Tôhoku Math. Journ.
21(1969), 539-547.

PSEUDO-JACOBI FIELDS ON MINIMAL VARIETIES

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Dedicated to Professor Hitoshi Hombu on his sixtieth birthday.

(Received October 2, 1968, Revised March 22, 1969)

Introduction. A Jacobi field was originaly defined for a geodesic in a Riemannian manifold. It has been generalized for a minimal variety in a Riemannian manifold by some authors. (Simons [4]). Recently intersting papers concerning this problem have been published ([2], [3]). In the present paper we shall shortly generalize the Jacobi field on the minimal variety and give a sufficient condition on which the generalized one becomes trivial. In the last section we shall give a theorem concerning the conjugate boundary on a minimal hypersurface.

1. First we explain the notations adopted in this paper. Let X_n be an n-dimensional Riemannian manifold. For simplicity we assume that X_n be of class C^∞. We denote by $(x^1, \cdots, x^n)$ a local coordinate system of X_n. The fundamental quadratic form of X_n is denoted by

$$ds^2 = g_{\lambda\mu} dx^\lambda dx^\mu .$$

Hereafter the Greek indices range over $1, \cdots, n$. The Christoffel symbols and the curvature tensor are given by

$$(1.1) \qquad \left\{ \begin{matrix} \lambda \\ \mu\omega \end{matrix} \right\} = \frac{1}{2} g^{\lambda\sigma}(g_{\mu\sigma,\omega} + g_{\omega\sigma,\mu} - g_{\mu\omega,\sigma}), \qquad g^{\lambda\mu} g_{\mu\omega} = \delta^\lambda_\omega ,$$

$$(1.2) \qquad R^{\lambda}_{\cdot\mu\omega\pi} = \left\{ \begin{matrix} \lambda \\ \mu\omega \end{matrix} \right\}_{,\pi} - \left\{ \begin{matrix} \lambda \\ \mu\pi \end{matrix} \right\}_{,\omega} + \left\{ \begin{matrix} \sigma \\ \mu\omega \end{matrix} \right\}\left\{ \begin{matrix} \lambda \\ \sigma\pi \end{matrix} \right\} - \left\{ \begin{matrix} \sigma \\ \mu\pi \end{matrix} \right\}\left\{ \begin{matrix} \lambda \\ \sigma\omega \end{matrix} \right\},$$

where for example $g_{\mu\sigma,\omega}$ denotes $\partial g_{\mu\sigma}/\partial x^\omega$. We write

$$(1.3) \qquad R_{\lambda\mu\omega\pi} = g_{\lambda\sigma} R^{\sigma}_{\cdot\mu\omega\pi} .$$

It is well-known that

$$(1.4) \qquad R_{\lambda\mu\omega\pi} = -R_{\mu\lambda\omega\pi} = -R_{\lambda\mu\pi\omega} = R_{\omega\pi\lambda\mu} .$$

Let X_m be an m-dimensional submanifold of X_n. We assume that $1 < m < n$ and X_m be of class C^∞ and differentiably imbedded in X_n. We denote by $(y^{\dot 1}, \cdots, y^{\dot m})$ a local coordinate system of X_m. The fundamental quadratic form of X_m is denoted by

(1.5) $$ds^2 = g_{ij}\, dy^i\, dy^j\,,$$

where we put

(1.6) $$\begin{cases} g_{ij} = g_{\lambda\mu} B_i^\lambda B_j^\mu\,, \\ B_i^\lambda = \partial x^\lambda / \partial y^i\,. \end{cases}$$

Hereafter the Latin indices range over $\dot 1, \cdots, \dot m$. For simplicity we assume that x^λ's are functions of class C^∞ with regard to y^i's. We put

(1.7) $$\begin{cases} g^{ij} g_{jk} = \delta_k^i\,, \\ \left\{ {i \atop j\ k} \right\} = \dfrac{1}{2}\, g^{ia}(g_{ja,k} + g_{ka,j} - g_{jk,a})\,, \\ B_\lambda^i = g^{ij} g_{\lambda\mu} B_j^\mu\,, \end{cases}$$

where for example $g_{jk,a}$ denotes $\partial g_{jk}/\partial y^a$. The Euler-Schouten's tensor is given by

(1.8) $$H_{ij}^\lambda = B_{i,j}^\lambda + \left\{ {\lambda \atop \mu\omega} \right\} B_i^\mu B_j^\omega - \left\{ {a \atop i\ j} \right\} B_a^\lambda = B_{i;j}^\lambda\,,$$

where the semicolon denotes the covariant differentiation along X_m. In this case B_i^λ is a contrvariant vector in the sense of X_n and is a covariant vector in the sense of X_m. Therefore we use $\left\{ {\lambda \atop \mu\omega} \right\} B_j^\omega$ or $\left\{ {a \atop i\ j} \right\}$ for the indices of X_n or X_m respectively. Hereafter we shall adopt this convention for the covariant differentiation along X_m. It is well-known that

(1.9) $$H_{ij}^\lambda = H_{ji}^\lambda$$

and

(1.10) $$g_{\lambda\mu} H_{ij}^\lambda B_k^\mu = 0\,.$$

We write

(1.11) $$H_{\lambda ij} = g_{\lambda\mu} H_{ij}^\mu\,, \qquad H^{\lambda ij} = H_{ab}^\lambda g^{ia} g^{jb}\,.$$

2. Let G be a bounded orientable domain of X_m and let ∂G be its boundary. First we consider the case where G is covered by a pair of local coordinate systems $(y^1, \cdots, y^m)$ and $(x^1, \cdots, x^n)$. However we can easily see that our results hold when G is covered by several coordinate systems. The area of G is given by

$$(2.1) \qquad \int_G |g_{ij}|^{1/2} dy^1 \cdots dy^m ,$$

where $|g_{ij}|$ denotes the determinant whose elements are g_{ij}'s. We consider an infinitesimal transformation

$$(2.2) \qquad \bar{x}^\lambda = x^\lambda(y) + \varepsilon v^\lambda(y)$$

where ε denotes an infinitesimal constant and $v^\lambda(y)$ denotes any vector along X_m which is normal to X_m and vanishes on ∂G and is of class C^1 with regard to y^i's. It is well-known that the first variation of the integral (2.1) by the infinitesimal transformation (2.2) is given by

$$(2.3) \qquad \varepsilon \int_G \left(\frac{\partial L}{\partial x^\lambda} - \frac{\partial}{\partial y^i} \left(\frac{\partial L}{\partial B_i^\lambda} \right) \right) v^\lambda dy^1 \cdots dy^m ,$$

where

$$(2.4) \qquad L(x^\lambda, B_i^\lambda) = |g_{ij}|^{1/2} .$$

Therefore in order that the area of G be minimal it is necessary that

$$(2.5) \qquad \frac{\partial L}{\partial x^\lambda} - \frac{\partial}{\partial y^i} \left(\frac{\partial L}{\partial B_i^\lambda} \right) = 0$$

which leads to

$$(2.6) \qquad H^\lambda \equiv H_{ij}^\lambda g^{ij} = 0 .$$

If (2.6) holds everywhere, such an X_m is called a "minimal variety".

3. Let X_m be a minimal variety. Let us shift it slightly by an infinitesimal transformation of the form (2.2) where we assume that v^λ is of class C^2 and normal to X_m. Let us calculate the first variation of H^λ. By the assumption we have

$$
(3.1)\qquad \begin{cases} H^{\lambda} \equiv H^{\lambda}_{ij} g^{ij} = 0 , \\ g_{\lambda\mu} B^{\lambda}_{i} v^{\mu} = 0 . \end{cases}
$$

First we compute the first variation of $\left\{ {\lambda \atop \mu\omega} \right\}$, $B^{\lambda}_{i,j}$, g_{ij}, g^{ij} and $\left\{ {i \atop j\,k} \right\}$. For example we have $\delta B^{\lambda}_{i,j} = \varepsilon v^{\lambda}{}_{,i,j}$. We have from (1.8) and these variations

$$
(3.2)\qquad \begin{aligned} \delta H^{\lambda} = -\varepsilon[2H^{\lambda}_{ij} g_{\mu\omega} B^{\omega}_{a} v^{\mu}{}_{;b}\; g^{ia} g^{jb} + B^{\lambda}_{k} g^{ak} g^{ij} \\ (-B^{\omega}_{i} v^{\mu}_{;j;a} + B^{\omega}_{j} v^{\mu}_{;a;i} + B^{\omega}_{a} v^{\mu}_{;j;i}) g_{\mu\omega} \\ -g^{ij}(v^{\lambda}_{;i;j} + R^{\lambda}_{.\omega\mu\pi} B^{\mu}_{i} B^{\omega}_{j} v^{\pi})] , \end{aligned}
$$

where δH^{λ} denotes the first variation of H^{λ}. Our result coincides with that of Duschek ([1]) which was obtained by the parametric method. Meanwhile we have from (3.1)

$$
(3.3)\qquad g_{\mu\omega} v^{\mu}_{;b} B^{\omega}_{a} + g_{\mu\omega} v^{\mu} H^{\omega}_{ab} = 0 .
$$

Hence we have from (3.2) and (3.3)

$$
(3.4)\qquad \begin{cases} \delta H^{\lambda} = \varepsilon(\delta^{\lambda}_{\sigma} - B^{\lambda}_{k} B^{k}_{\sigma})\{\Delta v^{\sigma} + (R^{\sigma}_{.\omega\mu\pi} B^{\omega}_{i} B^{\mu}_{j} g^{ij} + 2H^{\sigma}_{ji} H^{ij}_{\pi}) v^{\pi}\} , \\ \Delta v^{\sigma} \equiv v^{\sigma}_{;i;j} g^{ij} . \end{cases}
$$

If

$$
(3.5)\qquad (\delta^{\lambda}_{\sigma} - B^{\lambda}_{k} B^{k}_{\sigma})\{\Delta v^{\sigma} + (R^{\sigma}_{.\omega\mu\pi} B^{\omega}_{i} B^{\mu}_{j} g^{ij} + 2H^{\sigma}_{ij} H^{ij}_{\pi}) v^{\pi}\} = 0 ,
$$

i.e., the normal component of the vector

$$
(3.6)\qquad J^{\sigma} \equiv \Delta v^{\sigma} + (R^{\sigma}_{.\omega\mu\pi} B^{\omega}_{i} B^{\mu}_{j} g^{ij} + 2H^{\sigma}_{ij} H^{ij}_{\pi}) v^{\pi}
$$

vanishes, then we say that the infinitesimal transformation (2.2) preserves the minimal property of the variety.

REMARK. In our notations the second variation of the intergral (2.1) given by Duschek ([1]) becomes

$$
(3.7)\qquad \begin{aligned} -\frac{1}{2}\varepsilon^{2} \int_{G} \{(\delta^{\lambda}_{\sigma} - B^{\lambda}_{k} B^{k}_{\sigma})(\Delta v^{\sigma} + R^{\sigma}_{.\mu\omega\pi} B^{\omega}_{i} B^{\mu}_{j} g^{ij} v^{\pi}) \\ -v_{\sigma;i}(2H^{\lambda ij} B^{\sigma}_{j} + H^{\sigma j}_{j} B^{\lambda}_{k} g^{ik})\}\, v_{\lambda} |g_{rs}|^{1/2} dy^{1} \cdots dy^{m} , \end{aligned}
$$

where $v_\lambda = g_{\lambda\mu} v^\mu$. If v^λ is normal to X_m, then (3.7) becomes

$$(3.8) \qquad -\frac{1}{2}\varepsilon^2 \int_G J^\sigma v_\sigma |g_{rs}|^{1/2} dy^1 \cdots dy^m .$$

The same result was obtained by Simons in a different manner ([2]) p. 73). When v^λ satisfies $J^\lambda = 0$ and is normal to X_m, it is called a "Jacobi field".

When v^λ satisfies (3.5) and is normal to the minimal variety X_m, we call it a "pseudo-Jacobi field". From (3.5) and (3.8) we see that the second variation of the area (2.1) is zero when v^λ is a pseudo-Jacobi field. Let G be a bounded orientable domain of a minimal variety and let ∂G be its boundary. A pseudo-Jacobi field which vanishes on ∂G is called a "pseudo-Jacobi field on G". Let v^λ be a pseudo-Jacobi field on G. Then we have from (3.5)

$$\begin{aligned}(3.9) \quad 0 &= \int_G (\delta^\lambda_\sigma - B^\lambda_k B^k_\sigma)\{\Delta v^\sigma + (R^\sigma_{.\omega\mu\pi} B^\omega_i B^\mu_j g^{ij} + 2H^\sigma{}_{ij} H_\pi{}^{ij}) v^\pi\} v_\lambda d\sigma \\ &= \int_G \{\Delta v^\lambda + (R^\lambda_{.\omega\mu\pi} B^\omega_i B^\mu_j g^{ij} + 2H^\lambda{}_{ij} H_\pi{}^{ij}) v^\pi\} v_\lambda d\sigma \\ &= \int_G \Big[\frac{1}{2} g^{ij}(g_{\lambda\mu} v^\lambda v^\mu)_{;i;j} - g_{\lambda\mu} v^\lambda_{;i} v^\mu_{;j} g^{ij} \\ &\quad + (R_{\lambda\omega\mu\pi} B^\omega_i B^\mu_j g^{ij} + 2H_{\lambda ij} H_\pi{}^{ij}) v^\lambda v^\pi\Big] d\sigma ,\end{aligned}$$

where $d\sigma = |g_{ij}|^{1/2} dy^1 \cdots dy^m$. The first term of the last integral vanishes by the theorem of Stokes and the second term is negative definete. Hence we have the

THEOREM 1. *If the quadratic form*

$$(3.10) \qquad (R_{\lambda\omega\mu\pi} B^\omega_i B^\mu_j g^{ij} + 2H_{\lambda ij} H_\pi{}^{ij}) X^\lambda X^\pi$$

is everywhere negative semi-definite on G, where X^λ denotes any vector normal to G, then the pseudo-Jacobi field on G is identically zero.

PROOF. We have from (3.9) and the assumption

$$v^\lambda_{;i} = 0 .$$

Since v^λ vanishes on ∂G, v^λ must vanish everywhere on G. Q.E.D.

REMARK. When $R_{\lambda\mu\omega\pi} = K(g_{\mu\omega} g_{\lambda\pi} - g_{\lambda\omega} g_{\mu\pi})$, i. e., the space is of constant curvature, then the condition of the above theorem becomes "$(mKg_{\lambda\mu} + 2H_{\lambda ij}H_{\mu}{}^{ij}) X^\lambda X^\mu$ is everywhere negative semi-definite on G".

4. Let us consider the case where $m = n - 1$, i. e. X_m is a hypersurface. Let n^λ be the unit normal vector to a minimal hypersurface X_{n-1}. Putting $v^\lambda = \rho n^\lambda$ we have from (3. 5)

$$\Delta\rho + \rho(h_{ij} h^{ij} + R_{\sigma\mu\omega\pi} n^\sigma n^\pi B_i^\mu B_j^\omega g^{ij}) = 0 , \tag{4. 1}$$

where we put

$$\begin{cases} B^\lambda_{i;j} = H^\lambda_{ij} = n^\lambda h_{ij}, & \Delta\rho = \rho_{,i;j} g^{ij}, \quad n^\lambda_{;j} = -B^\lambda_i h^i_{.j} , \\ h^i_{.j} = g^{ik} h_{kj}, & h^{ij} = g^{ia} g^{jb} h_{ab}, \quad h_{ij} = h_{ji} . \end{cases} \tag{4. 2}$$

Meanwhile we have

$$\Delta\rho^2 = 2\rho\Delta\rho + 2\rho_{,i}\rho_{,j} g^{ij} . \tag{4. 3}$$

Hence (4. 1) leads to

$$\frac{1}{2}\Delta\rho^2 - \rho_{,i}\rho_{,j} g^{ij} + \rho^2(h_{ij} h^{ij} + R_{\sigma\mu\omega\pi} n^\sigma n^\pi B_i^\mu B_j^\omega g^{ij}) = 0 . \tag{4. 4}$$

Since

$$B_i^\mu B_j^\omega g^{ij} = g^{\mu\omega} - n^\mu n^\omega \tag{4. 5}$$

we have from (4. 4)

$$\frac{1}{2}\Delta\rho^2 - \rho_{,i}\rho_{,j} g^{ij} + \rho^2(h_{ij} h^{ij} + R_{\mu\lambda} n^\lambda n^\mu) = 0 . \tag{4. 6}$$

If $\rho = 0$ on the boundary of an orientable domain D on X_{n-1} and ρ satisfies (4. 1) in D, then we have from (4. 6) and Green's theorem

$$\int_D \{\rho_{,i}\rho_{,j} g^{ij} - \rho^2(h_{ij} h^{ij} + R_{\lambda\mu} n^\lambda n^\mu)\} \, d\sigma = 0 , \tag{4. 7}$$

where $d\sigma$ denotes the volume element of X_{n-1} and we assume that ∂D is smooth and orientable. If a relation

(4.8) $$h_{ij}h^{ij} + R_{\lambda\mu}n_\lambda n^\mu \leqq 0$$

holds everywhere in D, then we see from (4.7) that

(4.9) $$\rho_{,i} = 0$$

everywhere in D, i.e. $\rho = 0$ everywhere in D. Thus we have the

THEOREM 2. *Let D be an orientable domain of a minimal hypersurface in a Riemannian space and let ∂D be smooth and orientable. If*

$$h_{ij}h^{ij} + R_{\lambda\mu}n^\lambda n^\mu \leqq 0$$

holds everywhere in D, then there is no non-trivial pseudo-Jacobi field on D.

REMARK. When $R_{\lambda\mu} = \dfrac{R}{n} g_{\lambda\mu}$, i.e. the space is an Einstein space, then the inequality (4.8) becomes

(4.10) $$h_{ij}h^{ij} + \frac{R}{n} \leqq 0 .$$

5. In this section we shall generalize a theorem concerning the conjugate points in the classical differential geometry. Let g be a geodesic on a surface of the euclidean 3-space. Let P be a point on g and P be its first conjugate

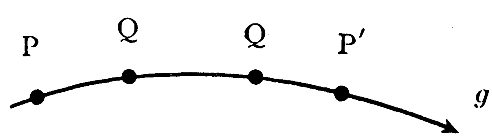

Fig .1

point. The following fact is well-known: It is impossible that a point Q′ on the geodesic arc PP′ be the first conjugate point of a point Q on PP′. The proof is done by the Jacobi equation

(5.1) $$\frac{d^2 Y}{ds^2} + K(s)\,Y = 0 ,$$

where K denotes the Gaussian curvature of the surface and s denotes the arc-length of g. In this case we assume that there exists a solution of (5.1) which vanishes at P and P′ and is not zero at any point on the arc PP′. The same thing doesn't hold for the pair of points Q,Q′. Let us generalize the above theorem to the case of a minimal hypersurface of a Riemannian manifold. We see from (1.4) the tensor

$$R_{\lambda\omega\mu\pi} B_i^\omega B_j^\mu g^{ij} + 2H_{\lambda ij} H_\pi^{\ ij} \tag{5.2}$$

is symmetric with regard to λ and π. Hence if there exist two pseudo-Jacobi fields v^λ and w^λ, then we have from (3.5)

$$g_{\lambda\mu} w^\lambda \Delta v^\mu - g_{\lambda\mu} v^\lambda \Delta w^\mu = \{(g_{\lambda\mu} v^\lambda_{;i} w^\mu - g_{\lambda\mu} w^\lambda_{;i} v^\mu) g^{ij}\}_{;j} = 0 . \tag{5.3}$$

In the case of X_{n-1} (5.3) becomes

$$\{(\varphi_{,i}\psi - \psi_{,i}\varphi) g^{ij}\}_{;j} = 0 , \tag{5.4}$$

where we put

$$\begin{aligned} v^\lambda &= \varphi n^\lambda , & w^\lambda &= \psi n^\lambda , \\ \varphi_{,i} &= \partial\varphi / \partial y^i , & \psi_{,i} &= \partial\psi / \partial y^i . \end{aligned} \tag{5.5}$$

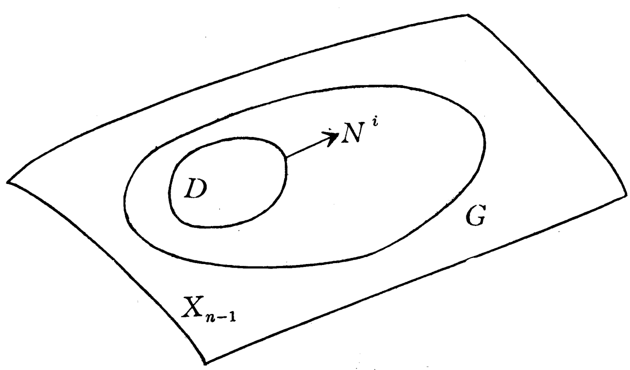

Fig.2

Let G be an orientable domain of a minimal hypersurface X_{n-1} and D be its sub-domain. We assume that ∂D be smooth and orientable. Let v^λ or w^λ in (5.5) be a pseudo-Jacobi field on X_{n-1} and vanish on ∂G or ∂D and be not zero at any point in G or D respectively. Integrating (5.4) over D we have

$$0 = \int_D \{(\varphi_{,i}\psi - \psi_{,i}\varphi) g^{ij}\}_{;j} d\sigma = \int_{\partial D} (\varphi_{,i}\psi - \psi_{,i}\varphi) N^i dS , \tag{5.6}$$

where $d\sigma$ denotes the volume element of X_{n-1} and dS denotes the surface element of ∂D and N^i denotes the unit normal vector to ∂D in X_{n-1}. The positive direction of N^i is directed outwards. Since $\psi = 0$ on ∂D we have from (5.6)

$$\int_{\partial D} (\psi_{,i} N^i)\, \varphi dS = 0\,. \tag{5.7}$$

Since ψ is not zero at any point of D, if for example ψ is positive in D, then $\psi_{,i} N^i$ is not positive on ∂D. We assume that $\psi_{,i} N^i$ is not zero at some point on ∂D. Considering that φ is not zero at any point on ∂D, we see that the left hand side of (5.7) is not zero. Thus we arrive at a contradiction. Hence we have the

THEOREM 3. *Let $X_{n-1} (n > 2)$ be a minimal hypersurface of a Riemannian space X_n. It is impossible that the following* (i)~(v) *hold simultaneously*:

(i) *G and D are domains of X_{n-1} and $G \supset D$ and $\partial G \cap \partial D = \phi$,*

(ii) *G is orientable,*

(iii) *∂D is smooth and orientable,*

(iv) *v^λ is a pseudo-Jacobi field on X_{n-1} which vanishes on ∂G and is not zero at any point of G,*

(v) *$w^\lambda = \psi n^\lambda$ is a pseudo-Jacobi field on X_{n-1} which vanishes on ∂D and is not zero at any point of D and $\psi_{,i} N^i$ is not zero at some point on ∂D, where n^λ denotes the unit normal vector to X_{n-1} and N^i denotes the unit normal vector to ∂D in X_{n-1} whose positive direction is directed outwards.*

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