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PSEUDO-INJECTIVE MODULES WHICH ARE NOT QUASI-INJECTIVE

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ABSTRACT. For certain rings with infinitely many nonisomorphic simple left modules, a method is given for constructing pseudo-injective modules which are not quasi-injective. This method is used to produce examples of such modules over a commutative ring.

Let R be a ring with unity. All modules considered here will be unital left R-modules. A module M is called quasi-injective (pseudo-injective) if, for every submodule N of M, every R-homomorphism (R-monomorphism) from N to M can be extended to an R-endomorphism of M [5] ([6]). Every quasi-injective module is pseudo-injective. In previous papers (e.g. [4], [6], [7]), most of the results on pseudo-injective modules are of the form, "if R satisfies a suitable hypothesis, then certain pseudo-injective modules are quasi-injective." The intent of most of the work, then, was to show that pseudo-injectives were generally always quasi-injective (e.g. see the comment at the end of the Introduction to [7]). Indeed, the only two examples of pseudo-injective modules which are not quasi-injective have recently appeared in the literature (see [2] and [4]). Both of these modules have precisely five submodules and have Loewy length 2.

In this note, we give a construction for forming pseudo-injective modules which are not quasi-injective. This construction yields examples which answer in the negative the following two questions of S. K. Jain [3] (see also [4]): (i) Is every pseudo-injective module over a commutative ring quasi-injective? (ii) Is every nonsingular pseudo-injective module quasi-injective? Using an example of Fuchs [1], we can also apply our construction to show that a pseudo-injective module which is not quasi-injective may have arbitrarily large Loewy length.

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We now introduce some notation which we will use throughout this note. Let I be an index set, and let $\{M_i\}_{i\in I}$ be a set of modules. For each $i \in I$, let $m_i \in M_i$. By $\langle m_i\rangle$ we mean the element of $\Pi_{i\in I}M_i$ whose ith coordinate is m_i for each $i \in I$. By $\langle m_j\rangle^*$ we mean the element of $\Pi_{i\in I}M_i$ with m_j as its jth coordinate and 0 for all other coordinates.

Now we can state the result which is a construction for pseudo-injective modules which are not quasi-injective.

Proposition. *Let I be an infinite set, and let $\{M_i\}_{i\in I}$ be a set of pseudo-injective R-modules each of which has nonzero socle. For each $i \in I$, assume that there exists $r_i \in R$ such that*

(a) *$r_i m = m$ for all $m \in M_i$, and*

(b) *$r_i m = 0$ for all $m \in M_j$ with $j \in I - \{i\}$.*

For each $i \in I$, let $m_i \in \operatorname{Soc} M_i$ such that $(0 : m_i)$ is a maximal left ideal. Define M to be the R-submodule of $\Pi_{i\in I}M_i$ generated by $\bigoplus\Sigma_{i\in I}M_i$ and $\langle m_i\rangle$. If $H = \{r \in R \mid r \in (0 : m_i)$ for all but finitely many $i \in I\}$ is a maximal left ideal of R, then the following statements are valid.

(1) *M is not quasi-injective.*

(2) *M is pseudo-injective if and only if the set $S = \{i \in I \mid$ there exists a monomorphism $f\colon Rm_i \to M_i$ such that $f(m_i) \neq m_i\}$ has finite cardinality.*

Before proving the proposition, we need two technical lemmas which use the notation of the proposition.

Lemma 1. *If $\langle y_i\rangle \in M - (\bigoplus\Sigma_{i\in I}M_i)$, then $y_i = 0$ for at most finitely many $i \in I$.*

Proof. Since H is a maximal left ideal of R, $M/\bigoplus\Sigma_{i\in I}M_i$ is a simple left R-module. Hence there exist $r \in R$ and $\langle d_i\rangle \in \bigoplus\Sigma_{i\in I}M_i$ such that $\langle ry_i + d_i\rangle = r\langle y_i\rangle + \langle d_i\rangle = \langle m_i\rangle$. It follows from the choice of the m_i that only finitely many of the y_i can be 0.

Lemma 2. *Let $f_i\colon Rm_i \to M_i$ for each $i \in I$. If $f\colon M \to M$ is an extension of $\bigoplus\Sigma_{i\in I}f_i$, then $f(\langle m_i\rangle) = \langle f_i(m_i)\rangle$.*

Proof. Let $\pi_i\colon M \to M_i$ be the projection map $(i \in I)$. Set $f(\langle m_i\rangle) = \langle k_i\rangle$. Then

$$k_i = \pi_i\langle k_i\rangle = \pi_i\langle k_i\rangle^* = \pi_i(r_i\langle k_i\rangle) = \pi_i(r_i f(\langle m_i\rangle))$$
$$= \pi_i f(r_i\langle m_i\rangle) = \pi_i f(\langle m_i\rangle^*) = f_i(m_i).$$

Proof of the proposition. (1) Partition I into two disjoint infinite sets, J and K. For each $i \in I$, define $f_i\colon Rm_i \to M_i$ to be the inclusion map if

$i \in J$ and the zero map if $i \in K$. If M were quasi-injective, then $\bigoplus\Sigma_{i \in I} f_i$ would extend to a homomorphism $f\colon M \to M$. By Lemma 2, $f(\langle m_i \rangle) = \langle f_i(m_i)\rangle \in M$. Since $f_i(m_i) = m_i \neq 0$ for $i \in J$ and $f_i(m_i) = 0$ if $i \in K$, this contradicts Lemma 1. Hence M is not quasi-injective.

(2) Let M be pseudo-injective. Suppose that the set S has infinite cardinality. Partition S into two disjoint infinite sets T and $S - T$. For each $i \in T$, let $f_i\colon Rm_i \to M_i$ be a monomorphism such that $f_i(m_i) \neq m_i$. For each $i \in I - T$, define f_i to be the inclusion map $f_i\colon Rm_i \to M_i$. Since M is pseudo-injective, $\bigoplus\Sigma_{i \in I} f_i$ extends to $f\colon M \to M$. By Lemma 2, $f(\langle m_i \rangle) = \langle f_i(m_i)\rangle \in M$. Hence

$$\langle m_i - f_i(m_i)\rangle = \langle m_i \rangle - \langle f_i(m_i)\rangle \in M.$$

By our choice of f_i $(i \in I)$, this contradicts the result of Lemma 1.

Conversely, suppose that S has finite cardinality. Let

$$M_0 = \ker\left(\sum_{i \in S} \pi_i\right),$$

where $\pi_i\colon M \to M_i$ is the canonical projection. Then $M = M_0 \oplus (\bigoplus\Sigma_{i \in S} M_i)$. By (a) and (b) any submodule of M is a direct sum of submodules of the M_i $(i \in S \cup \{0\})$, and no nonzero submodule of M_i is a homomorphic image of a submodule of M_j for $i \neq j$ $(i, j \in S \cup \{0\})$. Hence any monomorphism from a submodule of M to M must be a direct sum of monomorphisms from submodules of M_i to M_i $(i \in S \cup \{0\})$. Since M_i is pseudo-injective for each $i \in S$, then M will be pseudo-injective provided that M_0 is. But it is easy to see that showing M_0 is pseudo-injective is equivalent to showing M is pseudo-injective whenever S is the empty set. Therefore, we assume S is the empty set and prove that M is pseudo-injective.

Let $N \subseteq M$, and let $g\colon N \to M$ be a monomorphism. Let $W = \{i \in I \mid \langle m_i \rangle^* \notin N\}$. If $0 \neq \langle x_i \rangle \in \Sigma_{i \in W} R\langle m_i\rangle^* \cap N$, then for some $j \in W$, $0 \neq r_j\langle x_i\rangle \in R\langle m_j\rangle^* \cap N$. Since Rm_j is a simple module, there exists $r \in R$ such that $\langle m_j\rangle^* = rr_j\langle x_i\rangle \in N$, which is a contradiction to our choice of W. Hence $\Sigma_{i \in W} R\langle m_i\rangle^* \cap N = 0$. Similarly, if $0 \neq \langle y_i\rangle \in \Sigma_{i \in W} R\langle m_i\rangle^* \cap g(N)$, then for some $j \in W$, $\langle m_j\rangle^* \in g(N)$. From (a) and (b) it follows that $g^{-1}(\langle m_j\rangle^*) = \langle k_j\rangle^* \in N$ for some $k_j \in M_j$. But this allows us to define a monomorphism $g_j\colon Rm_j \to M_j\colon rm_j \to rk_j$. Since $j \in W$, this forces a contradiction to the assumption that S is the empty set. Hence

$$\sum_{i \in W} R\langle m_i\rangle^* \cap g(N) = 0.$$

Therefore we can define a monomorphism

$$h:\left(\sum_{i\in W} R\langle m_i\rangle^* \oplus N\right) \longrightarrow M: \sum s_i\langle m_i\rangle^* + n \longrightarrow \sum s_i\langle m_i\rangle^* + g(n),$$

where $s_i \in R$ and $n \in N$. For each $i \in I$, h induces a monomorphism

$$h_i: Rm_i + \pi_i N \longrightarrow M_i: rm_i + \pi_i\langle n_j\rangle \longrightarrow \pi_i h(\langle rm_i\rangle^* + \langle n_i\rangle^*),$$

where $r \in R$ and $\langle n_j\rangle \in N$. Since each M_i $(i \in I)$ is pseudo-injective, each h_i extends to a homomorphism f_i: $M_i \longrightarrow M_i$.

It remains to show that

$$f: M \longrightarrow M: \langle a_i\rangle \longrightarrow \langle f_i(a_i)\rangle$$

defines an extension of g. Clearly

$$f \in \operatorname{Hom}_R\left(M, \prod_{i\in I} M_i\right), \quad \text{and} \quad f\left(\bigoplus \sum_{i\in I} M_i\right) \subseteq M.$$

Since S is the empty set, we also have $f_i(m_i) = h_i(m_i) = m_i$; hence $f(\langle m_i\rangle) = \langle m_i\rangle \in M$. Thus $f \in \operatorname{Hom}_R(M, M)$. Finally, if $\langle n_i\rangle \in N$ and $g(\langle n_i\rangle) = \langle y_i\rangle$, then

$$\begin{aligned} y_j &= \pi_j(\langle y_i\rangle) = \pi_j(\langle y_j\rangle^*) = \pi_j(r_j\langle y_i\rangle) = \pi_j(r_j g(\langle n_i\rangle)) \\ &= \pi_j g(r_j\langle n_i\rangle) = \pi_j h(r_j\langle n_i\rangle) = h_j\pi_j(\langle n_j\rangle^*) = h_j(n_j) = f_j(n_j). \end{aligned}$$

Thus $g(\langle n_i\rangle) = \langle y_i\rangle = \langle f_i(n_i)\rangle = f(\langle n_i\rangle)$, and hence f extends g.

We now use the proposition to construct examples of R-modules M which are pseudo-injective, but not quasi-injective. In our first example, M is a torsion module over a commutative integral domain. The reader may wish to compare Example 1 with [4, Theorem 6] and [7, Theorem 2].

Example 1. Let Z_2 be the field with two elements, and let $R = Z_2[x_1, x_2, \cdots]$ be the commutative polynomial ring in countably many indeterminants. For each positive integer i, let

$$P_i = (x_1, x_2, \cdots, x_{i-1}, 1 - x_i, x_{i+1}, \cdots),$$

let $M_i = R/P_i$, let $m_i = x_i + P_i$, and let $r_i = x_i$. Then $H = (x_1, x_2, \cdots)$ is a maximal ideal of R; so this situation satisfies the hypothesis of the proposition. Since the additive group of each M_i is Z_2, it follows that the module M constructed in the proposition is pseudo-injective in this case.

In our second example, we see that M can be a nonsingular module over a Boolean ring R; indeed, R itself is a pseudo-injective module (i.e. R is a self-pseudo-injective ring).

Example 2. Let F be a finite field, and let $F^{(n)} \cong F$ for each positive integer n. Let R be the subring of $\prod_{i=1}^{\infty} F^{(i)}$ generated by $\bigoplus\sum_{i=1}^{\infty} F^{(i)}$ and

$\langle 1_i \rangle$, where 1_i is the identity element of $F^{(i)}$. For each positive integer i, let $M_i = R\langle 1_i \rangle^*$ and $r_i = m_i = \langle 1_i \rangle^*$. Then $H = \bigoplus \sum_{i=1}^{\infty} F^{(i)}$ is a maximal left ideal of R. Hence the proposition implies that M is a pseudo-injective module if and only if $F \cong Z_2$, the field with two elements. We also note that $M \cong_R R$ is a nonsingular module (as R is a von Neumann regular ring).

The modules constructed in the first two examples have Loewy length 2 (see [1, p. 174] for the definition). Our last example shows that M can be a (nonsingular) module with arbitrarily large Loewy length.

Example 3. Let $\gamma \geq 2$ be an ordinal, and let K be the field of two elements. In the first part of the proof of [1, Theorem 6], Fuchs constructs a commutative K-algebra L (Fuchs calls it R) such that the Loewy length of L is $\lambda = \gamma + 1$. Moreover, every ideal of L contains an idempotent element; so no two distinct modules in the socle of L are isomorphic.

For each positive integer n, let $L \cong L^{(n)}$. Define R to be the subring of $\prod_{n=1}^{\infty} L^{(n)}$ generated by $\bigoplus \sum_{n=1}^{\infty} L^{(n)}$ and $\langle 1_n \rangle$, where 1_n is the identity element of L_n. Define $M_i = L_\gamma(E(R\langle 1_i \rangle^*))$, where $E(R\langle 1_i \rangle^*)$ is the injective hull of $R\langle 1_i \rangle^*$ and L_γ denotes the γth step in the ascending Loewy series (see [1, p. 174] for definition). Then M_i is an invariant submodule of $E(\langle 1_i \rangle^*)$, and hence M_i is quasi-injective. Let $r_i = \langle 1_i \rangle^*$, and let m_i be defined such that Rm_i is a simple submodule of M_i. By the structure of L, any R-monomorphism $Rm_i \longrightarrow M_i$ is the inclusion map. Since $H = \bigoplus \sum_{n=1}^{\infty} L^{(n)}$ is a maximal ideal of R, the proposition implies M is pseudo-injective. Since any minimal ideal of L is projective, then any minimal ideal of R is also projective and hence nonsingular. Since M is isomorphic to an essential extension of the socle of R, M is nonsingular. Since each M_i has Loewy length γ and since $R\langle m_i \rangle$ has Loewy length 2, it follows that M has Loewy length γ.

Remark. (i) In Example 3, the module M of Loewy length $\gamma \geq 2$ can be made singular by taking

$$M_i = L_\gamma(E(R\langle 1_i \rangle^* / \operatorname{Soc} R\langle 1_i \rangle^*))$$

and using facts developed by Fuchs [1] to show that any monomorphism $Rm_i \longrightarrow M_i$ is the inclusion map.

(ii) Using [1, Examples 3 and 4], the reader may also construct interesting examples of pseudo-injective modules M over a noncommutative ring in a manner somewhat similar to Example 3.

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