

Let

$$F_{a_1,a_2}(x_1, x_2, \xi_1, \xi_2) = a_1(x_1, \xi_1) a_2(x_2, \xi_2)$$

and

$$\omega(x_1, x_2, \xi_1, \xi_2) = \omega_1(x_1, \xi_1) \omega_2(x_2, \xi_2).$$

By the definitions it follows that the map T_1 which takes (a_1, a_2) into F_{a_1,a_2} is continuous from $\Gamma_{(\omega_1)}^{\sigma,s}(\mathbf{R}^{2d}) \times \Gamma_{(\omega_2)}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Gamma_{(\omega)}^{\sigma,s}(\mathbf{R}^{4d})$.

Theorem 3.3 declares that the map T_2 which takes $F(x_1, x_2, \xi_1, \xi_2)$ to $e^{i\langle D_{\xi_1}, D_{x_2}\rangle} F(x_1, x_2, \xi_1, \xi_2)$ is continuous on $\Gamma_{(\omega)}^{\sigma,s}(\mathbf{R}^{4d})$. Hence, if T_3 is the trace operator which takes $F(x_1, x_2, \xi_1, \xi_2)$ into $F_0(x, \xi) \equiv F(x, x, \xi, \xi)$, Lemma 3.20 shows that $T \equiv T_3 \circ T_2 \circ T_1$ is continuous from $\Gamma_{(\omega_1)}^{\sigma,s}(\mathbf{R}^{2d}) \times \Gamma_{(\omega_2)}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Gamma_{(\omega_1\omega_2)}^{\sigma,s}(\mathbf{R}^{2d})$.

By [22, Theorem 18.1.8] we have $T(a_1, a_2) = a_1\#_0 a_2$ when $a_1, a_2 \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$. If instead $a_j \in \Gamma_{(\omega_j)}^{\sigma,s}(\mathbf{R}^{2d})$, $j = 1, 2$, then we take $T(a_1, a_2)$ as the definition of $a_1\#_0 a_2$. By the continuity of T it follows that $(a_1, a_2) \mapsto a_1\#_0 a_2$ is continuous from $\Gamma_{(\omega_1)}^{\sigma,s}(\mathbf{R}^{2d}) \times \Gamma_{(\omega_2)}^{\sigma,s}(\mathbf{R}^{2d})$ to $\Gamma_{(\omega_1\omega_2)}^{\sigma,s}(\mathbf{R}^{2d})$.

Since $\Gamma_{(\omega_j)}^{\sigma,s}(\mathbf{R}^{2d}) \subseteq \Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d})$, we get $\operatorname{Op}_0(a_1\#_0 a_2) = \operatorname{Op}_0(a_1) \circ \operatorname{Op}_0(a_2)$ and that $a_1\#_0 a_2$ is uniquely defined as an element in $\Gamma_{s,\sigma;0}^{\sigma,s}(\mathbf{R}^{2d})$, in view of Theorem 3.18. Hence $a_1\#_0 a_2$ is uniquely defined in $\Gamma_{(\omega_1\omega_2)}^{\sigma,s}(\mathbf{R}^{2d})$, since all these symbol classes are subspaces of $C^\infty(\mathbf{R}^{2d})$. □

Appendix A.

In this appendix we explain some facts on moderate weights and in particular give a proof of Proposition 1.6.

If $d = d_1 + \cdots + d_n$ and $s = (s_1, \ldots, s_n) \in \mathbf{R}_+^n$ with $d_j \geq 0$ being integers, $j = 1, \ldots, n$, then we let

$$\mathscr{P}_s(\mathbf{R}^d) = \mathscr{P}_{s_1,\ldots,s_n}(\mathbf{R}^{d_1+\cdots+d_n}) \quad (\mathscr{P}_s^0(\mathbf{R}^d) = \mathscr{P}_{s_1,\ldots,s_n}^0(\mathbf{R}^{d_1+\cdots+d_n}))$$

be the set of all weight functions ω on $\mathbf{R}^{d_1+\cdots+d_n}$ such that

$$\omega(x_1 + y_1, \ldots, x_n + y_n) \lesssim \omega(x_1, \ldots, x_n) e^{r(|y_1|^{\frac{1}{s_1}} + \cdots + |y_n|^{\frac{1}{s_n}})}, \qquad x_j, y_j \in \mathbf{R}^{d_j},\ j = 1, \ldots, n, \tag{1.12$'$}$$

for some $r > 0$ (for every $r > 0$). In particular, if $\omega \in \mathscr{P}_{s_1,\ldots,s_n}(\mathbf{R}^{d_1+\cdots+d_n})$ ($\omega \in \mathscr{P}_{s_1,\ldots,s_n}^0(\mathbf{R}^{d_1+\cdots+d_n})$), then

$$e^{-r(|x_1|^{\frac{1}{s_1}} + \cdots + |x_n|^{\frac{1}{s_n}})} \lesssim \omega(x_1, \ldots, x_n) \lesssim e^{r(|x_1|^{\frac{1}{s_1}} + \cdots + |x_n|^{\frac{1}{s_n}})} \tag{1.13$'$}$$

for some $r > 0$ (for every $r > 0$).

The proof of Proposition 1.6 in Sect. 1 is based on the following.

Proposition A.1. *Let $\omega \in \mathscr{P}_E(\mathbf{R}^d)$. Then ω is v-moderate for some submultiplicative weight v which satisfies*

$$v(x + y) \leq v(x)v(y), \qquad x, y \in \mathbf{R}^d.$$

Proposition A.1 follows by letting $v(x) = \max(v_0(x), v_0(-x))$, where

$$v_0(x) = \sup_{y \in \mathbf{R}^d} \left(\frac{\omega(x + y)}{\omega(y)} \right).$$

It follows that v satisfies all required properties. (See also (2.14) and (2.15) in [31].)

The following proposition is a multi-linear version of Proposition 1.6 in Sect. 1.

Proposition 1.6′. *Let* $d = d_1 + \cdots + d_n$ *and* $s, t \in \mathbf{R}_+^n$ *be such that* $d_j \ge 0$ *are integers and* $t_j = \max(1, s_j)$, $j = 1, \ldots, n$. *Then the set* $\mathscr{P}_s(\mathbf{R}^d)$ *is non-increasing with respect to* $s_1, \ldots, s_n$,

$$\mathscr{P}_{s_1,\ldots,s_n}(\mathbf{R}^{d_1+\cdots+d_n}) = \mathscr{P}_{t_1,\ldots,t_n}(\mathbf{R}^{d_1+\cdots+d_n}), \tag{1.14$'$}$$

and

$$\mathscr{P}_{1,\ldots,1}(\mathbf{R}^{d_1+\cdots+d_n}) = \mathscr{P}_E(\mathbf{R}^{d_1+\cdots+d_n}). \tag{1.15$'$}$$

Proof. It is evident that $\mathscr{P}_{s_1,\ldots,s_n} \subseteq \mathscr{P}_{s_{0,1},\ldots,s_{0,n}}$ when $s_{0,j} \le s_j$, $j = 1, \ldots, n$, which shows that $\mathscr{P}_{s_1,\ldots,s_n}(\mathbf{R}^{d_1+\cdots+d_n})$ is non-decreasing with respect to s.

By the definition it follows that if $\omega \in \mathscr{P}_{s_1,\ldots,s_n}(\mathbf{R}^{d_1+\cdots+d_n})$, then ω is moderate with respect to

$$v(x_1, \ldots, x_n) = e^{r(|x_1|^{\frac{1}{s_1}} + \cdots + |x_n|^{\frac{1}{s_n}})}$$

for some $r > 0$. Hence, $\mathscr{P}_{s_1,\ldots,s_n}(\mathbf{R}^{d_1+\cdots+d_n}) \subseteq \mathscr{P}_E(\mathbf{R}^d)$.

In order to prove (1.15)′, suppose that $\omega \in \mathscr{P}_{s_1,\ldots,s_n}(\mathbf{R}^{d_1+\cdots+d_n})$. In view of Proposition A.1, ω is v-moderate for some submultiplicative v. Hence, [20, Lemma 4.2] shows that

$$v(x_1, \ldots, x_n) \lesssim e^{r(|x_1| + \cdots + |x_n|)}$$

for some $r > 0$, which gives

$$\omega(x_1 + y_1, \ldots, x_n + y_n) \lesssim \omega(x_1, \ldots, x_n) e^{r(|y_1| + \cdots + |y_n|)}, \tag{A.1}$$

and we obtain $\mathscr{P}_E \subseteq \mathscr{P}_{1,1}$. Since reversed inclusion was given above, (1.15)′ follows.

Next we prove (1.14)′. First we observe that $\mathscr{P}_t \subseteq \mathscr{P}_s$ because $s_j \le t_j$. In order to prove the opposite inclusion, let $\omega \in \mathscr{P}_s$. A combination of (1.12)′ and (A.1) gives

$$\begin{aligned} &\omega(x_1, \ldots, x_j + y_j, \ldots, x_j) \lesssim \omega(x_1, \ldots, x_j, \ldots, x_j) e^{r|y_j|^{\frac{1}{t_j}}}, \\ &\quad x_j, y_j \in \mathbf{R}^{d_j}, \ j = 1, \ldots, n, \end{aligned}$$

for some $r > 0$, and we obtain by repeating use of this estimate and induction,

$$\begin{aligned} \omega(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) &\lesssim \omega(x_1, x_2 + y_2, \ldots, x_n + y_n) e^{r|y_1|^{\frac{1}{t_1}}} \\ &\lesssim \omega(x_1, x_2, \ldots, x_n + y_n) e^{r|y_2|^{\frac{1}{t_2}}} e^{r|y_1|^{\frac{1}{t_1}}} \\ &= \omega(x_1, x_2, \ldots, x_n + y_n) e^{r(|y_1|^{\frac{1}{t_1}} + |y_2|^{\frac{1}{t_2}})} \\ &\lesssim \cdots \lesssim \omega(x_1, x_2, \ldots, x_n) e^{r(|y_1|^{\frac{1}{t_1}} + \cdots + |y_n|^{\frac{1}{t_n}})}. \end{aligned}$$

This implies that $\omega \in \mathscr{P}_t$, and hence, $\mathscr{P}_s \subseteq \mathscr{P}_t$. Since opposite inclusion was already achieved, we get (1.14)′. □

Appendix B.

In this appendix we present a short proof of Proposition 2.5.

Proof of Proposition 2.5. Let $X = (x,\xi) \in \mathbf{R}^{2d}$, $Y = (y,\eta) \in \mathbf{R}^{2d}$ and $\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})\backslash 0$. Suppose that $\omega \in \mathscr{P}_{s,\sigma}(\mathbf{R}^{2d})$ and that (2.9) holds for all $h > 0$. If

$$F_X(Y) \equiv \omega(X)^{-1} a(Y+X)\overline{\phi(Y)}$$

then the fact that $\omega(X) \lesssim \omega(Y+X)e^{r_0(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})}$ gives that $Y \mapsto F_X(Y)$ is bounded in $\Sigma_{s,\sigma}^{\sigma,s}$ with respect to $X \in \mathbf{R}^{2d}$. Hence

$$|\partial_y^\alpha \partial_\eta^\beta F_X(y,\eta)| \lesssim h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s e^{-r(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})},$$

for every $h, r > 0$. In particular,

$$|V_\phi a(x,\xi,\eta,y)| = |(\mathscr{F} F_X)(\eta,y)\omega(X)| \lesssim \omega(X)e^{-r(|y|^{\frac{1}{s}}+|\eta|^{\frac{1}{\sigma}})},$$

for every $r > 0$. This gives (1) in one of the cases. The other case follows by similar arguments and is left for the reader.

Next we prove (2) in the case when $\omega \in \mathscr{P}_{s,\sigma}$ and $\phi \in \Sigma_{s,\sigma}^{\sigma,s}$. The other case follows by similar arguments and is left for the reader. Therefore, suppose (2.10) holds for all $r > 0$. By differentiation, the facts that

$$\omega(Z) \lesssim \omega(X)e^{r_0(|x-z|^{\frac{1}{s}}+|\xi-\zeta|^{\frac{1}{\sigma}})}, \quad Z = (z,\zeta) \in \mathbf{R}^{2d}$$

and $\phi \in \Sigma_{s,\sigma}^{\sigma,s}$ for some $r_0 > 0$, and (2.6) give

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim \sum_{\gamma \le \alpha}\sum_{\delta \le \beta} \binom{\alpha}{\gamma}\binom{\beta}{\delta} h^{|\alpha+\beta-\gamma-\delta|}(\alpha-\gamma)!^\sigma (\beta-\delta)!^s I_{\gamma,\delta}(X),$$

where

$$\begin{aligned} I_{\gamma,\delta}(X) &= \iint_{\mathbf{R}^{4d}} \omega(Z)|\eta^\gamma y^\delta| e^{-(r+r_0)(|x-z|^{\frac{1}{s}}+|y|^{\frac{1}{s}}+|\xi-\zeta|^{\frac{1}{\sigma}}+|\eta|^{\frac{1}{\sigma}})}\, dY\, dZ \\ &\lesssim \omega(X)\iint_{\mathbf{R}^{4d}} |\eta^\gamma y^\delta| e^{-r(|z|^{\frac{1}{s}}+|y|^{\frac{1}{s}}+|\zeta|^{\frac{1}{\sigma}}+|\eta|^{\frac{1}{\sigma}})}\, dY\, dZ \lesssim h^{|\gamma+\delta|}\gamma!^\sigma \delta!^s \omega(X) \end{aligned}$$

for every $h, r > 0$. It follows that (2.9) holds for every $h > 0$ by using the estimates above and similar computations as in (2.8). □

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