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Pseudo-differential operator associated with the fractional Fourier transform

Akhilesh Prasad[1,*] and Praveen Kumar[1]

[1] *Department of Applied Mathematics, Indian School of Mines, Dhanbad-826 004, India*

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Abstract. The main goal of this paper is to study properties of the fractional Fourier transform on Schwartz type space $\mathscr{S}_\theta$. Symbol class $S^{m,\theta}_{\rho,\sigma}$ is introduced. The fractional pseudo-differential operators (f.p.d.o.) associated with the symbol $a(x,\xi)$ are a continuous linear mapping of $\mathscr{S}$ into $\mathscr{S}_\theta$. Kernel and integral representations of f.p.d.o are obtained. The boundedness property of f.p.d.o. is studied. Application of the fractional Fourier transform in solving a generalized Fredholm integral equation is also given.
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1. Introduction

The motive of a generalization of the Fourier transform (FT) into the fractional Fourier transform (FrFT) in the mathematical literature was formally first introduced by Victor Namias in 1980 [4]. As a generalization of the FT, the FrFT is a linear operator in the time-frequency plane. It depends on an angle θ and can be interpreted as a rotation by θ in the time-frequency plane. It has many applications in solving differential equations, quantum mechanics, signal processing, image processing and other fields. The FrFT [1–6,12] with an angle θ of $\phi(x) \in L^2(\mathbb{R})$ denoted by $(\mathcal{F}^\theta\phi)(\omega) = \hat{\phi}^\theta(\omega)$ is given by

$$(\mathcal{F}^\theta\phi)(\omega) = \hat{\phi}^\theta(\omega) = \int_{\mathbb{R}} K^\theta(x,\omega)\phi(x)dx, \tag{1}$$

where the kernel is

$$K^\theta(x,\omega) = \begin{cases} C^\theta \; e^{i(x^2+\omega^2)\frac{\cot\theta}{2} - ix\omega\csc\theta}, & \theta \neq n\pi, \\ \frac{1}{\sqrt{2\pi}}\; e^{-ix\omega}, & \theta = \frac{\pi}{2}, \\ \delta(x-\omega), & \theta = 2n\pi, \\ \delta(x+\omega), & \theta = (2n-1)\pi, n \in \mathbb{Z}, \end{cases}$$

and $C^\theta = (2\pi i \sin\theta)^{-1/2} e^{i\theta/2} = \sqrt{\frac{1-i\cot\theta}{2\pi}}$.

*Corresponding author. *Email addresses:* apr_bhu@yahoo.com (A. Prasad), praveen502mzn@gmail.com (P. Kumar)

If $\hat{\phi}^\theta(\omega) \in L^2(\mathbb{R})$, the corresponding inversion formula is given by

$$\phi(x) = \int_{\mathbb{R}} \overline{K^\theta(x,\omega)}\ \hat{\phi}^\theta(\omega)d\omega, \tag{2}$$

where

$$\overline{K^\theta(x,\omega)} = \overline{C^\theta} e^{-i(x^2+\omega^2)\frac{\cot\theta}{2}+ix\omega\csc\theta} = K^{-\theta}(x,\omega),$$
$$\overline{C^\theta} = \frac{(2\pi i \sin\theta)^{1/2}e^{-i\theta/2}}{2\pi\sin\theta} = \sqrt{\frac{1+i\cot\theta}{2\pi}} = C^{-\theta}.$$

From this, we conclude that the inverse of a FrFT with an angle θ is the FrFT with the angle $-\theta$.

Definition 1. *The Schwartz space $\mathscr{S}$ is a set of rapidly decreasing complex-valued infinitely differentiable functions ϕ on $\mathbb{R}$ such that for every choice of α and β of non-negative integers it satisfies*

$$\gamma_{\alpha,\beta}(\phi) = \sup_{x\in\mathbb{R}} \left|x^\alpha\ D^\beta\phi(x)\right| < \infty. \tag{3}$$

If f is of polynomial growth and a locally integrable function on $\mathbb{R}$, then f generates a distribution in $\mathscr{S}'$ as follows:

$$\langle f,\phi\rangle = \int_{\mathbb{R}} f(x)\phi(x)dx, \quad \phi\in\mathscr{S}.$$

The space $\mathscr{S}(\mathbb{R})$ is equipped with the topology generated by the collection of semi-norms $\{\gamma_{\alpha,\beta}\}$; it is a Fréchet space. The dual of $\mathscr{S}$ is denoted by $\mathscr{S}'$; its elements are called tempered distributions.

Definition 2. *The Schwartz type space $\mathscr{S}_\theta$ is defined as follows: ϕ is a member of $\mathscr{S}_\theta$ iff it is a complex valued C^∞- function on $\mathbb{R}$ and for every choice of α and β of non-negative integers it satisfies*

$$\Gamma^\theta_{\alpha,\beta}(\phi) = \sup_{x\in\mathbb{R}} \left|x^\alpha\ (\Delta^*_x)^\beta\phi(x)\right| < \infty, \tag{4}$$

where $\Delta^*_x = -\left(\frac{d}{dx} + ix\cot\theta\right)$.

Definition 3. *The continuous fractional convolution of two continuous functions $\phi(x), \psi(x) \in L^2(\mathbb{R})$ is defined as*

$$(\phi \star_\theta \psi)(z) = \int_{-\infty}^{\infty} \phi(x)\psi(z-x)e^{-\frac{i}{2}(z^2-x^2)\cot\theta}dx, \tag{5}$$

where $\star_\theta$ denotes the fractional convolution operator.

Lemma 1. *Let $\phi(x), \psi(x) \in L^2(\mathbb{R})$ be two continuous functions; then the fractional Fourier transform of their convolution operator is given by*

$$\mathcal{F}^\theta(\phi\star_\theta\psi)(\omega) = \frac{1}{C^\theta}e^{-\frac{i}{2}\omega^2\cot\theta}\hat{\phi}^\theta(\omega)\mathcal{F}^\theta\left(e^{-\frac{i}{2}(.)^2\cot\theta}\psi\right)(\omega). \tag{6}$$

Proof. Using equations (1) and (5), we have

$$\begin{aligned}
\mathcal{F}^\theta(\phi \star_\theta \psi)(\omega) &= \int_{-\infty}^{\infty} K^\theta(z,\omega)(\phi \star_\theta \psi)(z)dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^\theta(z,\omega)\phi(x)\psi(z-x)e^{-\frac{i}{2}(z^2-x^2)\cot\theta}dx\ dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^\theta(x+y,\omega)\phi(x)\psi(y)e^{-\frac{i}{2}[(x+y)^2-x^2)\cot\theta}dx\ dy \\
&= \frac{1}{C^\theta}e^{-\frac{i}{2}\omega^2\cot\theta}\int_{-\infty}^{\infty} K^\theta(x,\omega)\phi(x)dx\int_{-\infty}^{\infty} K^\theta(y,\omega)\psi(y)e^{-\frac{i}{2}y^2\cot\theta}dy \\
&= \frac{1}{C^\theta}e^{-\frac{i}{2}\omega^2\cot\theta}\hat{\phi}^\theta(\omega)\ \mathcal{F}^\theta\big(e^{-\frac{i}{2}(.)^2\cot\theta}\psi\big)(\omega).
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2. *Let $\phi, \psi \in \mathscr{S}_\theta \subset \mathscr{S}$ and $K^\theta(x,\omega)$ be the kernel of the fractional Fourier transform. Then*

$$\begin{aligned}
&(i)\ \int_{-\infty}^{\infty}\big(\overline{\Delta_x^*}\big)^r\phi(x)\psi(x)dx = \int_{-\infty}^{\infty}\phi(x)\big(\overline{\Delta_x}\big)^r\psi(x)dx,\\
&(ii)\ \big(\Delta_x^*\big)^r\overline{K^\theta(x,\omega)} = (-i\omega\csc\theta)^r\overline{K^\theta(x,\omega)},\\
&(iii)\ \big(\overline{\Delta_x^*}\big)^r K^\theta(x,\omega) = (i\omega\csc\theta)^r K^\theta(x,\omega),\\
&(iv)\ \big(\mathcal{F}^\theta\big(\overline{\Delta_x}\big)^r\phi(x)\big)(\omega) = (i\omega\csc\theta)^r\hat{\phi}^\theta(\omega),\quad \forall r\in\mathbb{N}_0,
\end{aligned}$$

where Δ_x^* is defined as above and $\Delta_x = \big(\frac{d}{dx} - ix\cot\theta\big)$.

Proof. The proof is similar to [7, pp. 357-358]. □

Lemma 3 (Peetre)**.** *For any real number s and $\forall, \xi, \eta \in \mathbb{R}$, the upper boundary*

$$\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)^s \le 2^{|s|}\big(1+|\xi-\eta|^2\big)^{|s|}$$

is satisfied.

Proof. See [11, p. 97]. □

Lemma 4. *If $f(x),\ \phi(x) \in \mathscr{S}_\theta(\mathbb{R})$. Then*

$$(\Delta_x^*)^k[f(x)\ \phi(x)] = \sum_{r=0}^{k} A_{k,r}\ (\Delta_x^*)^r\phi(x)\ D_x^{k-r}f(x),\qquad k\in\mathbb{N}_0,$$

where Δ_x^* is the same as defined above and $A_{k,r}$ are constants.

Proof. See [8]. □

Theorem 1. *The fractional Fourier transform $\hat{\phi}^\theta$ is a continuous linear map of $\mathscr{S}$ into $\mathscr{S}_\theta$.*

Proof. See a similar proof of Proposition 2.2 (*iii*) of [6, p. 242]. □

2. Fractional pseudo-differential operators

In this section, we discuss a special class $S^{m,\theta}_{\rho,\sigma}$, which is a generalization of Hörmander class $S^{m}_{\rho,\sigma}$ [10]. For $\theta=\pi/2$, the class $S^{m,\theta}_{\rho,\sigma}$ reduces to the Hörmander class.

Definition 4. *Let $a(x,\xi)$ be a complex valued smooth function on $C^\infty(\mathbb{R}\times\mathbb{R})\to\mathbb{C}$. It belongs to symbol class $S^{m,\theta}_{\rho,\sigma}, m\in\mathbb{R}, 0\le\sigma,\rho\le 1$ iff $\forall q,\alpha,\beta\in\mathbb{N}_0$, there exist $A_{\alpha,\beta,m,q,\theta}>0$ such that*

$$(1+|x\csc\theta|)^q\left|D_x^\beta(\Delta_\xi^*)^\alpha a(x,\xi)\right|\le A_{\alpha,\beta,m,q,\theta}(1+|\xi\csc\theta|)^{m-\rho|\alpha|+\sigma|\beta|}, \tag{7}$$

where Δ_ξ^* is defined as above.

Definition 5. *Let the symbol $a(x,\xi)\in S^{m,\theta}_{\rho,\sigma}$. Then a fractional pseudo-differential operator T^θ_a associated with $a(x,\xi)$ is defined by*

$$\left(T_a^\theta\phi\right)(x)=\int_{-\infty}^{\infty}\overline{K^\theta(x,\xi)}a(x,\xi)\hat{\phi}^\theta(\xi)d\xi,\quad \phi\in\mathscr{S}^\theta. \tag{8}$$

Theorem 2. *Let the symbol $a(x,\xi)\in S^{m,\theta}_{\rho,\sigma}$. Then f.p.d.o. T^θ_a is a continuous linear mapping from $\mathscr{S}$ into $\mathscr{S}_\theta$.*

Proof. Let $\phi\in\mathscr{S}_\theta$ and $k,q\in\mathbb{N}_0$. Then using definitions (7)-(8), Lemma 4, Theorem 1 and applying the technique of [6, p. 247] and [11, p. 66], we have

$$\begin{aligned}
&\left|x^q(\Delta_x^*)^\beta\left(T_a^\theta\phi\right)(x)\right|\\
&\quad=\left|x^q(\Delta_x^*)^\beta\int_{-\infty}^{\infty}\overline{K^\theta(x,\xi)}a(x,\xi)\hat{\phi}^\theta(\xi)\ d\xi\right|\\
&\quad=\left|x^q\int_{-\infty}^{\infty}(\Delta_x^*)^\beta[\overline{K^\theta(x,\xi)}a(x,\xi)]\hat{\phi}^\theta(\xi)\ d\xi\right|\\
&\quad=\left|x^q\int_{-\infty}^{\infty}\sum_{\alpha=0}^{\beta}B_{\beta,\alpha}(\Delta_x^*)^\alpha\overline{K^\theta(x,\xi)}D_x^{\beta-\alpha}a(x,\xi)\hat{\phi}^\theta(\xi)\ d\xi\right|\\
&\quad=\left|x^q\int_{-\infty}^{\infty}\sum_{\alpha=0}^{\beta}B_{\beta,\alpha}(-i\xi\csc\theta)^\alpha\overline{K^\theta(x,\xi)}D_x^{\beta-\alpha}a(x,\xi)\hat{\phi}^\theta(\xi)\ d\xi\right|\\
&\quad\le\sum_{\alpha=0}^{\beta}B_{\beta,\alpha}|\overline{C^\theta}|\int_{-\infty}^{\infty}(1+|x\csc\theta|)^q|D_x^{\beta-\alpha}a(x,\xi)|\ |\xi\csc\theta|^\alpha|\hat{\phi}^\theta(\xi)|\ d\xi\\
&\quad\le\sum_{\alpha=0}^{\beta}B_{\beta,\alpha}|\overline{C^\theta}|\sup_{\xi\in\mathbb{R}}|\hat{\phi}^\theta(\xi)|\int_{-\infty}^{\infty}A_{\beta-\alpha,m,q,\theta}(1+|\xi\csc\theta|)^{m+\sigma\beta+\alpha(1-\sigma)}\ d\xi\\
&\quad\le A'_{\beta,m,q,\theta}\int_{-\infty}^{\infty}(1+|\xi\csc\theta|)^{m+\beta}\ d\xi.
\end{aligned}$$

The integral is convergent for $m+\beta+1<0$. Hence

$$\Gamma^\theta_{q,\beta}\left(T_a^\theta\phi\right)<\infty.$$

This completes the proof of the theorem. □

3. Kernel and integral representations

In this section, we can write fractional pseudo-differential operators $(T_a^\theta \phi)$ in different ways: For all $\phi \in \mathscr{S}_\theta$, we have

$$\begin{aligned}\left(T_a^\theta \phi\right)(x) &= \int_{-\infty}^{\infty} \overline{K^\theta(x,\xi)} a(x,\xi) \hat{\phi}^\theta(\xi) d\xi \\ &= \int_{-\infty}^{\infty} \overline{K^\theta(x,\xi)} a(x,\xi) \Big(\int_{-\infty}^{\infty} K^\theta(y,\xi)\phi(y) dy \Big) d\xi \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{i}{2}(x^2-y^2)\cot\theta + i(x-y)\xi\csc\theta} a(x,\xi)\phi(y)\ dy\ d\xi.\end{aligned}$$

Let $x - y = z$; then

$$\begin{aligned}\left(T_a^\theta \phi\right)(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(z^2-2xz)\cot\theta + iz\xi\csc\theta} a(x,\xi)\phi(x-z)\ dz\ d\xi \\ &= \int_{-\infty}^{\infty} k_\theta(x,z)\phi(x-z)\ dz,\end{aligned}$$

where

$$k_\theta(x,z) = \int_{-\infty}^{\infty} e^{\frac{i}{2}(z^2-2xz)\cot\theta + iz\xi\csc\theta} a(x,\xi) d\xi.$$

Thus

$$\left(T_a^\theta \phi\right)(x) = \int_{-\infty}^{\infty} K_\theta(x,y)\phi(y)\ dy,$$

with the kernel $K_\theta(x,y) = k_\theta(x, x-y)$.
The function $a_{\xi,\theta}(\eta)$ associated with the symbol $a(x,\xi)$ and defined by

$$a_{\xi,\theta}(\eta) = \mathcal{F}^\theta \left[\overline{K^\theta(x,\xi)} a(x,\xi)\right](\eta), \tag{9}$$

will play a fundamental role in our investigation. An upper boundary for $a_{\xi,\theta}(\eta)$ is given by the following lemma.

Lemma 5. *Let the symbol $a(x,\xi) \in S^{m,\theta}_{\rho,\sigma}$. Then the function $a_{\xi,\theta}(\eta)$ defined by* (9) *satisfies the inequality*

$$\left|a_{\xi,\theta}(\eta)\right| \le B_{t,m,q,\theta}(1+|\eta\csc\theta|)^{-t}(1+|\xi\csc\theta|)^{m+t}, \quad \forall\ t > 1 \in \mathbb{N}. \tag{10}$$

Proof. For $r \in \mathbb{N}_0$, using Lemma 2 (ii) and definition (7), we have

$$\begin{aligned}
a_{\xi,\theta}(\eta) &= \int_{-\infty}^{\infty} K^\theta(x,\eta)\frac{(1-\Delta_x^*)^t}{(1+|\eta\csc\theta|)^t}[\overline{K^\theta(x,\xi)}a(x,\xi)]dx \\
&= \int_{-\infty}^{\infty} K^\theta(x,\eta)\sum_{r=0}^{t}(-1)^r\binom{t}{r}\frac{(\Delta_x^*)^r}{(1+|\eta\csc\theta|)^t}[\overline{K^\theta(x,\xi)}a(x,\xi)]dx \\
&= \sum_{r=0}^{t}(-1)^r\binom{t}{r}\frac{1}{(1+|\eta\csc\theta|)^t}\int_{-\infty}^{\infty}K^\theta(x,\eta)\sum_{s=0}^{r}B_{r,s}(\Delta_x^*)^s\overline{K^\theta(x,\xi)} \\
&\quad\times D_x^{r-s}a(x,\xi)dx \\
&= \sum_{r=0}^{t}(-1)^r\binom{t}{r}\frac{1}{(1+|\eta\csc\theta|)^t}\int_{-\infty}^{\infty}K^\theta(x,\eta)\sum_{s=0}^{r}B_{r,s}(-i\xi\csc\theta)^s\overline{K^\theta(x,\xi)} \\
&\quad\times D_x^{r-s}a(x,\xi)dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|a_{\xi,\theta}(\eta)| &\leq \sum_{r=0}^{t}\sum_{s=0}^{r}\binom{t}{r}B_{r,s}(1+|\eta\csc\theta|)^{-t}\int_{-\infty}^{\infty}|\xi\csc\theta|^s|D_x^{r-s}a(x,\xi)|dx \\
&\leq \sum_{r=0}^{t}\sum_{s=0}^{r}\binom{t}{r}B_{r,s}(1+|\eta\csc\theta|)^{-t}\int_{-\infty}^{\infty}A_{r-s,m,q,\theta}(1+|x\csc\theta|)^{-q} \\
&\quad\times(1+|\xi\csc\theta|)^{m+\sigma(r-s)+s}dx \\
&\leq B_{t,m,q,\theta}(1+|\eta\csc\theta|)^{-t}(1+|\xi\csc\theta|)^{m+t}\int_{-\infty}^{\infty}(1+|x\csc\theta|)^{-q}dx.
\end{aligned}$$

The integral is convergent for $q > 1$. Hence we have

$$\left|a_{\xi,\theta}(\eta)\right| \leq B_{t,m,q,\theta}(1+|\eta\csc\theta|)^{-t}(1+|\xi\csc\theta|)^{m+t}.$$

This completes the proof of the lemma. □

Theorem 3. *For any symbol $a(x,\xi) \in S_{\rho,\sigma}^{m,\theta}$, the associated operator T_a^θ can be represented by*

$$(T_a^\theta\phi)(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{K^\theta(x,\eta)}a_{\xi,\theta}(\eta)\hat{\phi}^\theta(\xi)d\eta\, d\xi, \quad \phi \in \mathscr{S}_\theta,$$

where all involved integrals are convergent.

Proof. Since

$$a_{\xi,\theta}(\eta) = \int_{-\infty}^{\infty}K^\theta(x,\eta)[\overline{K^\theta(x,\xi)}a(x,\xi)]dx,$$

by the inversion formula, we have

$$\overline{K^\theta}(x,\xi)a(x,\xi) = \int_{-\infty}^{\infty}\overline{K^\theta(x,\eta)}a_{\xi,\theta}(\eta)d\eta.$$

Therefore,

$$\begin{aligned}(T_a^\theta\phi)(x) &= \int_{-\infty}^{\infty} \overline{K^\theta(x,\xi)} a(x,\xi)\hat{\phi}^\theta(\xi)d\xi \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{K^\theta(x,\eta)} a_{\xi,\theta}(\eta)\hat{\phi}^\theta(\xi)d\eta\ d\xi.\end{aligned}$$

Since $\hat{\phi}^\theta(\xi) \in \mathscr{S}$, we have

$$|\hat{\phi}^\theta(\xi)| \leq E_\theta\ (1+|\xi\csc\theta|)^{-q},\quad \forall\ q>0.$$

Now using the above upper boundary and Lemma 5, we have

$$\begin{aligned}&|(T_a^\theta\phi)(x)| \\ &\quad\leq |\overline{C^\theta}| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |a_{\xi,\theta}(\eta)|\ |\hat{\phi}^\theta(\xi)|d\eta\ d\xi \\ &\quad\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{t,m,q,\theta}(1+|\eta\csc\theta|)^{-t}(1+|\xi\csc\theta|)^{m+t}E_\theta\ (1+|\xi\csc\theta|)^{-q}d\eta\ d\xi \\ &\quad\leq B'_{t,m,q,\theta}\int_{-\infty}^{\infty}(1+|\xi\csc\theta|)^{m+t-q}d\xi\int_{-\infty}^{\infty}(1+|\eta\csc\theta|)^{-t}d\eta,\end{aligned}$$

for choosing q sufficiently large such that $q>m+t+1$ and $t>1$, we can make the integral convergent. □

4. Boundedness property of fractional pseudo-differential operators

Definition 6 (see Prasad et al. [9]). *A tempered distribution $\phi \in \mathscr{S}'_\theta(\mathbb{R})$ is said to belong to the generalized Sobolev spaces $H^{s,\theta}(\mathbb{R})$, $s\in\mathbb{R}$, if its fractional Fourier transform $\hat{\phi}^\theta$ satisfies*

$$||\phi||_{H^{s,\theta}(\mathbb{R})} = \Big(\int_{-\infty}^{\infty}\big|(1+|\xi\csc\theta|^2)^{s/2}\hat{\phi}^\theta(\xi)\big|^2 d\xi\Big)^{1/2} < \infty. \tag{11}$$

Lemma 6. *Let the symbol $a(x,\xi)\in S^{m,\theta}_{\rho,\sigma}$. Then we have*

$$\begin{aligned}&(i)\ \ \hat{a}^\theta(\xi,\eta) = \frac{1}{(1+i\xi\csc\theta)^t}\int_{-\infty}^{\infty} K^\theta(x,\xi)(1+\bar{\Delta}_x)^t a(x,\eta)dx, \\ &(ii)\ |\hat{a}^\theta(\xi,\eta)| \leq C_{t,m,q,\theta}(1+|\eta\csc\theta|)^{m+t\sigma}(1+\xi^2\csc^2\theta)^{-t/2}.\end{aligned}$$

Proof. (i) Using Lemma 2 (i) and (iii) and applying the technique of [6, p. 247] and [11, p. 66], we have

$$\begin{aligned}
&\frac{1}{(1+i\xi\csc\theta)^t}\int_{-\infty}^{\infty}K^\theta(x,\xi)(1+\bar{\Delta}_x)^t a(x,\eta)dx\\
&\quad=\frac{1}{(1+i\xi\csc\theta)^t}\int_{-\infty}^{\infty}K^\theta(x,\xi)\sum_{r=0}^{t}\binom{t}{r}(\bar{\Delta}_x)^r a(x,\eta)dx\\
&\quad=\sum_{r=0}^{t}\binom{t}{r}\frac{1}{(1+i\xi\csc\theta)^t}\int_{-\infty}^{\infty}(\overline{\Delta_x^*})^r K^\theta(x,\xi)\ a(x,\eta)dx\\
&\quad=\sum_{r=0}^{t}\binom{t}{r}\frac{1}{(1+i\xi\csc\theta)^t}\int_{-\infty}^{\infty}(i\xi\csc\theta)^r K^\theta(x,\xi)\ a(x,\eta)dx\\
&\quad=\frac{1}{(1+i\xi\csc\theta)^t}\int_{-\infty}^{\infty}K^\theta(x,\xi)\sum_{r=0}^{t}\binom{t}{r}(i\xi\csc\theta)^r\ a(x,\eta)dx\\
&\quad=\frac{1}{(1+i\xi\csc\theta)^t}\int_{-\infty}^{\infty}K^\theta(x,\xi)(1+i\xi\csc\theta)^t\ a(x,\eta)dx\\
&\quad=\hat{a}^\theta(\xi,\eta).
\end{aligned}$$

(ii) Using definition (7) and the technique of [7, p. 359], we have

$$\begin{aligned}
|\hat{a}^\theta(\xi,\eta)| &= \Big|\frac{1}{(1+i\xi\csc\theta)^t}\int_{-\infty}^{\infty}K^\theta(x,\xi)\sum_{r=0}^{t}\binom{t}{r}(\bar{\Delta}_x)^r a(x,\eta)dx\Big|\\
&= \Big|\frac{1}{(1+i\xi\csc\theta)^t}\int_{-\infty}^{\infty}K^\theta(x,\xi)\sum_{r=0}^{t}\binom{t}{r}(-1)^r(\Delta_x^*)^r a(x,\eta)dx\Big|\\
&\le \sum_{r=0}^{t}\binom{t}{r}|\overline{C^\theta}|(1+\xi^2\csc^2\theta)^{-t/2}\int_{-\infty}^{\infty}|(\Delta_x^*)^r a(x,\eta)|dx\\
&\le \sum_{r=0}^{t}\binom{t}{r}|\overline{C^\theta}|(1+\xi^2\csc^2\theta)^{-t/2}\int_{-\infty}^{\infty}\sum_{s=0}^{r}\sum_{l=0}^{s}|d_l|\ |x|^l|D_x^s a(x,\eta)|dx\\
&\le \sum_{r=0}^{t}\binom{t}{r}\sum_{s=0}^{r}\sum_{l=0}^{s}\ |d_l|\ |\overline{C^\theta}|(1+\xi^2\csc^2\theta)^{-t/2}\\
&\quad\times A_{s,m,q,\theta}(1+|\eta\csc\theta|)^{m+s\sigma}\int_{-\infty}^{\infty}(1+|x\csc\theta|)^{-q+t}dx\\
&\le A'_{t,m,q,l,\theta}(1+\xi^2\csc^2\theta)^{-t/2}(1+|\eta\csc\theta|)^{m+t\sigma}\\
&\quad\times\int_{-\infty}^{\infty}(1+|x\csc\theta|)^{-q+t}dx.
\end{aligned}$$

The integral is convergent for $q > t+1$. Hence there exists a constant $C_{t,m,q,l,\theta} > 0$ such that

$$|\hat{a}^\theta(\xi,\eta)| \le C_{t,m,q,l,\theta}(1+\xi^2\csc^2\theta)^{-t/2}(1+|\eta\csc\theta|)^{m+t\sigma}.$$

This completes the proof of the lemma. □

Theorem 4. *Let the symbol* $a(x,\xi) \in S^{m,\theta}_{\rho,\sigma}$. *Then the fractional Fourier transform of* $T^\theta_a\phi$ *can be represented as:*

$$\mathcal{F}^\theta\big[e^{\frac{i}{2}(.)^2\cot\theta}(T^\theta_a\phi)\big](\xi) = \overline{C^\theta}\ g_\theta(\xi), \quad \phi \in \mathscr{S}_\theta(\mathbb{R}),$$

where

$$g_\theta(\xi) = \int_{-\infty}^{\infty} \hat{\phi}^\theta(\eta) e^{i\eta(\xi-\eta)\cot\theta} \hat{a}^\theta(\xi-\eta,\eta)d\eta.$$

Proof. Using definitions (5) and (8), we have

$$\begin{aligned}
&(T^\theta_a\phi)(x)\\
&= \int_{-\infty}^{\infty} \overline{K^\theta(x,\eta)} a(x,\eta)\hat{\phi}^\theta(\eta)d\eta\\
&= \overline{C^\theta} e^{-\frac{i}{2}x^2\cot\theta} \int_{-\infty}^{\infty} e^{-\frac{i}{2}\eta^2\cot\theta + ix\eta\csc\theta}\Big(\int_{-\infty}^{\infty}\overline{K^\theta(x,\zeta)}\hat{a}^\theta(\zeta,\eta)d\zeta\Big)\hat{\phi}^\theta(\eta)d\eta\\
&= \overline{C^\theta} e^{-\frac{i}{2}x^2\cot\theta} \int_{-\infty}^{\infty} \overline{K^\theta(x,\xi-\eta)} e^{\frac{i}{2}(\eta^2-2\xi\eta)\cot\theta + ix\eta\csc\theta} g_\theta(\xi)d\xi\\
&= \overline{C^\theta} e^{-\frac{i}{2}x^2\cot\theta} \int_{-\infty}^{\infty} \overline{K^\theta(x,\xi)}\ g_\theta(\xi)d\xi\\
&= \overline{C^\theta} e^{-\frac{i}{2}x^2\cot\theta} \mathcal{F}^{-\theta}\big[g_\theta\big](x).
\end{aligned}$$

Therefore

$$\mathcal{F}^\theta\big[e^{\frac{i}{2}(.)^2\cot\theta}(T^\theta_a\phi)\big](\xi) = \overline{C^\theta}\ g_\theta(\xi).$$

This completes the proof of the theorem. □

Theorem 5. *Let the symbol* $a(x,\xi) \in S^{m,\theta}_{\rho,\sigma}$ *and* $T^\theta_a\phi$ *be associated with f.p.d.o. Then there exists* $M^\theta_{m,q,s,t,\sigma} > 0$ *s.t.*

$$||T^\theta_a\phi||_{H^{s,\theta}(\mathbb{R})} \le M^\theta_{m,q,s,t,\sigma}||\phi||_{H^{m+s+t\sigma,\theta}(\mathbb{R})}, \qquad \phi \in \mathscr{S}_\theta(\mathbb{R}),$$

$\forall s \in \mathbb{R}$ *and* m, q, t, σ *as above.*

Proof. From Theorem 4, we have

$$\begin{aligned}
&(1+|\xi\csc\theta|^2)^{s/2}\mathcal{F}^\theta\big[e^{\frac{i}{2}(.)^2\cot\theta}(T^\theta_a\phi)\big](\xi)\\
&= \overline{C^\theta}(1+|\xi\csc\theta|^2)^{s/2} g_\theta(\xi)\\
&= \overline{C^\theta}(1+|\xi\csc\theta|^2)^{s/2} \int_{-\infty}^{\infty} \hat{\phi}^\theta(\eta) e^{i\eta(\xi-\eta)\cot\theta}\hat{a}^\theta(\xi-\eta,\eta)d\eta\\
&= \overline{C^\theta} \int_{-\infty}^{\infty} \frac{(1+|\xi\csc\theta|^2)^{s/2}}{(1+|\eta\csc\theta|^2)^{s/2}}(1+|\eta\csc\theta|^2)^{s/2}\hat{\phi}^\theta(\eta) e^{i\eta(\xi-\eta)\cot\theta}\hat{a}^\theta(\xi-\eta,\eta)d\eta.
\end{aligned}$$

Using Lemma 3, we have

$$\begin{aligned}
&|(1+|\xi\csc\theta|^2)^{s/2}\mathcal{F}^\theta\big[e^{\frac{i}{2}(.)^2\cot\theta}(T^\theta_a\phi)\big](\xi)|\\
&\le |\overline{C^\theta}|2^{|s|/2} \int_{-\infty}^{\infty} (1+|(\xi-\eta)\csc\theta|^2)^{s/2}(1+|\eta\csc\theta|^2)^{s/2}|\hat{\phi}^\theta(\eta)|\ |\hat{a}^\theta(\xi-\eta,\eta)|d\eta.
\end{aligned}$$

We note that

$$(1+|\xi\csc\theta|)^{m+t\sigma} \leq L_{m,t,\sigma,\theta}(1+|\xi\csc\theta|^2)^{(m+t\sigma)/2}, \xi\in\mathbb{R}, \tag{12}$$

where $L_{m,t,\sigma,\theta} = \max[1, 2^{m+t\sigma/2}]$.
Moreover, from (12) and Lemma 6 (ii), we get

$$|\hat{a}^\theta(\xi-\eta,\eta)| \leq L_{m,t,\sigma,\theta}C_{t,m,q,\theta}(1+|\eta\csc\theta|^2)^{(m+t\sigma)/2}(1+|(\xi-\eta)\csc\theta|^2)^{-t/2}. \tag{13}$$

Therefore

$$\begin{aligned}
&|(1+|\xi\csc\theta|^2)^{s/2}\mathcal{F}^\theta\big[e^{\frac{i}{2}(.)^2\cot\theta}(T_a^\theta\phi)\big](\xi)| \\
&\quad\leq L_{m,t,\sigma,\theta}C_{t,m,q,\theta}|\overline{C^\theta}|2^{|s|/2}\int_{-\infty}^{\infty}(1+|\eta\csc\theta|^2)^{(m+s+t\sigma)/2}|\hat{\phi}^\theta(\eta)| \\
&\quad\times(1+|(\xi-\eta)\csc\theta|^2)^{(s-t)/2}d\eta \\
&\quad=\int_{-\infty}^{\infty}f^\theta(\eta)g^\theta(\xi-\eta)d\eta \quad \text{(say)} \\
&\quad=(f^\theta\star g^\theta)(\xi).
\end{aligned}$$

Now if $t > s+2$, $g^\theta = (1+|(\xi-\eta)\csc\theta|^2)^{(s-t)/2} \in L^1(\mathbb{R})$. Also, since $\hat{\phi}^\theta \in \mathscr{S}$, $f^\theta = (1+|\eta\csc\theta|^2)^{(m+s+t\sigma)/2}|\hat{\phi}^\theta(\eta)| \in L^2(\mathbb{R})$. Then $(f^\theta\star g^\theta)(\xi)$, $\forall\xi\in\mathbb{R}$ exists and belongs to $L^2(\mathbb{R})$. Hence

$$||(1+|\xi\csc\theta|^2)^{s/2}\mathcal{F}^\theta\big[e^{\frac{i}{2}(.)^2\cot\theta}(T_a^\theta\phi)\big](\xi)||_{L^2(\mathbb{R})} \leq M^\theta_{m,q,s,t,\sigma}||(f^\theta\star g^\theta)||_{L^2(\mathbb{R})}.$$

Therefore

$$||e^{\frac{i}{2}(.)^2\cot\theta}(T_a^\theta\phi)(\xi)||_{H^{s,\theta}(\mathbb{R})} \leq M^\theta_{m,q,s,t,\sigma}||f^\theta||_{L^2(\mathbb{R})}||g^\theta||_{L^1(\mathbb{R})}.$$

This implies that

$$||(T_a^\theta\phi)(\xi)||_{H^{s,\theta}(\mathbb{R})} \leq M'^\theta_{m,q,s,t,\sigma}||\phi||_{H^{m+s+t\sigma,\theta}(\mathbb{R})}.$$

This completes the proof of the theorem. □

5. Application of the fractional Fourier transform to a generalized Fredholm integral equation

Fredholm integral equations of the second kind occur in a natural way while solving a large class of boundary value problems of mathematical physics. It is very rare that exact solutions of such integral equations can be determined completely. Before describing the generalization, we need to note that it has recently been recognized that Fredholm's solution can afford a tool for obtaining analytic solutions.
In this section, we discuss the method of the fractional Fourier transform that can be used to solve a generalized Fredholm integral equation of the second kind in fractional convolution form.

We consider the generalized Fredholm integral equation of the second kind with the fractional convolution kernel in the form

$$\int_{-\infty}^{\infty} \mathsf{K}_\theta(x,t)f(t)dt + \lambda f(x) = u(x), \quad x \in \mathbb{R}, \tag{14}$$

where kernel $\mathsf{K}_\theta(x,t) = g(x-t)e^{-\frac{i}{2}(x^2-t^2)\cot\theta}$ is in $L^2(\mathbb{R}^2)$, while a known function $u(x)$ and an unknown function $f(x)$ are in $L^2(\mathbb{R})$ and λ is a known parameter. Equation (14) can be re-written in the convolution form as

$$(f \star_\theta g)(x) + \lambda f(x) = u(x). \tag{15}$$

Applying the FrFT to both sides of equation (15) and using (6), we have

$$\begin{aligned} \mathcal{F}^\theta(f \star_\theta g)(\omega) + \lambda \hat{f}^\theta(\omega) &= \hat{u}^\theta(\omega) \\ \frac{1}{C^\theta} e^{-\frac{i}{2}\omega^2\cot\theta} \hat{f}^\theta(\omega)\mathcal{F}^\theta\big(e^{-\frac{i}{2}(.)^2\cot\theta} g\big)(\omega) + \lambda \hat{f}^\theta(\omega) &= \hat{u}^\theta(\omega). \end{aligned}$$

Thus

$$\hat{f}^\theta(\omega) = \frac{\hat{u}^\theta(\omega)}{\frac{1}{C^\theta} e^{-\frac{i}{2}\omega^2\cot\theta}\mathcal{F}^\theta\big(e^{-\frac{i}{2}(.)^2\cot\theta} g\big)(\omega) + \lambda}.$$

The inverse fractional Fourier transform leads to a formal solution

$$f(x) = \int_{-\infty}^{\infty} \frac{\overline{K^\theta(x,\omega)}\hat{u}^\theta(\omega)d\omega}{\frac{1}{C^\theta} e^{-\frac{i}{2}\omega^2\cot\theta}\mathcal{F}^\theta\big(e^{-\frac{i}{2}(.)^2\cot\theta} g\big)(\omega) + \lambda}. \tag{16}$$

In particular, if $\lambda = 1$ and $g(x) = \frac{1}{2}\left(\frac{x}{|x|}\right)$, so that

$$\mathcal{F}^\theta\big(e^{-\frac{i}{2}(.)^2\cot\theta} g\big)(\omega) = \frac{C^\theta}{i\omega\csc\theta} e^{\frac{i}{2}\omega^2\cot\theta},$$

then equation (16) reduces to the form

$$f(x) = \int_{-\infty}^{\infty} \frac{\overline{K^\theta(x,\omega)}\ i\omega\csc\theta\ \hat{u}^\theta(\omega)d\omega}{1+i\omega\csc\theta}. \tag{17}$$

Now

$$\mathcal{F}^\theta\big(e^{-\frac{i}{2}(.)^2\cot\theta} e^{-(.)}\big)(\omega) = \frac{1}{1+i\omega\csc\theta} C^\theta e^{\frac{i}{2}\omega^2\cot\theta}. \tag{18}$$

Using Lemma 2 (iv) and (18),

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \overline{K^\theta(x,\omega)} \frac{1}{C^\theta} e^{-\frac{i}{2}\omega^2\cot\theta}\mathcal{F}^\theta\big(e^{-\frac{i}{2}(.)^2\cot\theta} e^{-(.)}\big)(\omega)\ \mathcal{F}^\theta[\bar{\Delta}u](\omega)d\omega \\ &= \int_{-\infty}^{\infty} \overline{K^\theta(x,\omega)}\mathcal{F}^\theta\big[\bar{\Delta}u \star_\theta e^{-\frac{i}{2}(.)^2\cot\theta} e^{-(.)}\big](\omega)d\omega \\ &= \big[\bar{\Delta}u \star_\theta e^{-\frac{i}{2}(.)^2\cot\theta} e^{-(.)}\big](x) \\ &= \int_{-\infty}^{\infty} \bar{\Delta}u(y)\ e^{-[(x-y)+\frac{i}{2}(x-y)^2\cot\theta]}\ e^{-\frac{i}{2}(x^2-y^2)\cot\theta} dy \\ &= \int_{-\infty}^{\infty} \bar{\Delta}u(y)\ exp[(y-x)(1+ix\cot\theta)]dy. \end{aligned}$$

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