Pseudo-Parabolic Partial Differential Equations

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Outline



The Initial-Boundary-Value Problems

- Parabolic Diffusion Equation
- Pseudo-Parabolic Equation
- Origins

Operators in L²

- Elliptic Boundary-Value Problem
- Evolution Equations in $L^2(G)$
- ODE and an Elliptic BVP

Parabolic Diffusion Equation Pseudo-Parabolic Equation Drigins

PDE are just ODE in an appropriate function space.

Here we treat simple partial differential equations as evolution equations (ordinary differential equations) in the space $L^2(G)$.

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Parabolic Diffusion Equation Pseudo-Parabolic Equation Origins

Parabolic equation

u = u(x, t): Initial-Boundary-Value Problem

$$egin{aligned} &rac{\partial u}{\partial t}-
abla\cdot k
abla u&=0,\ x\in\Omega,\ t>0,\ u(s,t)&=0,\ s\in\partial\Omega,\ t>0,\ u(x,0)&=u_0(x),\ x\in\Omega. \end{aligned}$$

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Parabolic Diffusion Equation Pseudo-Parabolic Equation Origins

Pseudo-Parabolic Equation

$$\begin{split} \frac{\partial u}{\partial t} &- \varepsilon \nabla \cdot k \nabla \frac{\partial u}{\partial t} - \nabla \cdot k \nabla u = 0, \ x \in \Omega, \ t > 0, \\ u(s,t) &= 0, \quad s \in \partial \Omega, \ t > 0, \\ u(x,0) &= u_0(x), \quad x \in \Omega. \end{split}$$

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Parabolic Diffusion Equation Pseudo-Parabolic Equation Origins

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Parabolic Diffusion Equation Pseudo-Parabolic Equation Origins

Origins

- 1926 Milne ... time delay, gas diffusion
- 1948 Rubinstein ... heat conduction in composite medium
- 1960 Barenblatt ... fluid flow in fissured medium
- 1960 Coleman-Noll ... heat conduction
- 1968 Chen-Gurtin
- 1966 Lighthill ... fluid
- 1966 Peregrine ... long waves (semilinear)
- 1972 Benjamin-Bona-Mahoney
- 1979 Aifantis ... highly-diffusive paths
- 1980 Gilbert ... Slightly-compressible Stokes velocity

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Elliptic Boundary-Value Problem Evolution Equations in $L^2(G)$ ODE and an Elliptic BVP

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Elliptic Boundary-Value Problem

The spatial derivatives are given by the operator

$$egin{aligned} &\mathcal{A}u = -
abla\cdot k
abla u(\cdot) ext{ in } L^2(G), \ &\mathcal{D}(\mathcal{A}) = \{u\in H^2(G): \ u=0 ext{ on } \partial G\} \end{aligned}$$

Eigen-functions: $\{v_j(\cdot) : j \ge 1\}$ is an ortho-normal basis for $L^2(G)$

$$A(v_j) = \lambda_j v_j, \ j \ge 1, \quad 0 < \lambda_j \to +\infty$$

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Elliptic Boundary-Value Problem Evolution Equations in $L^2(G)$ ODE and an Elliptic BVP

The Parabolic Equation

$$u'(t) + Au(t) = 0, \quad t > 0,$$

 $u(0) = u_0.$

$$u(t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} (u_0, v_j) v_j$$
$$= S(t) u_0 = e^{-At} u_0$$

- Analytic semigroup
- Regularity increasing for t > 0
- Unbounded decay rate of coefficients

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The Pseudo-Parabolic Equation

$$u'(t) + \varepsilon A u'(t) + A u(t) = 0, \quad t > 0,$$

 $u(0) = u_0.$

$$u(t) = \sum_{j=1}^{\infty} e^{\frac{-\lambda_j t}{1+\varepsilon\lambda_j}} (u_0, v_j) v_j$$
$$= S_{\varepsilon}(t)u_0 = e^{-(l+\varepsilon A)^{-1}At}u_0$$

- C⁰-group
- Regularity preserving for $-\infty < t < \infty$
- Decay rate bounded below by -

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The Pseudo-Parabolic Equation

$$egin{aligned} u'(t)+arepsilon {\it A}u'(t)+{\it A}u(t)&=0, \quad t>0\,, \ u(0)&=u_0\,. \end{aligned}$$

$$u(t) = \sum_{j=1}^{\infty} e^{\frac{-\lambda_j t}{1+\varepsilon\lambda_j}} (u_0, v_j) v_j$$
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$$egin{aligned} u'(t)+arepsilon {\it A}u'(t)+{\it A}u(t)&=0, \quad t>0\,, \ u(0)&=u_0\,. \end{aligned}$$

$$u(t) = \sum_{j=1}^{\infty} e^{\frac{-\lambda_j t}{1+\varepsilon\lambda_j}} (u_0, v_j) v_j$$
$$= S_{\varepsilon}(t)u_0 = e^{-(I+\varepsilon A)^{-1}At}u_0$$

- C⁰-group
- Regularity preserving for $-\infty < t < \infty$
- Decay rate bounded below by $\frac{1}{\epsilon}$

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ODE in $L^2(G)$

$$A_{\varepsilon} = (I + \varepsilon A)^{-1} A = A(I + \varepsilon A)^{-1} = \frac{1}{\varepsilon} (I - (I + \varepsilon A)^{-1})$$

is a bounded operator on $L^2(G)$.

 $u'(t) + A\varepsilon u(t) = 0$

is an Ordinary Differential Equation in $L^2(G)$.

Elliptic Boundary-Value Problem Evolution Equations in $L^2(G)$ ODE and an Elliptic BVP

... a little algebra ...

The pseudo-parabolic equation

$$u'(t) + \varepsilon Au'(t) + Au(t) = 0, \quad t > 0$$

can be written

$$u'(t) + \frac{1}{\varepsilon}u(t) = \frac{1}{\varepsilon}(I + \varepsilon A)^{-1}u(t) \in D(A)$$

The saltus or jump along an interface, [*u*](*t*), satisfies

$$[u]'(t) + \frac{1}{\varepsilon}[u](t) = 0,$$

so
$$[u](t) = e^{-\frac{t}{\varepsilon}}[u_0].$$

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... a little algebra ...

The pseudo-parabolic equation

$$u'(t) + \varepsilon Au'(t) + Au(t) = 0, \quad t > 0$$

can be written

$$u'(t) + \frac{1}{\varepsilon}u(t) = \frac{1}{\varepsilon}(I + \varepsilon A)^{-1}u(t) \in D(A)$$

The saltus or jump along an interface, [u](t), satisfies

$$[u]'(t) + \frac{1}{\varepsilon}[u](t) = 0,$$

so
$$[u](t) = e^{-\frac{t}{\varepsilon}}[u_0].$$