Pseudo-differential operators and symmetries

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HAPDE

1 October 2009

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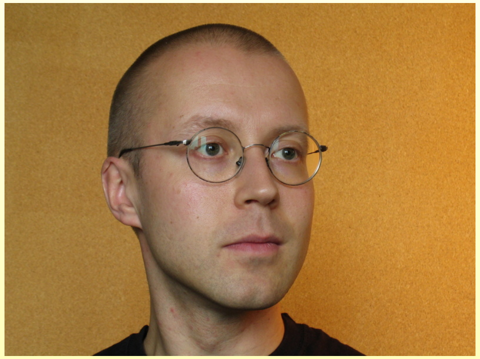

Joint work with **Ville Turunen**

(Helsinki University of Technology)

Some preliminary information

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• In this talk we discuss some aspects ($\mathbb{T}^n$, $\mathbb{S}^3$ and $\mathrm{SU}(2)$, compact groups).

M. Ruzhansky, V. Turunen:

Pseudo-Differential Operators and Symmetries

Contents:

(To be continued...)

M. Ruzhansky, V. Turunen:

Pseudo-Differential Operators and Symmetries

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Some observations

A linear elliptic PDE Dirichlet problem in a bounded smooth $\Omega \subset \mathbb{R}^n$ leads to a **pseudo-differential** equation on $M = \partial\Omega$ (e.g. Calderon projections).

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Suppose $M = \partial\Omega$ is "symmetric" (or diffeomorphic to "symmetric")
$\Rightarrow$ efficient **global** calculus on $\partial\Omega$
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Homogeneous spaces: A transitive action $G \times M \to M$ of a Lie group G on a manifold M; calculus on M as a "shadow" from that on G.
Chapter 13: ΨDOs on $M \longleftrightarrow \Psi$DOs on G (in this case $M = G/K$ or $K\backslash G$).
Interesting observation: harmonic analysis on Lie groups and the theory of ΨDOs on $\mathbb{R}^n$ seem to be "notationally incompatible".

Examples: $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ [Agranovich 1990], [Turunen & Vainikko 1998], [Turunen 2000], [R. & Turunen 2008]. Also, the spheres,
$\mathrm{SO}(n) \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, $\mathrm{SU}(n) \times M \to M$ with $M = \{x \in \mathbb{C}^n : \ \|x\|_{\mathbb{C}^n} = 1\}$.

Chapters 1–2 on $\mathbb{R}^n$ [Kohn & Nirenberg, Hörmander 1965]:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\ \mathsf{e}^{-\mathsf{i}2\pi x\cdot\xi}\ \mathrm{d}x, \quad Af(x) = \int_{\mathbb{R}^n} \mathsf{e}^{\mathsf{i}2\pi x\cdot\xi}\ \sigma_A(x,\xi)\ \widehat{f}(\xi)\ \mathrm{d}\xi,$$

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right| \leq C_{\alpha\beta}\ \langle\xi\rangle^{m-|\alpha|}$$

Short overview

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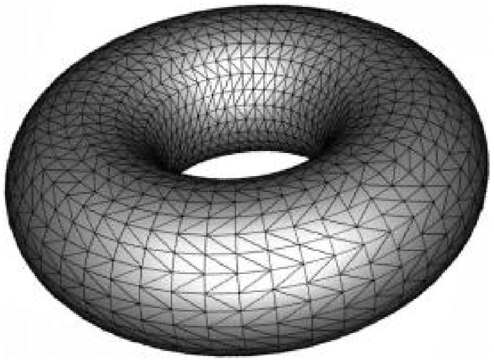
Chapters 5–13 on a compact Lie group G:

$$\widehat{f}(\xi) = \int_G f(x)\ \xi(x)^*\ \mathrm{d}x,\ Af(x) = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \mathsf{Tr}\left(\xi(x)\ \sigma_A(x,\xi)\ \widehat{f}(\xi)\right),$$

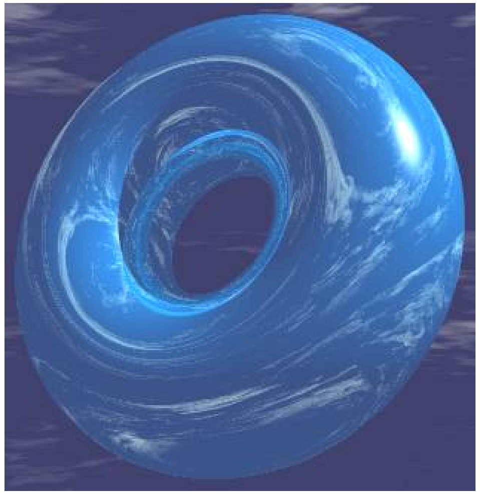
$$\left\|\triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right\| \le C_{\alpha\beta}\ \langle\xi\rangle^{m-|\alpha|},\ \ldots$$

Torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$



The idea behind:

$\mathbb{T}^n$ as manifold $\longrightarrow$ $\mathbb{T}^n$ as group



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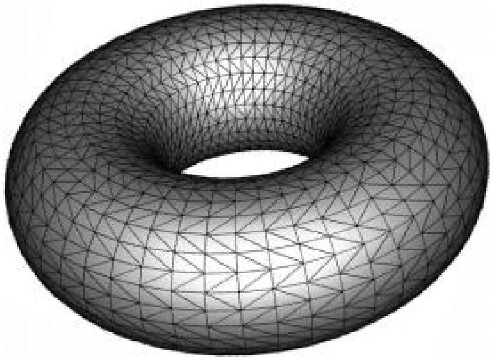

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$e_\xi : \mathbb{T}^n \to U(1)$ hence $\widehat{\mathbb{T}^n} \simeq \mathbb{Z}^n$

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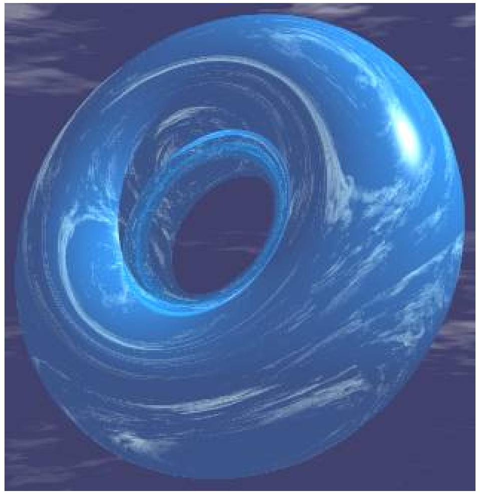
(note: same idea for $\widehat{\mathbb{R}^n} \simeq \mathbb{R}^n$)



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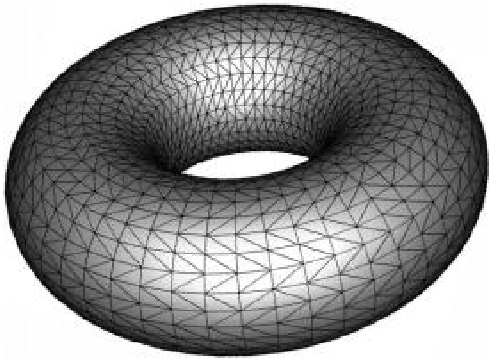

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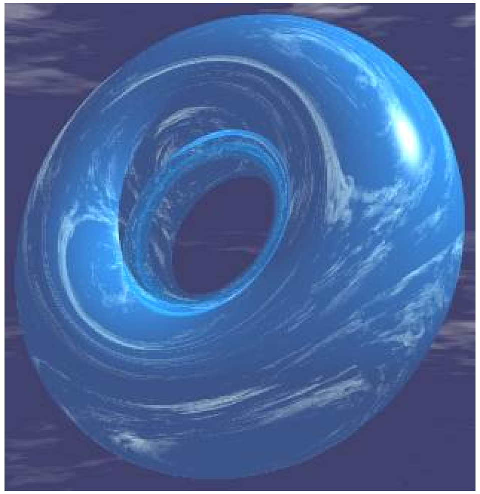

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"Toroidal" pseudo-differential operators

$$\boxed{(Au)(x) = \sum_{\xi\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}\, \sigma_A(x,\xi)\, \widehat{u}(\xi)}$$

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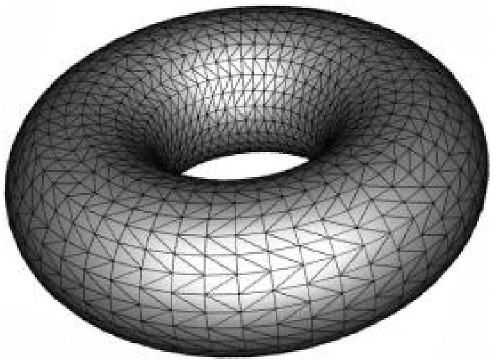

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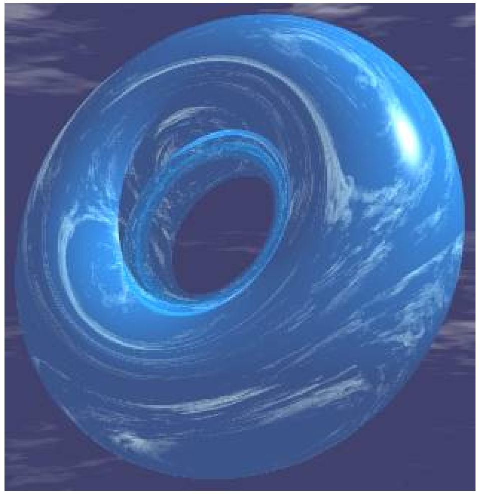

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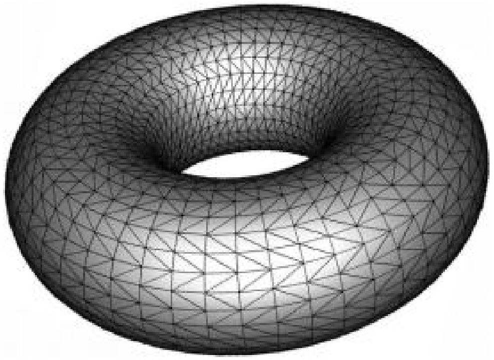

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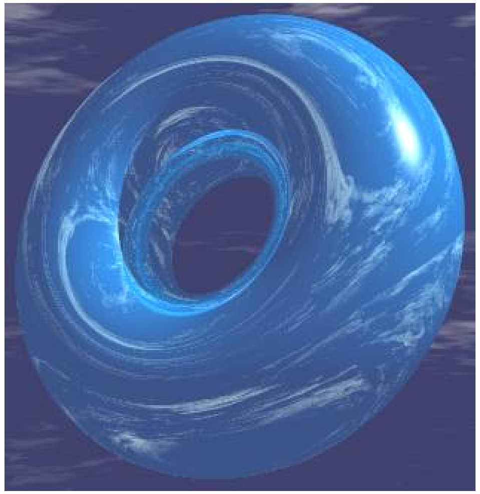

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[Agranovich '90], [McLean '91], [Turunen '00]; [R-Turunen '06]-periodisation: the set of such operators gives a "toroidal" quantisation of the usual Hörmander's class $\mathrm{Op}S^m_{\rho,\delta}(\mathbb{T}^n)$ defined by local coordinates – more later.

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Theorem (Taylor expansion on $\mathbb{Z}^n$) *Let $p : \mathbb{Z}^n \to \mathbb{C}$. Then*

$$\boxed{p(\xi + \theta) = \sum_{|\alpha|<M} \frac{1}{\alpha!} \, \theta^{(\alpha)} \triangle_\xi^\alpha p(\xi) + r_M(\xi, \theta)}$$

where $\theta^{(\alpha)} := \theta_1^{(\alpha_1)} \cdots \theta_n^{(\alpha_n)}$, $\theta_j^{(\alpha_j)} := \theta_j \, (\theta_j - 1) \cdots (\theta_j - (\alpha_j + 1))$ *and*

$$\left| \triangle_\xi^\omega r_M(\xi, \theta) \right| \leq \sum_{|\alpha|=M} \frac{1}{\alpha!} \, \left| \theta^{(\alpha)} \right| \, \max_{\nu \in Q(\theta)} \left| \triangle_\xi^{\alpha+\omega} p(\xi + \nu) \right|,$$

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Using this theorem, one develops all the calculus of globally defined toroidal symbols on $\mathbb{T}^n$. Formulae are same as usual, but with ∂_ξ–derivatives replaced by differences Δ_ξ. By periodisation theorems it is equivalent to the standard calculus on $\mathbb{T}^n$ (as a manifold), but here we have full symbols (thus also FFT).

For any operator $A : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$, consider its *toroidal quantisation*:

$$\boxed{A\varphi(x) = \sum_{\xi\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}\ \sigma_A(x,\xi)\ \widehat{f}(\xi)}$$

where its *toroidal symbol* $\sigma_A \in C^\infty(\mathbb{T}^n \times \mathbb{Z}^n)$ is uniquely defined by the formula

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Pseudo-differential operators on $\mathbb{T}^n$

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Let $m \in \mathbb{R}$. Define *toroidal symbol class* $S^m(\mathbb{T}^n \times \mathbb{Z}^n)$ to consist of functions $a(x,\xi)$ which are smooth in x for all $\xi \in \mathbb{Z}^n$, and which satisfy

$$\left|\triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right| \leq C_{a\alpha\beta m}\ \langle\xi\rangle^{m-|\alpha|}, \text{ for all } x \in \mathbb{T}^n, \alpha, \beta \in \mathbb{N}_0^n, \xi \in \mathbb{Z}^n.$$

Theorem (Agranovich, McLean): *On $\mathbb{T}^n$, Hörmander's usual (also (ρ,δ)) class of pseudo-differential operators $\mathrm{Op}S^m(\mathbb{R}^n \times \mathbb{R}^n)$ of order $m \in \mathbb{R}$ which are 2π-periodic in x coincides with the class $\mathrm{Op}S^m(\mathbb{T}^n \times \mathbb{Z}^n)$, i.e. we have*

$$\mathrm{Op}S^m(\mathbb{T}^n \times \mathbb{R}^n) = \mathrm{Op}S^m(\mathbb{T}^n \times \mathbb{Z}^n).$$

Relations between symbols

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Chapter 4. Pseudo-differential operators on $\mathbb{T}^n$ in toroidal quantization:

- periodisation operators, Poisson summation formula;
- relation between toroidal and Euclidean symbols and the corresponding operators;
- toroidal calculus: compositions, adjoints, compound symbols, ellipticity, ...
- boundedness on $L^2(\mathbb{T}^n)$, $L^p(\mathbb{T}^n)$, and on Sobolev spaces $W^{p,s}(\mathbb{T}^n)$; toroidal wave front sets;
- Fourier series operators, calculus of FSO's, boundedness of FSO's on $L^2(\mathbb{T}^n)$;
- Applications to hyperbolic problems and integral operators;

To develop similar things on general compact Lie groups we rely on the representation theory. Let us look at the example of $\mathbb{S}^3$. It is much more convenient for us to look at $\mathbb{S}^3$ as SU(2) because then we have lots of things that are known about representations of SU(2).

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The quaternion space $\mathbb{H}$ is (the associative $\mathbb{R}$-algebra) with a vector space basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, where $\mathbf{1} \in \mathbb{H}$ is the unit and

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$\mathbb{S}^3$ in $\mathbb{H}$ and SU(2) are isomorphic and diffeomorphic (there is a bijective differentiable mapping between them). This gives ΨDOs on sphere $\mathbb{S}^3$ parallel to ΨDOs on SU(2).

Thus, $\boxed{\mathbb{S}^3 \xrightarrow[\text{group isomorphism}]{C^\infty} SU(2)}$ This gives ΨDOs on sphere $\mathbb{S}^3$.

Note that by using the Poincaré conjecture, we can also extend everything to arbitrary closed (simply connected) 3–manifolds.

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Let us define (these are irreducible unitary representations of $\mathbb{S}^3$ (of SU(2) actually) $t^l_{mn} \in C^\infty(\mathbb{S}^3)$, where $l \in \frac{1}{2}\mathbb{N}$ and $-l \leq m, n \leq +l$ such that $l - m, l - n \in \mathbb{Z}$. In Euler's angles

$$t^l_{mn}(\phi, \theta, \psi) = \mathrm{e}^{-\mathrm{i}(m\phi + n\psi)}\; P^l_{mn}(\cos(\theta)),$$

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For simplicity, we forget for now about underlying representation theory and just present some outcomes.

Thus, on $\mathbb{S}^3$, we have a family of group homomorphisms
$t^l : \mathbb{S}^3 \to U(2l+1) \subset \mathbb{C}^{(2l+1)\times(2l+1)}$, $l \in \frac{1}{2}\mathbb{N}_0$.

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Then we have

$$\boxed{Af(x) = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \operatorname{Tr}\left(t^l(x)\ \sigma_A(x,l)\ \widehat{f}(l)\right)}$$

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We also note that if

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Questions:

• how to define symbols to recover Hörmander's classes $\Psi^m(\mathbb{S}^3)$?

e.g. do matrices $\sigma_A(x,l)$ have some structure?

• what are difference operators in symbolic inequalities?

Answers: very interesting!

Euclidean space $\mathbb{R}^n$:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \quad Af(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \sigma_A(x,\xi) \widehat{f}(\xi) d\xi,$$

with $|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| \leq C_{\alpha\beta} \langle\xi\rangle^{m-|\alpha|}$.

Torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$:

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Sphere $\mathbb{S}^3$:

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Compositions in $\mathbb{R}^n$ [Mikhlin, Calderon & Zygmung, Kohn & Nierenberg]:

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha\geq 0} \frac{1}{\alpha!} \, (\partial_\xi^\alpha \sigma_A)(x,\xi) \, (D_x^\alpha \sigma_B)(x,\xi).$$

Compositions on $\mathbb{T}^n$ [Turunen & Vainikko 1998 for $S^m_{1,0}(\mathbb{R})$, R. & Turunen 2007 for $S^m_{\rho,\delta}(\mathbb{R}^n)$ & also for FIOs/FSOs]:

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha\geq 0} \frac{1}{\alpha!} \, (\triangle_\xi^\alpha \sigma_A)(x,\xi) \, (D_x^{(\alpha)} \sigma_B)(x,\xi),$$

$D_x^{(\alpha)} = D_{x_1}^{(\alpha_1)} \cdots D_{x_n}^{(\alpha_n)}$, where $D_{x_j}^{(0)} = I$ and
$D_{x_j}^{(k+1)} = D_{x_j}^{(k)} \left(\frac{\partial}{\mathrm{i}\partial x_j} - kI\right) = \frac{\partial}{\mathrm{i}\partial x_j} \left(\frac{\partial}{\mathrm{i}\partial x_j} - I\right) \cdots \left(\frac{\partial}{\mathrm{i}\partial x_j} - kI\right).$

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$$\sigma_{AB}(x,l) \sim \sum_{\alpha\geq 0} \frac{1}{\alpha!} \, (\triangle_l^\alpha \sigma_A)(x,l) \, (D_x^{(\alpha)} \sigma_B)(x,l),$$

with appropriate definitions of differences $\triangle_l^\alpha$ and derivatives $D_x^{(\alpha)}$.

Theorem. [R. & Turunen 2008]
*$A : C^\infty(\mathbb{S}^3) \to C^\infty(\mathbb{S}^3)$ belongs to the usual Hörmander's class $\Psi^m(\mathbb{S}^3)$ if and only if its Lie group symbol $\sigma_A \in S^m(\mathbb{S}^3)$
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Definition. Symbol $\sigma_A \in S^m(\mathbb{S}^3)$ if and only if for every $N \geq 0$ we have

$$\left|\triangle_l^\alpha \partial_x^\beta \sigma_{A_u}(x,l)_{ij}\right| \leq C_{A\alpha\beta mN} \, (1+|i-j|)^{-N} (1+l)^{m-|\alpha|},$$

where for $u \in \mathrm{SU}(2)$ we define

$$\sigma_{A_u}(x,l) = t^l(u)^* \sigma_A(x,l)\; t^l(u)$$

and kernel $K_A(x,y)$ of A is smooth outside the diagonal $x=y$.

There are 3 difference operators $\Delta_+, \Delta_-, \Delta_0$ and $\Delta_l^\alpha = \Delta_+^{\alpha_1} \Delta_-^{\alpha_2} \Delta_0^{\alpha_3}$.
Operators $\Delta_+, \Delta_-, \Delta_0$ act on symbols on $\mathbb{S}^3$ and there are explicit formulae for them.

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If G is a compact Lie group, its unitary dual $\widehat{G}$ is defined as

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$m = \dim H$ is called the dimension of the representation ϕ $(\dim \phi := m)$.

If the group is commutative (e.g. $\mathbb{R}^n$, $\mathbb{T}^n$), its representations are one-dimensional $(m = 1)$ – will be important!

Peter–Weyl theorem: $\sqrt{\dim\psi}\,\psi_{ij}$ is an orthonormal basis of $L^2(G,\mu_G)$, where $\psi=\{\psi_{ij}\}_{i,j=1}^m$ and $[\psi]\in\widehat{G}$.

Examples:
$e^{2\pi i x\cdot k}$, $k\in\mathbb{Z}^n$, is an orthonormal basis of $\mathbb{T}^n$.
$e^{2\pi i x\cdot \xi}$, $\xi\in\mathbb{R}^n$, is an "orthonormal basis" of $\mathbb{R}^n$.

Since these groups are commutative, 1×1 representations are just complex valued functions, and they simply give the basis.

This also implies that familiar symbols on $\mathbb{R}^n$ and $\mathbb{T}^n$ are just complex valued (and they are matrix-valued for non-commutative groups, e.g. on $\mathbb{S}^3$).

Unitary dual $\widehat{G}$ consists of equivalence classes $[\xi]$ of irreducible unitary representations ξ of G. Choosing a particular representation from $[\xi]$, we can think that $\xi(x) \in \mathbb{C}^{\dim \xi \times \dim \xi}$, where $\dim \xi$ is the dimension of representation ξ. Note: often there are explicit formulae for $\xi(x)$.

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Answers: very interesting!

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Torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ [Agranovich 1990, McLean 1991, Turunen 2000]:

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Compact Lie group G:

$$\widehat{f}(\xi) = \int_G f(x) \xi(x)^* dx, \quad Af(x) = \sum_{[\xi]\in\widehat{G}} \dim(\xi) \ \operatorname{Tr}\left(\xi(x) \sigma_A(x,\xi) \widehat{f}(\xi)\right),$$

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Question: what are difference operators Δ_ξ on $\widehat{G}$?

Space $L^2(\widehat{G})$

On $\widehat{G}$ we work with mappings

$$F : \widehat{G} \to \bigcup_{[\xi]\in\widehat{G}} \mathcal{L}(\mathcal{H}_\xi) \subset \bigcup_{m=1}^{\infty} \mathbb{C}^{m\times m},$$

satisfying $F([\xi]) \in \mathcal{L}(\mathcal{H}_\xi)$ for every $[\xi] \in \widehat{G}$. In matrix representations, we can view $F([\xi]) \in \mathbb{C}^{\dim(\xi)\times\dim(\xi)}$ as a $\dim(\xi) \times \dim(\xi)$ matrix.

The space $L^2(\widehat{G})$ consists of all mappings

$$\boxed{||F||^2_{L^2(\widehat{G})} := \sum_{[\xi]\in\widehat{G}} \dim(\xi) \ \|F([\xi])\|^2_{HS} < \infty}$$

where $||F([\xi])||_{HS} = \sqrt{\mathrm{Tr}(F([\xi]) \ F([\xi])^*)}$.

Parseval's identity Let $f, g \in L^2(G)$. Then we have

$$(f, g)_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} \dim(\xi) \ \mathrm{Tr}\left(\widehat{f}(\xi) \ \widehat{g}(\xi)^*\right) = (\widehat{f}(\xi), \widehat{g}(\xi))_{L^2(\widehat{G})}.$$

What is $\langle\xi\rangle$ on $\widehat{G}$?

For every $\xi \in \widehat{G}$ we can construct the eigenspace $\mathcal{H}^{\xi}$ of the Laplacian $\mathcal{L}_G$: $-\mathcal{L}_G|_{\mathcal{H}^{\xi}} = \lambda_{\xi}^2 I$, for some $\lambda_{\xi} \in \mathbb{R}$. We have $\dim \mathcal{H}^{\xi} = (\dim(\xi))^2$. We denote

$$\boxed{\langle\xi\rangle := (1 + \lambda_{[\xi]}^2)^{1/2}}$$

Proposition (Dimension and eigenvalues) *There exists a constant $C > 0$ such that the inequality* $\dim(\xi) \leq C\langle\xi\rangle^{\frac{\dim G}{2}}$ *holds for all $\xi \in \mathrm{Rep}(G)$.*

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The space $\mathcal{S}(\widehat{G})$ consists of all mappings H such that for all $k \in \mathbb{N}$ we have

$$\sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \langle\xi\rangle^k\ ||H(\xi)||_{HS} < \infty.$$

Proposition $\mathcal{S}(\widehat{G})$ *is a Montel nuclear space.*

The space $\mathcal{S}'(\widehat{G})$ is the space of all H for which there exists some $k \in \mathbb{N}$:

$$\sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \langle\xi\rangle^{-k}||H(\xi)||_{HS} < \infty.$$

Fourier transform: continuous bijection $C^\infty(G) \longleftrightarrow \mathcal{S}(\widehat{G})$, $\mathcal{D}'(G) \longleftrightarrow \mathcal{S}'(\widehat{G})$.

For $1 \leq p < \infty$, we will write $L^p(\widehat{G}) \equiv \ell^p\left(\widehat{G}, \dim^{p\left(\frac{2}{p}-\frac{1}{2}\right)}\right)$ for the space of all $H \in \mathcal{S}'(\widehat{G})$ such that

$$||H||_{L^p(\widehat{G})} := \left(\sum_{[\xi]\in\widehat{G}} (\dim(\xi))^{p\left(\frac{2}{p}-\frac{1}{2}\right)} \, ||H(\xi)||_{HS}^p\right)^{1/p} < \infty.$$

For $p = \infty$, we write $L^\infty(\widehat{G}) \equiv \ell^\infty\left(\widehat{G}, \dim^{-1/2}\right)$ for all $H \in \mathcal{S}'(\widehat{G})$:

$$||H||_{L^\infty(\widehat{G})} := \sup_{[\xi]\in\widehat{G}} (\dim(\xi))^{-1/2} \, ||H(\xi)||_{HS} < \infty.$$

Important cases of $L^2(\widehat{G}) = \ell^2\left(\widehat{G}, \dim^1\right)$ and $L^1(\widehat{G}) = \ell^1\left(\widehat{G}, \dim^{3/2}\right)$ are

$$||H||_{L^2(\widehat{G})} := \left(\sum_{[\xi]\in\widehat{G}} \dim(\xi) \, ||H(\xi)||_{HS}^2\right)^{1/2}, \quad ||H||_{L^1(\widehat{G})} := \sum_{[\xi]\in\widehat{G}} (\dim(\xi))^{3/2} \, ||H(\xi)||_{HS}.$$

Interpolation of $L^p(\widehat{G})$ spaces Let $1 \le p_0, p_1 < \infty$. Then

$$\left(L^{p_0}(\widehat{G}), L^{p_1}(\widehat{G})\right)_{\theta,p} = L^p(\widehat{G}),$$

where $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Fourier transforms on $L^1(G)$ and $L^1(\widehat{G})$ We have

$$||\widehat{f}||_{L^\infty(\widehat{G})} \le ||f||_{L^1(G)}, \quad ||\mathcal{F}_G^{-1} H||_{L^\infty(G)} \le ||H||_{L^1(\widehat{G})}.$$

Hausdorff–Young inequality Let $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(G)$ and $H \in L^p(\widehat{G})$. Then $||\widehat{f}||_{L^q(\widehat{G})} \le ||f||_{L^p(G)}$ and $||\mathcal{F}_G^{-1} H||_{L^q(G)} \le ||H||_{L^p(\widehat{G})}$.

Duality of $L^p(\widehat{G})$ Let $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $\left(L^p(\widehat{G})\right)' = L^q(\widehat{G})$.

Sobolev spaces $L^p_k(\widehat{G})$ For $k \in \mathbb{N}$ we can define

$$L^p_k(\widehat{G}) = \left\{H \in L^p(\widehat{G}) : \triangle^\alpha H \in L^p(\widehat{G}) \text{ for all } |\alpha| \le k\right\}.$$

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- Developments of many related aspects: standard questions of microlocal analysis, non-compact spaces, etc.

Thank you

Okini