Pseudo-Parabolic Partial Differential Equations

R.E. Showalter

Department of Mathematics
Oregon State University

Applied Mathematics & Computation Seminar

Outline

PDE are just ODE in an appropriate function space.

Here we treat simple partial differential equations as evolution equations (ordinary differential equations) in the space $L^2(G)$.

Outline

Parabolic equation

$u = u(x,t)$: Initial-Boundary-Value Problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla\cdot k\nabla u = 0,\ x \in \Omega,\ t > 0, \\ u(s,t) = 0, \quad s \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{aligned}$$

Outline

Pseudo-Parabolic Equation

$$\frac{\partial u}{\partial t} - \varepsilon \nabla \cdot k \nabla \frac{\partial u}{\partial t} - \nabla \cdot k \nabla u = 0, \ x \in \Omega, \ t > 0,$$
$$u(s,t) = 0, \quad s \in \partial\Omega, \ t > 0,$$
$$u(x,0) = u_0(x), \quad x \in \Omega.$$

Outline

Origins

- 1926 Milne ... time delay, gas diffusion
- 1948 Rubinstein ... heat conduction in composite medium
- 1960 Barenblatt ... fluid flow in fissured medium
- 1960 Coleman-Noll ... heat conduction
- 1968 Chen-Gurtin
- 1966 Lighthill ... fluid
- 1966 Peregrine ... long waves (semilinear)
- 1972 Benjamin-Bona-Mahoney
- 1979 Aifantis ... highly-diffusive paths
- 1980 Gilbert ... Slightly-compressible Stokes velocity

Outline

Elliptic Boundary-Value Problem

The spatial derivatives are given by the operator

$$Au = -\nabla \cdot k \nabla u(\cdot) \text{ in } L^2(G),$$
$$D(A) = \{u \in H^2(G) : \ u = 0 \text{ on } \partial G\}$$

Eigen-functions: $\{v_j(\cdot) : \ j \geq 1\}$ is an ortho-normal basis for $L^2(G)$

$$A(v_j) = \lambda_j v_j \, , \, j \geq 1 \, , \quad 0 < \lambda_j \to +\infty$$

Outline

The Parabolic Equation

$$u'(t) + Au(t) = 0, \quad t > 0\,,$$
$$u(0) = u_0\,.$$

$$u(t) = \sum_{j=1}^{\infty} e^{-\lambda_j t}(u_0, v_j)\, v_j$$
$$= S(t)u_0 = e^{-At}u_0$$

- Analytic semigroup
- Regularity increasing for $t > 0$
- Unbounded decay rate of coefficients

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$$u'(t) + \varepsilon A u'(t) + A u(t) = 0, \quad t > 0\,,$$
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$$u(t) = \sum_{j=1}^{\infty} e^{\frac{-\lambda_j t}{1+\varepsilon\lambda_j}} (u_0, v_j) v_j$$
$$= S_\varepsilon(t) u_0 = e^{-(I+\varepsilon A)^{-1} A t} u_0$$

- C^0-group
- Regularity preserving for $-\infty < t < \infty$
- Decay rate bounded below by $\frac{1}{\varepsilon}$

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ODE in $L^2(G)$

$$A_\varepsilon = (I + \varepsilon A)^{-1} A = A(I + \varepsilon A)^{-1} = \frac{1}{\varepsilon}(I - (I + \varepsilon A)^{-1})$$

is a bounded operator on $L^2(G)$.

$$u'(t) + A\varepsilon u(t) = 0$$

is an Ordinary Differential Equation in $L^2(G)$.

... a little algebra ...

The pseudo-parabolic equation

$$u'(t) + \varepsilon A u'(t) + A u(t) = 0, \quad t > 0$$

can be written

$$u'(t) + \frac{1}{\varepsilon} u(t) = \frac{1}{\varepsilon}(I + \varepsilon A)^{-1} u(t) \in D(A)$$

The *saltus* or *jump* along an interface, $[u](t)$, satisfies

$$[u]'(t) + \frac{1}{\varepsilon}[u](t) = 0\,,$$

so

$$[u](t) = e^{-\frac{t}{\varepsilon}}[u_0]\,.$$

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