J. DIFFERENTIAL GEOMETRY
13 (1978) 25-41

PSEUDO-HERMITIAN STRUCTURES ON A REAL HYPERSURFACE

S. M. WEBSTER

Introduction

The invariance properties of a real hypersurface M (of real dimension $2n + 1$) in complex $(n + 1)$ space C^{n+1} with respectt o the infinite pseudo-group of biholomorphic transformations are the object of study in pseudo-conformal geometry. The systematic study of such properties for hypersurfaces with nondegenerate Levi form was first made by Cartan [2] in 1932. More recently, the study of invariants for such M was taken up by S. S. Chern and J. Moser [6]. A main aspect of the theory is the existence of a complete system of local differential invariants.

In this paper we take a somewhat different point of view. Such a manifold M has an integrable, nondegenerate, Cauchy-Riemann structure. In particular, there is a subbundle $H(M)$ of the tangent bundle $T(M)$ each fiber of which has the structure of a complex n-dimensional vector space. We single out a real nonvanishing one-form θ annihilating $H(M)$ and consider invariants of the pair (M, θ). (M, θ) will be called a pseudo-hermitian manifold.

In § 1 we apply the Cartan method of equivalence [3] to find a compete system of invariants. This results in a connection and curvature forms on the coframe bundle of M. These are not, in general, pseudo-conformal invariants; they depend on the choice of θ. In § 3 we consider the relation between these two systems of invariants. (3.8) gives a formula for the fourth order curvature tensor of Chern and Moser. A similar formula was given by Bochner [1] as a formal analogue of the conformal curvature tensor for a Kähler manifold. Here a geometric interpretation of the formula is given. In § 4 we apply the theory to some examples. It is shown that an ellipsoid is not, in general, equivalent to a sphere.

Also, the author wishes to remark that the theory developed here provides a complete system of invariants for nondegenerate real hypersurfaces under volume-preserving biholomorphic transformations, when the ambient complex space is equipped with a volume form.

We will follow the notation adopted in [6]. Small Greek indices run from 1 to n, and the summation convention is used. The Levi form $g_{\alpha\bar{\beta}}$ and its inverse $g^{\bar{\beta}\alpha}$ are used to lower and raise indices, e.g.,

Communicated by D. C. Spencer, November 19, 1975.

$$\theta_\alpha = g_{\alpha\bar\beta}\theta^{\bar\beta}\ , \qquad A^{\bar\alpha}{}_\beta = g^{\bar\alpha\gamma}A_{\gamma\beta}\ .$$

Thus the vertical as well as the horizontal position of an index carries information. Also, complex conjugation will be reflected in the indices, e.g.,

$$\theta^{\bar\beta} = \bar\theta^\beta\ , \quad U_{\bar\beta}{}^{\bar\alpha} = \bar U_\beta{}^\alpha\ , \quad \bar A_{\alpha\bar\beta\gamma} = A_{\bar\alpha\beta\bar\gamma}\ .$$

The work presented in this paper was submitted as part of the author's thesis at the University of California at Berkeley in June of 1975.

1. The equivalence problem

Let (M, θ) denote a $(2n + 1)$-dimensional pseudo-hermitian manifold. θ is a fixed real one-form, and locally we can choose n complex one-forms θ^α, so that $(\theta, \theta^\alpha, \theta^{\bar\alpha})$ form a basis of complex covectors. They are determined up to

$$\theta = \theta'\ , \quad \theta^\alpha = \theta'^\beta U_\beta{}^\alpha + \theta v^\alpha\ , \quad \theta^{\bar\alpha} = \theta'^{\bar\beta} U_{\bar\beta}{}^{\bar\alpha} + \theta v^{\bar\alpha}\ . \tag{1.1}$$

We require our structure to be integrable in the sense that

$$d\theta \equiv d\theta^\alpha \equiv 0\ , \qquad \mathrm{mod}\ \theta, \theta^\gamma\ . \tag{1.2}$$

Because $\theta = \bar\theta$, we must have

$$d\theta = i g_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} + \theta \wedge (\eta_\alpha\theta^\alpha + \eta_{\bar\alpha}\theta^{\bar\alpha})\ , \tag{1.3}$$

where $\eta_{\bar\alpha} = \bar\eta_\alpha$, and $g_{\alpha\bar\beta}$ is hermitian:

$$g_{\alpha\bar\beta} = \bar g_{\beta\bar\alpha} = g_{\bar\beta\alpha}\ . \tag{1.4}$$

Under the change (1.1) we have

$$g_{\alpha\bar\beta} = U^{-1}{}_\alpha{}^\rho g'_{\rho\bar\sigma} U^{-1}{}_{\bar\beta}{}^{\bar\sigma}\ . \tag{1.5}$$

We will also assume that (M, θ) is nondegenerate in the sense that the matrix (1.4) is nonsingular at each point. It will have a signature, say p negative and q positive eigenvalues, $p + q = n$, which we will speak of as the signature of (M, θ). If $g_{\alpha\bar\beta}$ is negative definite, (M, θ) will be said to be strongly pseudoconvex. In the computations to follow $g_{\alpha\bar\beta}$ and its inverse $g^{\bar\beta\alpha}$ will be used to lower and raise indices.

In other words, we have a nondegenerate, integrable G-structure on M, G being the group of matrices

$$\begin{pmatrix} 1 & v^\alpha & v^{\bar\alpha} \\ 0 & U_\beta{}^\alpha & 0 \\ 0 & 0 & U_{\bar\beta}{}^{\bar\alpha} \end{pmatrix}, \quad v^\alpha \in \mathbf{C}\ , \quad (U_\beta{}^\alpha) \in GL(n, C)\ . \tag{1.6}$$

To study the equivalence problem we begin by reducing the group (1.6). Substituting (1.1) with $U_\beta{}^\alpha = \delta_\beta{}^\alpha$ into (1.3), we get

$$d\theta = ig_{\alpha\bar\beta}\theta'^\alpha \wedge \theta'^{\bar\beta} + \theta \wedge (\eta'_\alpha\theta'^\alpha + \eta'_{\bar\alpha}\theta'^{\bar\alpha}) ,$$

where

$$\eta'_\alpha = \eta_\alpha - ig_{\alpha\bar\gamma}v^{\bar\gamma} .$$

Since $g_{\alpha\bar\gamma}$ is nondegenerate we can choose v^γ so that $\eta'_\alpha = 0$, and if $\eta_\alpha = \eta'_\alpha = 0$, then $v^\alpha = 0$.

Hence by requiring

$$d\theta = ig_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} , \tag{1.7}$$

we can reduce our group (1.6) to $GL(n, C)$, that is, to changes

$$\theta^\alpha = \theta'^\beta U_\beta{}^\alpha , \qquad \theta^{\bar\alpha} = \theta'^{\bar\beta} U_{\bar\beta}{}^{\bar\alpha} . \tag{1.8}$$

By also requiring

$$g_{\alpha\bar\beta} = \text{const.} = \pm\delta_{\alpha\beta} , \tag{1.9}$$

we can reduce our group further to $U(p, q)$, the unitary group with signature (p, q). The conditions (1.7) and (1.9) are invariant under maps preserving our structure.

For a geometric interpretation of (1.7) let us consider the dual frame

$$X = \bar X , \quad X_\alpha , \quad X_{\bar\alpha} = \bar X_\alpha \tag{1.10}$$

to $(\theta, \theta^\alpha, \theta^{\bar\alpha})$. The transformation (1.1) gives

$$X' = X + v^\alpha X_\alpha + v^{\bar\alpha} X_{\bar\alpha} , \quad X_\alpha = U_\alpha{}^\beta X_\beta , \quad X_{\bar\alpha} = U_{\bar\alpha}{}^{\bar\beta} X_{\bar\beta} . \tag{1.11}$$

The condition (1.7) then singles out a unique transversal X to $H(M)$.

Our admissible coframes are now those $(\theta, \theta^\alpha, \theta^{\bar\alpha})$ for which (1.7) holds. We allow $g_{\alpha\bar\beta}$ to be variable. Let P be the bundle of such coframes with structure group $GL(n, C)$. On P we have globally defined functions $g_{\alpha\bar\beta}$ given locally by (1.5) and globally defined complex one-forms $\theta^\alpha, \theta^{\bar\alpha}$ defined by (1.8), where now the $U_\beta{}^\alpha$ are independent fibre coordinates on P. We also have the real one-form θ pulled up to P and can view (1.7) as an equation on P. Since the real dimension of P is $2n^2 + 2n + 1$, we must find $2n^2$ more independent, intrinsically defined one-forms on P.

We first differentiate (1.8) and see that locally

$$d\theta^\alpha = \theta^\beta \wedge (-U^{-1}{}_\beta{}^\gamma dU_\gamma{}^\alpha) + d\theta'^\beta U_\beta{}^\alpha . \tag{1.12}$$

Because of the integrability condition (1.2) for θ, θ'^α, we have

(1.13) $$d\theta'^{\beta}U_{\beta}{}^{\alpha} = \theta^{\beta} \wedge \xi_{\beta}{}^{\alpha} + \theta \wedge \xi^{\alpha}$$

for some one-forms $\xi_{\beta}{}^{\alpha}$, ξ^{α} satisfying

(1.14) $$\xi_{\beta}{}^{\alpha} \equiv \xi^{\alpha} \equiv 0\ , \qquad \operatorname{mod} \theta, \theta^{\gamma}, \theta^{\bar{\gamma}}\ .$$

It follows from (1.12), (1.13) ,(1.14), and Cartan's lemma that the most general such expression of type (1.12) is

(1.15) $$d\theta^{\alpha} = \theta^{\beta} \wedge \omega_{\beta}{}^{\alpha} + \theta \wedge \tau^{\alpha}\ ,$$

where $\omega_{\beta}{}^{\alpha}$ and τ^{α} are one-forms satisfying

(1.16) $$\omega_{\beta}{}^{\alpha} \equiv -U^{-1}{}_{\beta}{}^{\gamma} dU_{\gamma}{}^{\alpha}\ , \qquad \operatorname{mod} \theta, \theta^{\gamma}, \theta^{\bar{\gamma}}\ ,$$

(1.17) $$\tau^{\alpha} \equiv 0\ , \qquad \operatorname{mod} \theta, \theta^{\gamma}, \theta^{\bar{\gamma}}\ .$$

From the form of (1.15) we see that we may require

(1.18) $$\tau^{\alpha} \equiv 0\ , \qquad \operatorname{mod} \theta^{\bar{\gamma}}\ .$$

Now the $\omega_{\beta}{}^{\alpha}$ are determined up to a transformation of the form

(1.19) $$\omega_{\beta}{}^{\alpha} = \tilde{\omega}_{\beta}{}^{\alpha} + C_{\beta}{}^{\alpha}{}_{\gamma}\omega^{\gamma}\ , \qquad C_{\beta}{}^{\alpha}{}_{\gamma} = C_{\gamma}{}^{\alpha}{}_{\beta}\ ,$$

and the τ^{α} are completely determined. The condition (1.18) allows us to put

(1.20) $$\tau_{\alpha} = A_{\alpha\gamma}\theta^{\gamma}\ .$$

Now we differentiate (1.7), using (1.15), to get

(1.21) $$0 = i(dg_{\alpha\bar{\beta}} - \omega_{\alpha}{}^{\gamma}g_{\gamma\bar{\beta}} - g_{\alpha\bar{\gamma}}\omega_{\bar{\beta}}{}^{\bar{\gamma}}) \wedge \theta^{\alpha} \wedge \theta^{\bar{\beta}} + i\theta \wedge (\tau_{\bar{\alpha}} \wedge \theta^{\bar{\alpha}} + \theta^{\alpha} \wedge \tau_{\alpha})\ .$$

With (1.20) substituted into (1.21), we see that

(1.22) $$dg_{\alpha\bar{\beta}} - \omega_{\alpha\bar{\beta}} - \omega_{\bar{\beta}\alpha} = A_{\alpha\bar{\beta}\gamma}\theta^{\gamma} + B_{\alpha\bar{\beta}\bar{\gamma}}\theta^{\bar{\gamma}}\ ,$$

where

$$A_{\alpha\bar{\beta}\gamma} = A_{\gamma\bar{\beta}\alpha}\ , \qquad B_{\alpha\bar{\beta}\bar{\gamma}} = B_{\alpha\bar{\gamma}\bar{\beta}}\ ,$$

and that

(1.23) $$\tau_{\alpha} \wedge \theta^{\alpha} = 0\ , \qquad \text{or } A_{\alpha\gamma} = A_{\gamma\alpha}\ .$$

The hermitian condition (1.4) implies

$$B_{\alpha\bar{\beta}\bar{\gamma}} = A_{\bar{\beta}\alpha\bar{\gamma}}\ .$$

It therefore follows that the change

(1.23a) $$\omega_{\beta\bar{\alpha}} \rightarrow \omega_{\beta\bar{\alpha}} + A_{\beta\bar{\alpha}\gamma}\theta^{\gamma}$$

is of the form (1.19) and reduces (1.22) to

(1.24) $$dg_{\alpha\bar{\beta}} - \omega_{\alpha}{}^{\gamma}g_{\gamma\bar{\beta}} - g_{\alpha\bar{\gamma}}\omega_{\bar{\beta}}{}^{\bar{\gamma}} = 0 .$$

The condition (1.24) for both $\omega_{\beta}{}^{\alpha}$ and $\tilde{\omega}_{\beta}{}^{\alpha}$ implies that $C_{\beta}{}^{\alpha}{}_{\gamma} = 0$ in (1.19), so that the $\omega_{\beta}{}^{\alpha}$ are uniquely determined. We have derived the following theorem.

Theorem (1.1). *Let (M, θ) be a nondegenerate, integrable pseudohermitian manifold. Then in the bundle P over M described above there is an intrinsic basis of one-forms*

$$\{\theta, \theta^{\alpha}, \theta^{\bar{\alpha}}, \omega_{\beta}{}^{\alpha}, \omega_{\bar{\beta}}{}^{\bar{\alpha}}\} ,$$

one-forms τ^{α}, and functions $g_{\alpha\bar{\beta}}$ satisfying (1.7), (1.15), (1.18), *and* (1.24). *We also have the relations* (1.20) *and* (1.23).

Now that the one-forms $\omega_{\beta}{}^{\alpha}$ are determined, we want to compute their exterior derivatives. If we differentiate (1.15) and make use of (1.7) and (1.15) itself, we get

(1.25) $$0 = \theta^{\beta} \wedge \{d\omega_{\beta}{}^{\alpha} - \omega_{\beta}{}^{\gamma} \wedge \omega_{\gamma}{}^{\alpha} - i\theta_{\beta} \wedge \tau^{\alpha}\} + \theta \wedge \{d\tau^{\alpha} - \tau^{\beta} \wedge \omega_{\beta}{}^{\alpha}\} .$$

Next, we differentiate (1.24) to get

(1.26) $$0 = (d\omega_{\alpha}{}^{\gamma} - \omega_{\alpha}{}^{\mu} \wedge \omega_{\mu}{}^{\gamma})g_{\gamma\bar{\beta}} + g_{\alpha\bar{\gamma}}(d\omega_{\bar{\beta}}{}^{\bar{\gamma}} - \omega_{\bar{\beta}}{}^{\bar{\mu}} \wedge \omega_{\bar{\mu}}{}^{\bar{\gamma}}) .$$

Therefore, if we put

(1.27) $$\Omega_{\beta}{}^{\alpha} = d\omega_{\beta}{}^{\alpha} - \omega_{\beta}{}^{\gamma} \wedge \omega_{\gamma}{}^{\alpha} - i\theta_{\beta} \wedge \tau^{\alpha} + i\tau_{\beta} \wedge \theta^{\alpha} ,$$

(1.28) $$\Omega^{\alpha} = d\tau^{\alpha} - \tau^{\beta} \wedge \omega_{\beta}{}^{\alpha} ,$$

then we get from (1.25), noting (1.23),

(1.29) $$0 = \theta^{\beta} \wedge \Omega_{\beta}{}^{\alpha} + \theta \wedge \Omega^{\alpha} .$$

From (1.26) it follows that

(1.30) $$0 = \Omega_{\beta}{}^{\gamma}g_{\gamma\bar{\alpha}} + g_{\beta\bar{\gamma}}\Omega_{\bar{\alpha}}{}^{\bar{\gamma}} \equiv \Omega_{\beta\bar{\alpha}} + \Omega_{\bar{\alpha}\beta} .$$

For future use we can, via (1.24), write (1.28) as

(1.31) $$\Omega_{\alpha} = d\tau_{\alpha} - \omega_{\alpha}{}^{\beta} \wedge \tau_{\beta} .$$

(1.29) implies that

(1.32) $$\Omega_{\beta\bar{\alpha}} = \chi_{\beta\bar{\alpha}\rho} \wedge \theta^{\rho} + \lambda_{\beta\bar{\alpha}} \wedge \theta$$

for certain one-forms $\chi_{\beta\bar\alpha\rho}$ and $\lambda_{\beta\bar\alpha}$, which we may assume contain no terms in θ. From (1.30) and (1.32) we have

$$0 = \chi_{\beta\bar\alpha\rho} \wedge \theta^\rho + \chi_{\bar\alpha\beta\bar\sigma} \wedge \theta^{\bar\sigma} + (\lambda_{\beta\bar\alpha} + \lambda_{\bar\alpha\beta}) \wedge \theta ,$$

which implies

$$\chi_{\beta\bar\alpha\rho} = B_{\beta\bar\alpha\rho\gamma}\theta^\gamma - R_{\beta\bar\alpha\rho\bar\sigma}\theta^{\bar\sigma} ,$$

where

$$B_{\beta\bar\alpha\rho\gamma} = B_{\beta\bar\alpha\gamma\rho} ,$$

$$(1.33) \qquad R_{\beta\bar\alpha\rho\bar\sigma} = \bar R_{\alpha\bar\beta\sigma\bar\rho} = R_{\bar\alpha\beta\bar\sigma\rho} ,$$

and furthermore

$$(1.34) \qquad \lambda_{\beta\bar\alpha} + \lambda_{\bar\alpha\beta} = 0 .$$

Thus we have

$$(1.35) \qquad \Omega_{\beta\bar\alpha} = R_{\beta\bar\alpha\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma} + \lambda_{\beta\bar\alpha} \wedge \theta ,$$

which, substituted into (1.29), gives

$$(1.36) \qquad R_{\beta\bar\alpha\rho\bar\sigma} = R_{\rho\bar\alpha\beta\bar\sigma} ,$$

$$0 = \theta \wedge (\theta^\beta \wedge \lambda_\beta{}^\alpha + \Omega^\alpha) .$$

This last condition implies that

$$(1.37) \qquad \Omega^\alpha = -\theta^\beta \wedge \lambda_\beta{}^\alpha + \mu^\alpha \wedge \theta ,$$

in which μ^α is some one-form, which we assume to have no θ-term.

Now we differentiate (1.23) using (1.31) and (1.15). It follows that

$$(1.38) \qquad 0 = \Omega^\alpha \wedge \theta_\alpha + \theta \wedge \tau^\alpha \wedge \tau_\alpha .$$

Putting (1.37) into (1.38) gives

$$(1.39) \qquad 0 = \lambda_{\beta\bar\alpha} \wedge \theta^\beta \wedge \theta^{\bar\alpha} + \theta \wedge (\tau^\alpha \wedge \tau_\alpha - \mu_{\bar\alpha} \wedge \theta^{\bar\alpha}) .$$

Since $\lambda_{\beta\bar\alpha}$ was chosen to have no θ-term, (1.39) implies that

$$\lambda_{\beta\bar\alpha} = W_{\beta\bar\alpha\gamma}\theta^\gamma + N_{\beta\bar\alpha\bar\gamma}\theta^{\bar\gamma} ,$$

where

$$(1.40) \qquad W_{\beta\bar\alpha\gamma} = W_{\gamma\bar\alpha\beta} ,$$

and, because of (1.34),

$$N_{\beta\alpha\bar{\gamma}} = -W_{\alpha\beta\bar{\gamma}} ,$$

We can now put

$$(1.41) \qquad \Omega_\beta{}^\alpha = R_\beta{}^\alpha{}_{\rho\bar{\sigma}}\theta^\rho \wedge \theta^{\bar{\sigma}} + W_\beta{}^\alpha{}_\rho\theta^\rho \wedge \theta - W^\alpha{}_{\beta\bar{\sigma}}\theta^{\bar{\sigma}} \wedge \theta ,$$

and the exterior derivatives $d\omega_\beta{}^\alpha$ are determined.

(1.39) and the expression (1.20) for τ_α also imply

$$0 = \theta \wedge \theta^\beta \wedge (A_{\beta\gamma}\tau^\gamma + \mu_\beta) ,$$

so that

$$\mu_\beta = -A_{\beta\gamma}\tau^\gamma + B_{\beta\gamma}\theta^\gamma ,$$

where

$$(1.42) \qquad B_{\beta\gamma} = B_{\gamma\beta} .$$

Finally, (1.37) becomes

$$(1.43) \qquad \Omega^\alpha = W^\alpha{}_{\rho\bar{\sigma}}\theta^\rho \wedge \theta^{\bar{\sigma}} - A^\alpha{}_{\bar{\gamma}}\tau^{\bar{\gamma}} \wedge \theta + B^\alpha{}_{\bar{\sigma}}\theta^{\bar{\sigma}} \wedge \theta ,$$

and we have also determined the derivatives $d\tau^\alpha$.

We sum these results up in the following:

Theorem (1.1a). *The exterior derivatives of the forms* $\omega_\beta{}^\alpha$ *and* τ^α *of Theorem* (1.1) *are given by* (1.27) *and* (1.28), *respectively, where* $\Omega_\beta{}^\alpha$ *and* Ω^α *are given by* (1.41) *and* (1.43), *respectively. The coefficients satisfy* (1.33), (1.36), (1.40), *and* (1.42).

The existence of the invariant forms $\omega_\beta{}^\alpha$ on the bundle P with structure group reduced to $U(p, q)$ gives the following.

Theorem (1.2). *The group* $PsH(M, \theta)$ *of all pseudo-hermitian transformations of the pseudo-hermitian space* (M, θ) *of dimension* $2n + 1$ *is a Lie transformation group of dimension not exceeding* $(n + 1)^2$, *with isotropy subgroups of dimension not exceeding* n^2. *If* M *is strongly pseudo-covex, then the isotropy groups are compact, and* $PsH(M, \theta)$ *is compact for compact* M.

2. Geometric interpretation

We shall interpret the $\omega_\beta{}^\alpha$ of Theorem (1.1) as connection forms of a connection on the complex vector bundle $H(M)$. If we choose local forms θ'^α on M, then according to (1.8) and (1.16) we can put

$$(2.1) \qquad U_\beta{}^\gamma\omega_\gamma{}^\alpha + dU_\beta{}^\alpha = \omega'_\beta{}^\gamma U_\gamma{}^\alpha ,$$

where

$$\omega'_\beta{}^\gamma \equiv 0 , \qquad \mathrm{mod}\, \theta, \theta'^\alpha, \theta'^{\bar{\alpha}} .$$

In the usual manner [3] we see that the coefficients of the ${\omega'}_\beta{}^\gamma$ are independent of $U_\rho{}^\sigma$ by differentiating (2.1). Using (2.1) to eliminate $dU_\beta{}^\alpha$ we get

$$(2.2) \qquad U_\alpha{}^\gamma(d\omega_\gamma{}^\beta - \omega_\gamma{}^\rho \wedge \omega_\rho{}^\beta) = (d{\omega'}_\alpha{}^\gamma - {\omega'}_\alpha{}^\rho \wedge {\omega'}_\rho{}^\gamma)U_\gamma{}^\beta .$$

By (1.27) and (1.41) we see that the left hand side of (2.2) is a two-form in $\theta, \theta^\alpha, \theta^{\bar\alpha}$, therefore so is $d{\omega'}_\alpha{}^\gamma$, and so ${\omega'}_\beta{}^\alpha$ is a one-form on M.

Now we consider θ^α, as well as θ'^α, as local one-forms on M and (1.8) as a change of coframe. Let $(X, X_\alpha, X_{\bar\alpha})$ be the dual frame to $(\theta, \theta^\alpha, \theta^{\bar\alpha})$, and let $V = U^{-1}$; then

$$(2.3) \qquad X_\alpha = V_\alpha{}^\beta {X'}_\beta .$$

Define an operator D locally by

$$(2.4) \qquad DX_\alpha = \omega_\alpha{}^\beta X_\beta , \qquad D\colon \Gamma(H(M)) \to \Gamma(T^*(M) \otimes H(M)) .$$

Under the change (2.3) we get from (2.1)

$$(2.5) \qquad \omega_\beta{}^\gamma V_\gamma{}^\alpha = dV_\beta{}^\alpha + V_\beta{}^\gamma {\omega'}_\gamma{}^\alpha ;$$

hence, (2.4) defines a connection on $H(M)$.

We can define an hermitian metric $(\, , \bar{\ })$ in the fibres of $H(M)$ by

$$(2.6) \qquad (X_\alpha, \bar{X}_\beta) = g_{\alpha\bar\beta} .$$

The condition (1.24) yields that D is a metric connection. τ^α in (1.15) can be viewed as a kind of torsion. The condition (1.18) on τ^α is analogous to the requirement in hermitian geometry that the torsion form be of a given type (i.e., of type (2, 0)) [5].

With these interpretations we can restate Theorem (1.1) as

Theorem (2.1). *Let (M, θ) be a nondegenerate, integrable pseudo-hermitian manifold. Then there are a unique hermitian metric* (2.6) *determined by the Levi form and a unique metric connection D on $H(M)$ with torsion form satisfying*

$$\tau^\alpha \equiv 0 , \qquad \mod \theta^{\bar\gamma} .$$

Under the change (1.8) (or (2.3)) we have

$$(2.7) \qquad {\theta'}_\beta = U_\beta{}^\alpha \theta_\alpha ,$$

$$(2.8) \qquad \tau'^\beta U_\beta{}^\alpha = \tau^\alpha , \qquad {\tau'}_\beta = U_\beta{}^\alpha \tau_\alpha .$$

By (2.2) the curvature matrix of $\omega_\beta{}^\alpha$,

$$(2.9) \qquad \Pi_\beta{}^\alpha = d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma \wedge \omega_\gamma{}^\alpha = \Omega_\beta{}^\alpha + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha ,$$

transforms by

(2.10) $$U_\alpha{}^\gamma \Pi_\gamma{}^\beta = \Pi'_\alpha{}^\gamma U_\gamma{}^\beta \, .$$

We also have

(2.11) $$U_\alpha{}^\gamma \Omega_\gamma{}^\beta = \Omega'_\alpha{}^\gamma U_\gamma{}^\beta \, .$$

The two curvature matrices are equal when the torsion τ^α vanishes.

The vanishing of the torsion has a more geometric interpretation. Let L_X be Lie derivation by the transversal X to $H(M)$. By the standard formula

$$L_X = \iota_X \circ d + d \circ \iota_X \, ,$$

(1.7) and (1.15) imply

(2.12) $$L_X\theta = 0 \, , \qquad L_X\theta^\alpha = -\phi_\beta{}^\alpha(X)\theta^\beta - \tau^\alpha(X)\theta + \tau^\alpha \, .$$

So if $\tau^\alpha = 0$, then X is an infinitesimal pseudo-conformal transformation.

Conversely, given a transverse infinitesimal pseudo-conformal transformation X, complete it to a basis by choosing X_α. On the dual coframe we have

(2.13) $$L_X\theta = u\theta \, , \qquad L_X\theta^\alpha = \theta^\beta U_\beta{}^\alpha + \theta v^\alpha \, .$$

From (1.3) it follows that

$$L_X\theta = \eta_\alpha\theta^\alpha + \eta_{\bar\alpha}\theta^{\bar\alpha} \, ;$$

hence $\eta_\alpha = u = 0$, and we have an admissible coframe with respect to θ. From (2.12) we see that $\tau^\alpha = 0$.

Hence we have shown

Proposition (2.2). *The torsion τ^α vanishes if and only if the transversal X determined by θ is an infinitesimal pseudo-conformal transformation.*

Proposition 2.2 gives the condition required by Tanaka in [9].

Using the curvature tensor $R_{\beta\bar\alpha\rho\bar\sigma}$ in (1.41), we can define a kind of curvature for holomorphic plane sections in $H(M)$ as follows: if

(2.14) $$Z = \xi^\alpha X_\alpha \, ,$$

then

(2.15) $$K(Z) = -\tfrac{1}{2}(R_{\beta\bar\alpha\rho\bar\sigma}\xi^\beta\xi^{\bar\alpha}\xi^\rho\xi^{\bar\sigma})/(g_{\alpha\bar\beta}\xi^\alpha\xi^{\bar\beta})^2 \, .$$

The coefficient $-\frac{1}{2}$ makes the unit hypersphere in $\boldsymbol{C}^{n+1}$ have constant curvature $+1$ (see § 4). We also define the Ricci tensor

(2.16) $$R_{\rho\bar\sigma} = R_\alpha{}^\alpha{}_{\rho\bar\sigma}$$

and the scalar curvature

$$R = g^{\rho\bar{\sigma}} R_{\rho\bar{\sigma}} . \tag{2.17}$$

Finally, we can define a Riemannian metric on $T(M)$ by

$$\begin{aligned} ds^2 &= \theta \otimes \theta - \operatorname{Re}(g_{\alpha\bar{\beta}} \theta^\alpha \otimes \theta^{\bar{\beta}}) \\ &= \theta \otimes \theta - \tfrac{1}{2}(g_{\alpha\bar{\beta}} \theta^\alpha \otimes \theta^{\bar{\beta}} + g_{\bar{\alpha}\beta} \theta^{\bar{\alpha}} \otimes \theta^\beta) . \end{aligned} \tag{2.18}$$

This metric is invariant under a pseudo-hermitian transformation.

3. Relation to pseudo-conformal invariants

The object of this section is to derive pseudo-conformal invariants from the curvature tensors introducted in part one. To do this we start with a local coframe field

$$\omega = \theta , \quad \omega^\alpha = \theta^\alpha , \quad \omega^{\bar{\alpha}} = \theta^{\bar{\alpha}} \tag{3.1}$$

adapted to the particular choice of θ. We then try to find local forms $\phi_\beta{}^\alpha$, ϕ^α, and ψ which will satisfy the structure equations [6, (A.1)–(A.6), p. 269] and [6, (4.21), p. 253]. Note that with our normalization

$$\phi = 0 . \tag{3.2}$$

Because of (3.2), (1.15), (1.23), and (1.24) the choice

$$\phi_\beta{}^\alpha = \omega_\beta{}^\alpha , \quad \phi^\alpha = \tau^\alpha , \quad \psi = 0$$

satisfies [6, (A.1), (A.2), (A.3), and (4.21)]. The transformation [6, (4.35)] indicates that we should try

$$\phi_\beta{}^\alpha = \omega_\beta{}^\alpha + D_\beta{}^\alpha \theta , \quad \phi^\alpha = \tau^\alpha + D_\gamma{}^\alpha \theta^\gamma , \quad \psi = 0 , \tag{3.3}$$

where

$$D_{\beta\bar{\alpha}} + D_{\bar{\alpha}\beta} = 0 . \tag{3.4}$$

By the procedure of [6, § 4] the $D_{\beta\bar{\alpha}}$ are determined by requiring that the contraction of equation [6, (A.4)] be trivial, mod θ. Substituting (3.3) into this contracted equation gives

$$\begin{aligned} \Phi_\alpha{}^\alpha &\equiv \Omega_\alpha{}^\alpha + i(Dg_{\rho\bar{\sigma}} + (n+2)D_{\rho\bar{\sigma}})\theta^\rho \wedge \theta^{\bar{\sigma}} \\ &\equiv (R_{\rho\bar{\sigma}} + i(Dg_{\rho\bar{\sigma}} + (n+2)D_{\rho\bar{\sigma}}))\theta^\rho \wedge \theta^{\bar{\sigma}} , \qquad \operatorname{mod} \theta , \end{aligned} \tag{3.5}$$

where

$$D = D_\alpha{}^\alpha ,$$

and we have made use of (1.23), (1.27), and (1.41).

To make (3.5) vanish, mod θ. we choose

$$D_{\rho\bar{\sigma}} = \frac{i}{n+2} R_{\rho\bar{\sigma}} - \frac{i}{2(n+1)(n+2)} R g_{\rho\bar{\sigma}} \,. \tag{3.6}$$

Then the $\phi_\beta{}^\alpha$ in (3.3) is the intrinsic (pseudo-conformal) connection form.

The substitution of (3.3) and (3.6) into [6, (A.4)] gives

$$\begin{aligned} \Phi_\beta{}^\alpha &\equiv \Omega_\beta{}^\alpha + i(D_\beta{}^\alpha g_{\rho\bar{\sigma}} + D_\rho{}^\alpha g_{\beta\bar{\sigma}} + \delta_\beta{}^\alpha D_{\rho\bar{\sigma}} + \delta_\rho{}^\alpha D_{\beta\bar{\sigma}})\theta^\rho \wedge \theta^{\bar{\sigma}} \\ &\equiv S_{\beta\rho}{}^\alpha{}_{\bar{\sigma}} \theta^\rho \wedge \theta^{\bar{\sigma}} \,, \qquad \bmod \theta \,. \end{aligned} \tag{3.7}$$

It now follows that Chern's pseudo-conformal curvature tensor is given by

$$\begin{aligned} S_{\beta\rho}{}^\alpha{}_{\bar{\sigma}} &= R_\beta{}^\alpha{}_{\rho\bar{\sigma}} - \frac{1}{n+2}(R_\beta{}^\alpha g_{\rho\bar{\sigma}} + R_\rho{}^\alpha g_{\beta\bar{\sigma}} + \delta_\beta{}^\alpha R_{\rho\bar{\sigma}} + \delta_\rho{}^\alpha R_{\beta\bar{\sigma}}) \\ &\quad + \frac{R}{(n+1)(n+2)}(\delta_\beta{}^\alpha g_{\rho\bar{\sigma}} + \delta_\rho{}^\alpha g_{\beta\bar{\sigma}}) \,. \end{aligned} \tag{3.8}$$

Formula (3.8) is similar to H. Weyl's formula for the conformal curvature tensor of a Riemannian manifold (see [7]). The trace of S with respect to β and α is zero, so S vanishes identically when $n = 1$. When $n > 1$, S vanishes if and only if M is locally equivalent to the real hypersphere in C^{n+1} (see [6] and [10]). Formula (3.8) will be used to compute S for specific hypersurfaces in the next section.

We could continue the procedure of [6] to determine further relations, however, when $n > 1$, the Bianchi identities [6] can be used to show that all higher order invariants are obtained from S by covariant differentation with respect to the pseudo-conformal connection [10]. It can then be shown, with the aid of (3.2), (3.3), (3.6), and (3.8), that these invariants can be expressed in terms of the curvatures of (M, θ) and their covariant derivatives with respect to the connection $\omega_\beta{}^\alpha$. Such expressions will be valid only with respect to coframes satisfying (3.2).

As a system of local functions on M, S transforms tensorially (explicit details are in [10]). Under the structure group (4.1) of [6] we have the changes

$$\tilde{\theta} = u\theta \,, \quad u g_{\alpha\bar{\beta}} = \tilde{g}_{\rho\bar{\sigma}} U_\alpha{}^\rho U_{\bar{\beta}}{}^{\bar{\sigma}} \,, \quad S_{\beta\rho\alpha\bar{\sigma}} = \tilde{S}_{\mu\nu\bar{\gamma}\bar{\varepsilon}} U_\beta{}^\mu U_\rho{}^\nu U_{\bar{\alpha}}{}^{\bar{\gamma}} U_{\bar{\sigma}}{}^{\bar{\varepsilon}} \,. \tag{3.9}$$

If we define the norm of S with respect to θ by

$$\|S\|_\theta{}^2 = g^{\alpha\bar{\beta}} g^{\rho\bar{\sigma}} g_{\gamma\bar{\mu}} g^{\nu\bar{\varepsilon}} S_{\alpha\rho}{}^\gamma{}_{\bar{\varepsilon}} S_{\bar{\beta}\bar{\sigma}}{}^{\bar{\mu}}{}_\nu \,, \tag{3.10}$$

then (3.9) gives

$$\|S\|_\theta = |u| \, \|S\|_{\tilde{\theta}} \,. \tag{3.11}$$

If M is strongly pseudo-convex, for example, we can restrict to changes (3.9) with $u > 0$. If, in addition, S does not vanish (3.11) shows that we can choose a unique θ^* with respect to which S has norm one. This θ^* and all the invariants of (M, θ^*) are intrinsic to the C-R structure of M. In particular, the corresponding transversal X (1.10) and its integral curves are intrinsic to M. The latter are called principal curves [2].

Let N be a Kähler manifold with Kähler form χ. Each point of N has a neighborhood U, with holomorphic coordinate vector Z, on which there is a positive function h satisfying

$$\chi = i\bar{\partial}\partial \log h \ .$$

On $U \times C$ define

$$r = h(Z, \bar{Z})w\bar{w} - 1 \ , \quad Z \in U \ , \quad w \in C \ ,$$

and let M be the real hypersurface on which r vanishes. Then χ is also the Levi form of $(M, \theta = i\partial r)$. It is easily seen that the torsion τ^α vanishes, and that $R_{\beta\bar{\alpha}\rho\bar{\sigma}}$ is also the curvature tensor of the Kähler metric associated to χ. $S_{\beta\rho}{}^{\alpha}{}_{\bar{\sigma}}$ is then the same tensor defined by Bochner [1].

4. The curvature for real hypersurfaces in C^{n+1}, spaces of constant curvature, & ellipsoids

In this section we will give a procedure for computing the torsion and curvature tensors for a real hypersurface (M, θ) in C^{n+1} defined as the zero set of a given real valued function r.

We have coordinates

$$Z = (z^1, \cdots, z^n) \ , \qquad w = z^{n+1} \ ,$$

and, for the applications we have in mind, will assume that the Z and w variables are separated in r, i.e.,

$$r(Z, w, \bar{Z}, \bar{w}) = p(Z, \bar{Z}) + q(w, \bar{w}) \ , \tag{4.1}$$

p and q being real valued. We choose the one-form

$$\theta = i\partial r = i(p_\alpha dz^\alpha + q_w dw) \ . \tag{4.2}$$

Throughout we shall use the abbreviations

$$p_\alpha = \partial p/\partial z^\alpha \ , \qquad q_w = \partial q/\partial w \ , \quad \text{etc.}$$

Then we have

$$d\theta = i\bar{\partial}\partial r = ig_{\alpha\bar{\beta}}dz^\alpha \wedge dz^{\bar{\beta}} + \eta_\alpha dz^\alpha \wedge \theta + \eta_{\bar{\alpha}} dz^{\bar{\alpha}} \wedge \theta = ig_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} \ , \tag{4.3}$$

where

$$g_{\alpha\bar\beta} = -p_{\alpha\bar\beta} - Qp_\alpha p_{\bar\beta}\,, \qquad Q = (q_{w\bar w})/(q_w q_{\bar w})\,, \tag{4.4}$$

$$\eta_\alpha = -Qp_\alpha\,, \qquad \eta^\alpha = g^{\alpha\bar\gamma}\eta_{\bar\gamma}\,, \tag{4.5}$$

$$\theta^\alpha = dz^\alpha + i\eta^\alpha\theta\,. \tag{4.6}$$

The coframe $(\theta, \theta^\alpha, \theta^{\bar\alpha})$ is admissible for (M, θ). Our computation will be valid where $q_{\bar w} \neq 0$. The dual frame, characterized by

$$df = Xf\theta + X_\alpha f\theta^\alpha + X_{\bar\alpha} f\theta^{\bar\alpha} \tag{4.7}$$

for any function f on M, is given by

$$\begin{aligned} X = {} & -i\eta^\alpha(\partial/\partial z^\alpha) + i\eta^{\bar\alpha}(\partial/\partial z^{\bar\alpha}) - i(1 - p_\mu\eta^\mu)(q_w)^{-1}(\partial/\partial w) \\ & + i(1 - p_{\bar\mu}\eta^{\bar\mu})(q_{\bar w})^{-1}(\partial/\partial \bar w)\,, \end{aligned} \tag{4.8}$$

$$X_\alpha = (\partial/\partial z^\alpha) - p_\alpha(q_w)^{-1}(\partial/\partial w)\,, \qquad X_{\bar\alpha} = \overline{X_\alpha}\,. \tag{4.9}$$

We first compute the connection and torsion forms $\omega_{\beta\bar\alpha}$, τ_α. Differentiating (4.6) gives

$$d\theta^\alpha = \theta^\beta \wedge (-\eta^\alpha\theta_\beta + iX_\beta\eta^\alpha\theta) + \theta \wedge (-iX_{\bar\gamma}\eta^\alpha\theta^{\bar\gamma}) = \theta^\beta \wedge \omega'_\beta{}^\alpha + \theta \wedge \tau^\alpha\,.$$

Next, we compute

$$dg_{\beta\bar\alpha} - \omega'_{\beta\bar\alpha} - \omega'_{\bar\alpha\beta} = (X_\gamma g_{\beta\bar\alpha} + \eta_\beta g_{\gamma\bar\alpha})\theta^\gamma + (X_{\bar\gamma} g_{\beta\bar\alpha} + \eta_{\bar\alpha} g_{\beta\bar\gamma})\theta^{\bar\gamma}\,,$$

where the θ-term vanishes by (1.22). Therefore the change (1.23a) yields

$$\omega_{\beta\bar\alpha} = B_{\beta\bar\alpha\gamma}\theta^\gamma + C_{\beta\bar\alpha\bar\gamma}\theta^{\bar\gamma} + E_{\beta\bar\alpha}\theta\,, \tag{4.10}$$

where

$$B_{\beta\bar\alpha\gamma} = X_\gamma g_{\beta\bar\alpha} + \eta_\beta g_{\bar\alpha\gamma}\,, \quad C_{\beta\bar\alpha\bar\gamma} = -\eta_{\bar\alpha} g_{\beta\bar\gamma}\,, \quad E_{\beta\bar\alpha} = ig_{\bar\alpha\gamma}X_\beta\eta^\gamma\,. \tag{4.11}$$

Also, the torsion form is

$$\tau_\alpha = A_{\alpha\gamma}\theta^\gamma\,, \tag{4.12}$$

where

$$A_{\alpha\gamma} = ig_{\alpha\bar\mu}X_\gamma\eta^{\bar\mu} = iX_\gamma\eta_\alpha - i\eta^{\bar\mu}X_\gamma g_{\alpha\bar\mu}\,. \tag{4.13}$$

To find the curvature tensor $R_{\beta\bar\alpha\rho\bar\sigma}$, we substitute (4.10) and (4.12) into

$$\Omega_{\beta\bar\alpha} = d\omega_{\beta\bar\alpha} - \omega_{\bar\alpha\gamma} \wedge \omega_\beta{}^\gamma - i\theta_\beta \wedge \tau_{\bar\alpha} + i\tau_\beta \wedge \theta_{\bar\alpha}\,,$$

and compute mod θ. We need to consider only the $\theta^\rho \wedge \theta^{\bar\sigma}$-term. The coefficient of this term is

$$R_{\beta\bar\alpha\rho\bar\sigma} = -X_{\bar\sigma}B_{\beta\bar\alpha\rho} + X_\rho C_{\beta\bar\alpha\bar\sigma} + B_\beta{}^\gamma{}_\rho B_{\bar\alpha\gamma\bar\sigma} + B_{\beta\bar\alpha\gamma}C_\rho{}^\gamma{}_{\bar\sigma} - C_{\beta\bar\alpha\bar\gamma}C_{\bar\sigma}{}^{\bar\gamma}{}_\rho - C_\beta{}^\gamma{}_{\bar\sigma}C_{\bar\alpha\gamma\rho} + iE_{\beta\bar\alpha}g_{\rho\bar\sigma} . \tag{4.14}$$

If we substitute (4.11) into (4.14) we get

$$\begin{aligned} R_{\beta\bar\alpha\rho\bar\sigma} = & -X_{\bar\sigma}X_\rho g_{\beta\bar\alpha} + g^{\gamma\bar\mu}X_\rho g_{\beta\bar\mu}\cdot X_{\bar\sigma}g_{\bar\alpha\gamma} + g_{\rho\bar\sigma}\eta^\gamma X_\beta g_{\bar\alpha\gamma} \\ & - g_{\rho\bar\sigma}\eta^\gamma X_\gamma g_{\beta\bar\alpha} - g_{\bar\alpha\rho}X_{\bar\sigma}\eta_\beta - g_{\beta\bar\sigma}X_\rho\eta_{\bar\alpha} - g_{\rho\bar\sigma}X_\beta\eta_{\bar\alpha} \\ & - \eta_\beta\eta_{\bar\alpha}g_{\rho\bar\sigma} - \eta_\gamma\eta^\gamma g_{\beta\bar\sigma}g_{\rho\bar\alpha} . \end{aligned} \tag{4.15}$$

Examples. *A. Spaces of constant curvature.* We will consider here three examples in C^{n+1} which are locally equivalent in the pseudo-conformal sense but differ according to the choice (4.2) of θ.

$$Q_0\colon\ r_0 = h_{\alpha\bar\beta}z^\alpha z^{\bar\beta} + \frac{i}{2}(w - \bar w) = 0 . \tag{4.16.1}$$

$$Q_+(c)\colon\ r_+ = h_{\alpha\bar\beta}z^\alpha z^{\bar\beta} + w\bar w = c . \tag{4.16.2}$$

$$Q_-(c)\colon\ r_- = h_{\alpha\bar\beta}z^\alpha z^{\bar\beta} - w\bar w = -c . \tag{4.16.3}$$

The constant c is positive, and $h_{\alpha\bar\beta}$ is a constant nonsingular hermitian matrix with signature p positive and q negative eigenvalues, $p + q = n$.

The transformation

$$w = c/w' , \qquad z^\alpha = \sqrt{c}\, z'^\alpha/w' \tag{4.17}$$

maps $Q_-(c)$ onto $Q_+(c)$ minus $\{w = 0\}$. A transformation mapping Q_0 onto $Q_+(c)$ minus a point is given in [6]. However, these transformations do not preserve the one-forms $\theta = i\partial r$.

(1) Q_0. Let G_0 be the group of $(n + 1) \times (n + 1)$ matrices

$$\begin{pmatrix} 1 & b^\beta & b \\ 0 & B_\alpha{}^\beta & b_\alpha \\ 0 & 0 & 1 \end{pmatrix} , \tag{4.18}$$

where

$$B_\alpha{}^\gamma h_{\gamma\bar\rho}B_{\bar\beta}{}^{\bar\rho} = h_{\alpha\bar\beta} , \quad b_\alpha = 2iB_\alpha{}^\rho h_{\rho\bar\gamma}b^{\bar\gamma} , \quad 0 = \frac{i}{2}(b - \bar b) + h_{\alpha\bar\beta}b^\alpha b^{\bar\beta} . \tag{4.19}$$

G_0 acts on C^{n+1} by

$$\tilde z^\alpha = b^\alpha + z^\beta B_\beta{}^\alpha , \qquad \tilde w = b + z^\beta b_\beta + w , \tag{4.20}$$

preserves the function r_0 defining Q_0, and hence preserves $\theta = i\partial r$.

The isotropy group of (0, 0) in Q_0 is the unitary group $U(p, q)$ of the hermitian form $h_{\alpha\bar\beta}$. It follows that Q_0 is homogeneous,

$$Q_0 = G_0/U(p, q) \, . \tag{4.21}$$

If we choose as our coframe

$$\theta, \theta^\alpha = dz^\alpha, \theta^{\bar\alpha} = dz^{\bar\alpha} ,$$

then

$$d\theta = -ih_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} ,$$

and $\omega_\beta{}^\alpha = \tau^\alpha = 0$ since $d\theta^\alpha = 0$. The curvature and torsion of (Q_0, θ) vanish identically.

(2) $Q_+(c)$. The function r_+ in (4.16.2) is an hermitian form of signature $(p + 1, q)$. The unitary group $U(p + 1, q)$ acts transitively on $Q_+(c)$ and preserves $\theta = i\partial r_+$. The isotropy group at $(Z = 0, w = \sqrt{c}\,)$ is $U(p, q)$; hence

$$Q_+(c) = U(p + 1, q)/U(p, q) \, . \tag{4.22}$$

(3) $Q_-(c)$. The function r_- in (4.16.3) is an hermitian form of signature $(p, q + 1)$; $\theta = i\partial r$ is invariant under $U(p, q + 1)$, and

$$Q_-(c) = U(p, q + 1)/U(p, q) \, . \tag{4.23}$$

Because $Q_+(c)$ and $Q_-(c)$ are homogeneous, it suffices to compute their curvature and torsion at a point where $Z = 0$. From (4.13), (4.5), and (4.9) we see that $A_{\alpha\gamma}$ vanishes when $Z = 0$. Also, substituting (4.4) and (4.5) into (4.15), we see that, when $Z = 0$,

$$R_{\beta\bar\alpha\rho\bar\sigma} = -\frac{\varepsilon}{c}(g_{\beta\bar\alpha}g_{\rho\bar\sigma} + g_{\rho\bar\alpha}g_{\beta\bar\sigma}) ,$$

where $\varepsilon = +1$ for $Q_+(c)$ and $\varepsilon = -1$ for $Q_-(c)$. From the definition of sectional curvature (2.15), we have $K \equiv 1/c$ for $Q_+(c)$ and $K \equiv -1/c$ for $Q_-(c)$.

Q_0, $Q_+(c)$, and $Q_-(c)$ each have a transformation group of dimension $(n + 1)^2$. It is easily seen from (3.8) that the tensor $S_{\beta\rho\bar\alpha\bar\sigma}$ vanishes identically in each case.

B. Ellipsoids. For a less trivial example we consider the general ellipsoid E in C^{n+1} defined by

$$\begin{aligned} r = {} & A_1(X^1)^2 + B_1(y^1)^2 + \cdots + A_n(x^n)^2 + B_n(y^n)^2 \\ & + A(u)^2 + B(v)^2 - 1 = 0 , \end{aligned} \tag{4.24}$$

where $x^\alpha + iy^\alpha = z^\alpha$, $u + iv = w$, and A, A_α, B, B_α are all positive constants.

We rewrite this as

$$(4.25)\quad r = \sum_{\alpha=1}^{n} (a_\alpha(z^\alpha)^2 + a_\alpha(z^{\bar\alpha})^2 + b_\alpha z^\alpha z^{\bar\alpha}) + a(w^2 + \bar{w}^2) + bw\bar{w} - 1 = 0 ,$$

where

$$(4.26)\quad \begin{aligned} a &= \tfrac{1}{4}(A - B) , & a_\alpha &= \tfrac{1}{4}(A_\alpha - B_\alpha) , \\ b &= \tfrac{1}{2}(A + B) > 0 , & b_\alpha &= \tfrac{1}{2}(A_\alpha + B_\alpha) > 0 . \end{aligned}$$

More generally, we take

$$(4.27)\quad r = p(Z, \bar{Z}) + q(w, \bar{w}) ,$$

where

$$(4.27a)\quad p = a_{\alpha\beta} z^\alpha z^\beta + a_{\bar\alpha\bar\beta} z^{\bar\alpha} z^{\bar\beta} + b_{\alpha\bar\beta} z^\alpha z^{\bar\beta} ,$$

$$(4.27b)\quad q = aw^2 + \bar{a}\bar{w}^2 + bw\bar{w} - 1 ,$$

all the coefficients are constant, $a_{\alpha\beta}$ is symmetric, $b_{\alpha\bar\beta}$ is positive definite hermitian, and b is positive.

We will compute the curvature tensor $S_{\beta\rho\bar\alpha\bar\sigma}$ for E along the curve $E \cap (Z = 0)$ by computing $R_{\beta\bar\alpha\rho\bar\sigma}$ and using (3.8). We let $|_0$ denote evaluation at $Z = 0$. We have

$$(4.28)\quad \begin{aligned} p_\alpha|_0 &= 0 , & q_w|_0 &\neq 0 , \\ p_{\alpha\bar\beta} &= b_{\alpha\bar\beta} , & p_{\alpha\gamma} &= 2a_{\alpha\gamma} . \end{aligned}$$

This, together with the expressions (4.4) and (4.5), gives

$$(4.29)\quad \begin{aligned} &X_\rho g_{\beta\bar\alpha} = \frac{Q_w}{q_w} p_\rho p_\beta p_{\bar\alpha} - Q p_\beta b_{\rho\bar\alpha} - 2Q a_{\beta\rho} p_{\bar\alpha} , \\ &-X_{\bar\sigma}|_0(X_\rho g_{\beta\bar\alpha}) = Q(b_{\beta\bar\sigma} b_{\rho\bar\alpha} + 4a_{\beta\rho} a_{\bar\alpha\bar\sigma}) , \\ &X_\rho|_0(g_{\beta\bar\alpha}) = 0 , \qquad X_{\bar\sigma}|_0(\eta_\beta) = -Q b_{\beta\bar\alpha} . \end{aligned}$$

Substituting (4.29) into (4.15) gives

$$(4.30)\quad R_{\beta\bar\alpha\rho\bar\sigma}|_0 = -Q(b_{\beta\bar\alpha} b_{\rho\bar\sigma} + b_{\rho\bar\alpha} b_{\beta\bar\sigma} - 4a_{\beta\rho} a_{\bar\alpha\bar\sigma}) ,$$

where $Q = Q|_0 \neq 0$. Let $b^{\bar\beta\alpha}$ be the inverse matrix of $b_{\beta\bar\alpha}$. Then

$$(4.31)\quad R_{\rho\bar\sigma}|_0 = Q((n + 1)b_{\rho\bar\sigma} - 4b^{\mu\bar\nu} a_{\mu\rho} a_{\bar\nu\bar\sigma}) ,$$

$$(4.32)\quad R|_0 = -Q(n(n + 1) - 4b^{\mu\bar\nu} b^{\varepsilon\bar\tau} a_{\mu\varepsilon} a_{\bar\nu\bar\tau}) .$$

Now, if we put (4.30), (4.31), and (4.32) into (3.8) with the index α lowered, we get, after simplification,

$$(4.33)\quad \begin{aligned} S_{\beta\rho\bar\alpha\bar\sigma}|_0 = {} & 4Qb^{\mu\bar\nu}b^{\varepsilon\bar\tau}a_{\mu\varepsilon}a_{\bar\nu\bar\tau}(b_{\beta\bar\alpha}b_{\rho\bar\sigma} + b_{\rho\bar\alpha}b_{\beta\bar\sigma}) \\ & + 4Qa_{\beta\rho}a_{\bar\alpha\bar\sigma} - \frac{4Q}{n+2}(b^{\mu\bar\nu}a_{\mu\beta}a_{\bar\nu\bar\alpha}b_{\rho\bar\sigma} \\ & + b^{\mu\bar\beta}a_{\mu\rho}a_{\bar\nu\bar\alpha}b_{\beta\bar\sigma} + b^{\mu\bar\nu}a_{\mu\rho}a_{\bar\nu\bar\sigma}b_{\beta\bar\alpha} + b^{\mu\bar\nu}a_{\mu\beta}a_{\bar\nu\bar\sigma}b_{\rho\bar\alpha}) \,. \end{aligned}$$

Now let us assume we have the form (4.25)–(4.26). Then

$$(4.34)\qquad S_{\alpha\alpha\bar\alpha\bar\alpha}|_0 = \frac{8Q}{(n+1)(n+2)}\Big(\sum_{r=1}^{n} a_r^2/b_r^2\Big)b_\alpha^2 + 4Q\frac{n-2}{n+2}a_\alpha^2 \,.$$

It follows that if $n = 1$, $S_{11\bar1\bar1}|_0 = 0$, as expected. However, if $n \geq 2$, then $S_{\alpha\alpha\bar\alpha\bar\alpha}|_0$ vanishes for some α if and only if $a_1 = \cdots = a_n = 0$. Since we can relable our variables, say $z^1 \leftrightarrow w$, we see that E has nonflat points if $a \neq 0$, or if $a_0 \neq 0$ for some α. Hence

Theorem (4.1). *Let $n \geq 2$. The ellipsoid E given by* (4.24) *is equivalent to the real hypersphere if and only if*

$$A_1 = B_1, \cdots, A_n = B_n, A = B \,.$$

In [8] Fefferman has shown that a biholomorphic map between two bounded strongly pseudo-convex domains with smooth boundaries extends smoothly to the boundaries. Theorem (4.1) then gives a necessary and sufficient condition for an ellipsoidal domain to be equivalent to the unit ball.

References

[1] S. Bochner, *Curvature and Betti numbers. II,* Ann. of Math. **50** (1949) 77–93.

[2] E. Cartan, *Sur la geometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complex.* I *and* II, Ann. Mat. Pura Appl. **11** (1932) 17–90, and Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. **1** (1932) 333–354.

[3] E. Cartan, *Lecons sur la geometrie des espaces de Riemann,* Gauthier-Villars, Paris, 1951.

[4] S. S. Chern, *Differentiable manifolds,* Lecture notes, University of Chicago, 1959.

[5] ——, *Complex manifolds without potential theory,* Math. Studies No. 15, Van Nostrand, Princeton, 1967.

[6] S. S. Chern & J. Moser, *Real hypersurfaces in complex manifolds,* Acta Math. **133** (1974) 219–271.

[7] L. P. Eisenhart, *Riemannian geometry,* Princeton University Press, Princeton, 1966.

[8] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudo-convex domains,* Invent. Math. **26** (1974) 1–65.

[9] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables,* J. Math. Soc. Japan **14** (1962) 397–429.

[10] S. Webster, *Real hypersurfaces in complex space,* Thesis, University of California, Berkeley, 1975.

PRINCETON UNIVERSITY