Pseudo-Riemannian metrics on tangent bundle and harmonic problems *

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Abstract

The problems studied in this paper are related to the harmonicity of the canonical projection $\pi : TM \longrightarrow M$, where (M, g) is a Riemannian space and TM is its tangent bundle, and to the harmonicity of the vector fields $\xi \in \chi(M)$ thought of as maps from M to TM. We have considered on TM the pseudo-Riemannian metrics G, g^c of lift-complete type defined by means of an arbitrary nonlinear connection on TM. We have also studied the harmonicity of a tensor field J of type $(1, 1)$ on M, where J is thought of as a map from TM into itself.

Introduction

A vector field ξ on a Riemannian manifold (M, g) can be thought of as a map $\xi : M \longrightarrow TM$, where $\pi : TM \longrightarrow M$ is the tangent bundle of the manifold M. The conditions under which ξ is an isometric immersion, a totally geodesic or harmonic map, have been studied in the cases where one considers on TM the Riemannian metrics defined by Sasaki, Cheeger-Gromoll or the pseudo-Riemannian metrics of complete lift type (see [7], [16], [12], [13], [14], [11]). The conditions under which the canonical projection $\pi : TM \longrightarrow M$ is a totally geodesic or harmonic map have been also studied.

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A tensor field J of type $(1,1)$ on M can be thought of as a map $J : TM \longrightarrow TM$. The conditions under which J is a harmonic map have been studied only in the case where the complete lift metric is considered on TM (see [4]).

In this paper we deal with the same problems of harmonicity of vector fields, the canonical projection and tensor fields of type $(1,1)$ on M. We consider on TM the pseudo-Riemannian metrics G and g^c defined by means of a nonlinear connection on TM. The main idea is to modify the nonlinear connection: we shall no longer use the nonlinear connection defined by the Levi-Civita connection ∇ of (M,g) but, more generally, an arbitrary nonlinear connection on TM. We can obtain some quite interesting results.

The manifolds, maps, vector fields etc. considered in this work are assumed to be smooth, i.e. differentiable of class C^∞. The well known summation convention is used throughout the paper. The ranges for the indices are $\{1,...,n\}$ for h,i,j,k,l and $\{1,...,m\}$ for $\alpha,\beta,\gamma,\sigma$; $\chi(M)$ stands for the Lie algebra of the smooth vector fields on M.

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1 The tangent bundle

Let (M,g) be an $n-$dimensional Riemannian manifold with $n > 1$ and let $\pi : TM \longrightarrow M$ be its tangent bundle. A local chart $(U, x^i), i = 1,...,n$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i), i = 1,...,n$ on TM, where we denote, by abuse, x^i instead of $\pi^* x^i = x^i \circ \pi$, and y^i are the vector space coordinates of the element $v \in \pi^{-1}(U) \subset TM$ with respect to the natural basis $(\frac{\partial}{\partial x^i})_{\pi(v)}, i = 1,...,n$, i.e. $v = y^i (\frac{\partial}{\partial x^i})_{\pi(v)}$. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have the vertical distribution VTM on TM, defined by $V_v(TM) = \ker \pi_{*,v}$, $v \in TM$. One must note that $V(TM)$ is an integrable distribution. We consider a nonlinear connection on TM defined by the distribution $H(TM)$ on TM, complementary to $V(TM)$, i.e. $H_v(TM) \oplus V_v(TM) = T_v(TM)$, $v \in TM$. The distribution $H(TM)$ is the horizontal distribution. For any induced local chart $(\pi^{-1}(U), x^i, y^i)$ we have a local adapted frame in $H(TM)$ defined by the vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x,y)\frac{\partial}{\partial y^j}\ ,\ \ i = 1,...,n\ , \tag{1}$$

where the local functions $N^j_i(x,y)$ are the connection coefficients of the nonlinear connection defined by $H(TM)$. The vector fields $(\frac{\partial}{\partial y^i})$, $i = 1,\ldots,n$ define a local frame for the vertical distribution $V(TM)$. Let $\xi = \xi^i \frac{\partial}{\partial x^i}$ be a (local) vector field on M. The horizontal and the vertical lifts of ξ are defined by

$$\xi^H = \xi^i \frac{\delta}{\delta x^i}, \quad \xi^V = \xi^i \frac{\partial}{\partial y^i}.$$

We have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$.

The system of local 1-forms (dx^i, Dy^i) defines the dual frame of the frame $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, where

$$Dy^i = dy^i + N^i_j(x,y)dx^j. \tag{2}$$

Denote

$$R^i_{jk} = \frac{\delta N^i_k}{\delta x^j} - \frac{\delta N^i_j}{\delta x^k} = \frac{\delta}{\delta x^j}(N^i_k) - \frac{\delta}{\delta x^k}(N^i_j). \tag{3}$$

Remark that $R^i_{kj} = -R^i_{jk}$. We have the following commutation formulae

$$\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^i_{kj}\frac{\partial}{\partial y^i}, \quad \left[\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\right] = \frac{\partial N^i_j}{\partial y^k}\frac{\partial}{\partial y^i}, \quad \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0. \tag{4}$$

It follows that the horizontal distribution $H(TM)$ is integrable if and only if $R^i_{jk} = 0$.

We define on TM the tensor field of type $(0,2)$ G of lift-complete type by

$$G(X^V, Y^V) = 0, \quad G(X^V, Y^H) = g(X,Y), \quad G(X^H, Y^H) = g(X,Y), \tag{5}$$

(see [14]). If $g = g_{ij}dx^i dx^j$ is the expression of g in local coordinates, then G is given locally by

$$G = g_{ij}dx^i dx^j + 2g_{ij}dx^i Dy^j. \tag{6}$$

The tensor G defines a pseudo-Riemannian metric on TM. The vertical distribution is maximally isotropic and G defines a pairing between the horizontal and vertical distribution. The signature of G is (n,n) and the fibres of horizontal distribution are isometric with the tangent spaces on M in the coresponding points.

Proposition 1 *The Levi-Civita connection ${}^G\nabla$ of G is given locally by*

$$\begin{cases} {}^G\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = 0, \\ {}^G\nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial N^l_k}{\partial y^i}g_{lj} - \frac{\partial N^l_j}{\partial y^i}g_{lk}\right)g^{kh}\frac{\partial}{\partial y^h}, \\ {}^G\nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial N^l_k}{\partial y^j}g_{li} + \frac{\partial N^l_i}{\partial y^j}g_{lk}\right)g^{kh}\frac{\partial}{\partial y^h}, \\ {}^G\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial N^l_j}{\partial y^k}g_{li} - \frac{\partial N^l_i}{\partial y^k}g_{lj}\right)g^{kh}\frac{\delta}{\delta x^h} + \\ +\frac{1}{2}\left(-\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial N^l_i}{\partial y^k}g_{lj} + \frac{\partial N^l_j}{\partial y^k}g_{li} + R^l_{ji}g_{lk} + R^l_{ik}g_{lj} - R^l_{kj}g_{li}\right)g^{kh}\frac{\partial}{\partial y^h}. \end{cases} \tag{7}$$

Remark that T_pM is a submanifold of TM and $T_v(T_pM) = span\{\frac{\partial}{\partial y^1}(v), ..., \frac{\partial}{\partial y^n}(v)\}$, where $v \in T_pM$. Since $H_v(TM) \oplus V_v(TM) = T_v(TM)$, we can project the linear connection ${}^G\nabla$ on T_pM. The induced connection is a flat connection and its geodesics are the straight lines of T_pM. They are geodesics of TM too, so T_pM is a totally geodesic submanifold of $(TM, {}^G\nabla)$.

In the following we shall consider the case $N^i_j(x,y) = \Gamma^i_{jh}y^h - T^i_j(x)$, where $(T^i_j(x))$ are the components of a tensor field of type $(1,1)$ on M.

By computing the curvature tensor field ${}^GR$ of ${}^G\nabla$ we obtain

Proposition 2 *The curvature tensor field ${}^GR$ is given locally by*

$$(8)\quad \begin{cases} {}^GR(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} ={}^G R(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\delta}{\delta x^k} ={}^G R(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k}=0, \\ {}^GR(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} ={}^G R(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k} = R^h_{kij}\frac{\partial}{\partial y^h}, \\ {}^GR(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = R^h_{kij}\frac{\delta}{\delta x^h}+ \\ +\{(\nabla_l R^h_{kij})y^l+\nabla_i Q^h_{jk}-\nabla_j Q^h_{ik}+R^h_{klj}T^l_i - R^h_{kli}T^l_j\}\frac{\partial}{\partial y^h}, \end{cases}$$

where R^h_{klj} are the components of the curvature tensor field R of ∇,

$$Q^h_{ij}(x)=\frac{1}{2}(\nabla_i T_{kj}-\nabla_j T_{ki}+\nabla_k T_{ji}-\nabla_i T_{jk}-\nabla_j T_{ik}+\nabla_k T_{ij})g^{kh},$$

and $T_{ij}=g_{ih}T^h_j$.

Remark that, if $\nabla_i T^k_j = 0$ or if $T_{ij}=-T_{ji}$, then (TM,G) is flat if and only if (M,g) is a flat manifold.

Denote by $K(v;\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})$, $i\neq j$, the sectional curvature of the two-dimensional subspace generated by $\frac{\delta}{\delta x^i}(v)$ and $\frac{\delta}{\delta x^j}(v)$. The restriction of G to $span\{\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\}$ is positive defined.

Theorem 1 *Assume that (M,g) have the constant sectional curvature c.*
(a) If $T_{ij}=-T_{ji}$ then $K(v;\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})=c$.
(b) If $T_{ij}=\frac{c_1}{2}g_{ij}$ then $K(v;\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})=c+c_1$.

Proposition 3 *The Ricci tensor field Ric of ${}^GR$ is given locally by*

$$Ric(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k})=Ric(\frac{\partial}{\partial y^j},\frac{\delta}{\delta x^k})=0, Ric(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})=2R_{kj},$$

where R_{ij} are the components of the Ricci tensor field of (M,g). So, (TM,G) is Ricci flat if and only if (M,g) is Ricci flat.

Let G_0 be the metric of type (5) which is obtained when $T^j_i=0$. By studying the harmonicity of G_0 with respect to G we obtain

Proposition 4 *The metric G is biharmonic with respect to G_0, i.e. the identity maps $1:(TM,G)\longrightarrow(TM,G_0)$, $1:(TM,G_0)\longrightarrow(TM,G)$ are harmonic.*

We remark that $G=G_0$ if and only if $T_{ij}=-T_{ji}$. In the case where $T_{ij}=-T_{ji}$ or $\nabla_i T^k_j=0$ then G and G_0 have the same geodesics. If $T_{ij}=T_{ji}$ we have generally ${}^{G_0}\nabla \neq {}^G \nabla$.

2 The harmonicity of the canonical projection $\pi: TM \longrightarrow M$

Let (M,g) and $(\widetilde{M},\widetilde{g})$ be two Riemannian manifolds of dimensions n and m, respectively, and let $f:M\longrightarrow\widetilde{M}$ be a smooth map. Denote by ${}^{\widetilde{M}}\Gamma^\alpha_{\beta\gamma}$ the Christoffel symbols of the metric $\widetilde{g}$ and by ${}^{\widetilde{M}}\nabla$ the Levi-Civita connection of $\widetilde{g}$. Let (U,x^i), $i=1,...,n$, be a local chart on M in $p\in M$ and let (V,u^α), $\alpha=1,...,m$, be also a local chart on $\widetilde{M}$ in $f(p)$. The second fundamental form of f in p, denoted by $\beta(f)_p$, is given in local coordinates by

$$\beta(f)_p(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) = (f^\alpha_{ij} - \Gamma^k_{ij} f^\alpha_k + {}^{\widetilde{M}}\,\Gamma^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j)\frac{\partial}{\partial u^\alpha}, \tag{9}$$

where $f^\alpha_k = \frac{\partial f^\alpha}{\partial x^k}$ and $f^\alpha_{ij} = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}$. The form $\beta(f)$ is $C^\infty(M)$ bilinear and symmetric. The map f is a totally geodesic map if an only if $\beta(f) = 0$.

The tension field $\tau(f)$ of f is defined by

$$\tau(f) = tr\ \beta(f) = g^{ij}\beta(f)(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}). \tag{10}$$

The map f is a harmonic map if $\tau(f) = 0$ (see [2]). If $X \in \chi(M)$ is f-related with $\widetilde{X} \in \chi(\widetilde{M})$, i.e. $f_{*,p}X = \widetilde{X}(f(p))$, $\forall p \in M$, and $Y \in \chi(M)$ is f-related with $\widetilde{Y} \in \chi(\widetilde{M})$, then we have

$$\beta(f)_p(X,Y) = ({}^{\widetilde{M}}\nabla_{\widetilde{X}}\widetilde{Y})_{f(p)} - f_{*,p}(\nabla_X Y). \tag{11}$$

Recall that if M is a compact and orientable manifold then $f : (M,g) \longrightarrow (\widetilde{M},\widetilde{g})$ is harmonic if and only if f is a critical map for the energy E, where $E(f) = \int_M e(f)dv_g$ and $e(f) = \frac{1}{2}g^{ij} f^\alpha_i f^\beta_j \widetilde{g}_{\alpha\beta}$.

From the relations (9) and (10) one can see that the notion of harmonicity can be extended to the case where M and $\widetilde{M}$ are not Riemannian manifolds. Consider (M,g) or $(\widetilde{M},\widetilde{g})$ as pseudo-Riemannian spaces or consider on $\widetilde{M}$ just a torsion free connection. If M is a compact and orientable manifold and $(\widetilde{M},\widetilde{g})$ is a pseudo-Riemannian manifold then f still remains a harmonic map if and only if f is a critical map for the energy E.

We shall study the extensions considered above in the case of the map $\pi : (TM,G) \longrightarrow (M,g)$. From (7) and (11) we obtain

$$\beta(\pi)(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}) = \beta(\pi)(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}) = 0,$$

$$\beta(\pi)(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}) = -\frac{1}{2}(\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial N^l_j}{\partial y^k} g_{li} - \frac{\partial N^l_i}{\partial y^k} g_{lj})g^{kh}\frac{\partial}{\partial x^h}.$$

Consequently we have the following

Theorem 2 *The following statements are equivalent:*
(a) The map $\pi : TM \longrightarrow M$ *is a totally geodesic map,*
(b) $\beta(\pi)(X^H, Y^H) = 0, \quad \forall X, Y \in \chi(M)$,
(c) $\frac{\partial N^h_i}{\partial y^j}(x,y) = \Gamma^h_{ij}(x) + \{(P^l_{jk} g_{li} + P^l_{ik} g_{lj})g^{kh} + P^h_{ij}\}$, *where* $P^l_{jk}(x,y)$ *are the components of a* M*-tensor field of type* $(1,2)$ *on* TM*, having the property:* $P^l_{jk} = -P^l_{kj}$.

Remarks:

1) Under the condition of complete integrability of the equation above, the M-tensor $(P^h_{ij}(x,y))$, with the property $P^h_{ij} = -P^h_{ji}$, must satisfy the following system of partial differential equation

$$\frac{\partial P^l_{jh}}{\partial y^k} g_{li} + \frac{\partial P^l_{ih}}{\partial y^k} g_{lj} + \frac{\partial P^l_{ij}}{\partial y^k} g_{lh} = \frac{\partial P^l_{kh}}{\partial y^j} g_{li} + \frac{\partial P^l_{ih}}{\partial y^j} g_{lk} + \frac{\partial P^l_{ik}}{\partial y^j} g_{lh}.$$

Remark that if (P^h_{ij}) is a tensor on M then this system is fulfilled automatically.

2) If the coeficients of the nonlinear connection are $N^i_j(x,y) = \Gamma^i_{jl}y^l + T^i_j(x) + Q^i_j(x,y)$, where $Q^i_j(x,y)$ are the components of a M-tensor of type $(1,1)$ on TM, having the property that the M-tensor of type $(0,3)$ on TM defined by $Q_{ijk}(x,y) = g_{il}\frac{\partial Q^l_j}{\partial y^k}$ is antisymmetric in the first two indices, then $\pi : (TM,G) \longrightarrow (M,g)$ is a totally geodesic map.

Obviously, if $Q^i_j(x,y) = 0$, i.e. $N^i_j(x,y) = \Gamma^i_{jl}y^l + T^i_j(x)$, then $\pi : (TM,G) \longrightarrow (M,g)$ is totally geodesic.

3) If $N^h_i(x,y) = \{\Gamma^h_{ij} + [(P^l_{jk}(x)g_{li} + P^l_{ik}(x)g_{lj})g^{kh} + P^h_{ij}(x)]\}y^j$, where $P^l_{jk}(x)$ is a tensor field of type $(1,2)$ on M with the property $P^l_{jk} = -P^l_{kj}$, then $\pi : (TM,G) \longrightarrow (M,g)$ is a totally geodesic map.

In the following we shall give some results concerning the property of π to be totally geodesic. The results are obtained giving various expressions for the nonlinear connection. We consider the following cases

A) $N^i_j(x,y) = -\frac{1}{2}\frac{\partial S^i}{\partial y^j}$, where $S(x,y) = y^i\frac{\partial}{\partial x^i} + S^i(x,y)\frac{\partial}{\partial y^i}$ is a semispray on TM. Then,

Proposition 5 *The map π is totally geodesic if and only if $S^h(x,y) = -\Gamma^h_{ij}y^iy^j + T^h_i(x)y^i + \xi^h(x)$, where $\xi \in \chi(M)$.*

Remark that, for $T^h_i = 0$ and $\xi^h = 0$, we obtain $S^h(x,y) = -\Gamma^h_{ij}y^iy^j$, i.e. S is the geodesic spray.

B) $N^i_j(x,y) = \{\Gamma^i_{jh} + P^i_{jh}(x)\}y^h$, where $(P^i_{jk}(x))$ is a tensor field on M; $\Gamma^i_{jh} + P^i_{jh}$ are the coefficients of the linear connection $\nabla + P$, where $P = (P^i_{jh}(x))$. In this case we have

Proposition 6 *The map π is totally geodesic if and only if $P_{ijk} = -P_{jik}$, where $P_{ijk} = g_{il}P^l_{jk}$.*

If $P^i_{jk} = -P^i_{kj}$, i.e. $\nabla + P$ and ∇ have the same geodesics, then π is totally geodesic if and only if $(P_{ijk}(x))$ is an exterior differential form of degree 3.

Proposition 7 *Let $P^i_{jk} = Q^i_{kj}$, where $(\nabla + Q)g = 0$, i.e. $\nabla + Q$ is a metric connection. Then π is a totally geodesic map. So if Q is a homogeneous structure then π is totally geodesic.*

Proposition 8 *a) Let (M,g,J) be an almost Hermitian manifold. If $P^i_{jk} = Q^i_{kj}$, where $Q(X,Y) = -\frac{1}{2}J(\nabla_X J)Y$, then π is totally geodesic.*

b) Let (M,g,J) be a nearly-Kähler manifold. If $P^i_{jk} = Q^i_{jk}$, where $Q(X,Y) = -\frac{1}{2}J(\nabla_X J)Y$, then π is a totally geodesic map.

We note that $(\nabla + Q)J = 0$, i.e. $\nabla + Q$ is an almost complex connection.

Also we remark that if $P^i_{jk} = A_k\nabla_j\xi^i$, $A_k \neq 0$, where $(A_k(x))$ is a tensor field on M of type $(0,1)$, then π is a totally geodesic map if and only if ξ is a Killing vector field.

By studying the harmonicity of the canonical projection we obtain

Theorem 3 *The map $\pi : (TM,G) \longrightarrow (M,g)$ is harmonic.*

3 The harmonicity of vector fields

In the following, we shall consider the case of $N^i_j(x,y) = \Gamma^i_{jh}y^h - T^i_j(x)$.

Let ξ be a vector field which is considered to be a map from (M,g) to (TM,G). Investigating the situations in which ξ is an isometric immersion, a totally geodesic or harmonic map, as well as the relation between them, we will obtain interesting results.

Taking into account the relation

$$\xi_*(\frac{\partial}{\partial x^i}) = \frac{\delta}{\delta x^i} + (\nabla_i \xi^h - T^h_i)\frac{\partial}{\partial y^h} \tag{12}$$

we obtain

Theorem 4 *The map $\xi : (M,g) \longrightarrow (TM,G)$ is an isometric immersion if and only if*

$$\nabla_i \xi_j + \nabla_j \xi_i = c_{ij}, \tag{13}$$

where $c_{ij} = T_{ij} + T_{ji}$, $T_{ij} = g_{ih}T^h_j$ and $\xi_i = g_{ih}\xi^h$.

Remarks:

1) If $T_{ij} = -T_{ji}$ then ξ is an isometric immersion if and only if ξ is a Killing vector field.

2) If $T_{ij} = T_{ji}$ then ξ is an isometric immersion if and only if $T_{ij} = \frac{1}{2}(\nabla_i\xi_j + \nabla_j\xi_i)$.

3) If $T_{ij} = \nabla_i\xi_j$ then ξ is an isometric immersion.

Thus we obtain

Proposition 9 *For any $\xi \in \chi(M)$ there is G a pseudo-Riemannian metric of type (5) on TM such that $\xi : (M,g) \longrightarrow (TM,G)$ is an isometric immersion.*

From (13) we obtain by a straightforward computation the following relation

$$\nabla_i\nabla_j\xi_h = R^h_{ijk}\xi_h + \frac{1}{2}(\nabla_j c_{ki} + \nabla_i c_{kj} - \nabla_k c_{ij}) \tag{14}$$

satisfied by ξ in the case of an isometric immersion $\xi : M \longrightarrow TM$.

Considering the condition of complete integrability of the partial differential system defined by (13) with its consequence (14), it follows that, under the condition of complete integrability, the manifold (M,g) must have constant sectional curvature and the tensor field c must satisfy the following system of partial differential equations

$$\nabla_i(\nabla_h c_{jk} - \nabla_k c_{jh}) + \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) = c_{il}R^l_{jkh} - c_{jl}R^l_{ikh}.$$

Remark that the tensor field $c_{ij} = 0$ satisfies this system and if $c = g$ the system is satisfied if and only if $R = 0$; for $c_{ij} = 0$ we obtain the condition of complete integrability for Killing vector fields.

By computing the second fundamental form $\beta(\xi)$ of ξ we obtain

$$\beta(\xi)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \{\nabla_i\nabla_j\xi^h - \nabla_i T^h_j + (P^h_{ij}\circ\xi)\}\frac{\partial}{\partial y^h}, \tag{15}$$

where

$$P^h_{ij}\circ\xi = \frac{1}{2}(2R_{iljk}\xi^l + \nabla_i T_{kj} - \nabla_j T_{ki} + \nabla_k T_{ji} - \nabla_i T_{jk} - \nabla_j T_{ik} + \nabla_k T_{ij})g^{kh}.$$

Proposition 10 *a) If $\nabla_k T_{ij} = 0$ or $T_{ij} = -T_{ji}$ then ξ is totally geodesic if and only if*

$$R_{iljk}\xi^l + \nabla_i \nabla_j \xi_k = 0,$$

where R_{iljk} are the components of the Riemann-Christoffel tensor. So $\xi : (M, g) \longrightarrow (TM, G)$ is a totally geodesic map if and only if $L_\xi \nabla = 0$, i.e. φ_t is a totally geodesic map $\forall t$, where $\{\varphi_t\}$ is the flow of ξ.

b) If $T_{ij} = T_{ji}$ then $\xi : (M, g) \longrightarrow (TM, G)$ is a totally geodesic map if and only if

$$\nabla_k T_{ij} = \frac{1}{2}(\nabla_k \nabla_i \xi_j + \nabla_k \nabla_j \xi_i).$$

Remarks:

1) If $T_{ij} = T_{ji}$, $\nabla_k T_{ij} \neq 0$ and $\xi : (M, g) \longrightarrow (TM, G)$ is totally geodesic then $tr\ T = div\xi + c$, where $tr\ T = T_{ij}g^{ij}$. So, if $div\xi = 0$ and $tr\ T$ is not constant, then ξ cannot be totally geodesic.

2) If $T_{ij} = T_{ji}$, $\nabla_k T_{ij} \neq 0$ and ξ is a Killing vector field then ξ is not a totally geodesic map.

3) If ξ is a Killing vector field then $L_\xi \nabla = 0$; if M is a compact and orientable manifold then ξ is Killing if and only if $L_\xi \nabla = 0$ (see [18]).

By studying the relation between ξ as isometric immersion map and ξ as totally geodesic map we obtain

Theorem 5 *If $\xi : (M, g) \longrightarrow (TM, G)$ is an isometric immersion then $\xi : (M, g) \longrightarrow (TM, G)$ is totally geodesic.*

Remark that for any vector field $\xi \in \chi(M)$ there is a pseudo-Riemannian metric G of type (5) such that $\xi : (M, g) \longrightarrow (TM, G)$ is a totally geodesic map.

Now computing the tension field $\tau(\xi)$ of ξ we obtain

Proposition 11 *If $\nabla_k T_{ij} = 0$ or $T_{ij} = -T_{ji}$ then $\xi : (M, g) \longrightarrow (TM, G)$ is a harmonic map if and only if ξ is a geodesic vector field, i.e. $g^{ij}(\nabla_i \nabla_j \xi^k) + R^k_l \xi^l = 0$, where $R^k_l = R_{li}g^{ik}$.*

We remark that if φ_t is harmonic $\forall t$, then ξ is a geodesic vector field.

From Theorem 5 it follows that, if $\xi : (M, g) \longrightarrow (TM, G)$ is an isometric immersion then ξ is a harmonic map. So, for any $\xi \in \chi(M)$ there is G of type (5) such that ξ is a harmonic map.

We consider now the case $T_{ij} = T_{ji}$. The tension field $\tau(\xi)$ is given by

$$\tau(\xi) = \{R^k_l \xi^l + g^{ij}(\nabla_i \nabla_j \xi^k) + [\nabla_h (tr\ T) - 2(\nabla_i T^i_h)]g^{hk}\}\frac{\partial}{\partial y^k}. \tag{16}$$

It is known that the symmetric tensor field (T_{ij}) is harmonic with respect to g if and only if

$$\nabla_i T^i_h - \frac{1}{2}\nabla_h (tr\ T) = 0$$

(see [1]). So we obtain

Proposition 12 *If (T_{ij}) is harmonic with respect to g then ξ is a harmonic map if and only if ξ is a geodesic vector field.*

We remark that if $T_{ij} = R_{ij}$ then (T_{ij}) is harmonic with respect to g.

By studying the harmonicity of the tensor field $\xi^\star G$ with respect to g we obtain

Theorem 6 *The symmetric tensor field $\xi^\star G$ is harmonic with respect to g if and only if $\xi : (M, g) \longrightarrow (TM, G)$ is a harmonic map.*

Finally, we shall study the energy $E(\xi)$ and the stress-energy tensor S_ξ of ξ. S_ξ is defined by $S_\xi = e(\xi)g - \xi^\star G$ (see [2]).

We have

$$e(\xi) = \frac{n}{2} + div\xi - tr\ T, E(\xi) = \frac{n}{2}\int_M 1dv_g - \int_M tr\ Tdv_g,$$

$$S_\xi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \frac{n-2}{2}g_{ij} + (div\xi)g_{ij} - (tr\ T)g_{ij} - (\nabla_i\xi_j + \nabla_j\xi_i) + T_{ij} + T_{ji},$$

$$divS_\xi(\frac{\partial}{\partial x^i}) = -\{g^{jk}\nabla_j\nabla_k\xi_i + R_{li}\xi^l - g^{jk}(\nabla_j T_{ki} + \nabla_j T_{ik}) + \nabla_i(tr\ T)\}.$$

We remark that if $T_{ij} = g_{ij}$ we have $E(\xi) < 0$ and if $T_{ij} = \frac{1}{2}g_{ij}$ we have $E(\xi) = 0$. We note that in the Riemannian case $E(\xi) > 0$. Also, we note that the energy $E(\xi)$ of ξ is independent of ξ. So, if $\xi \in \chi(M)$ is a harmonic map, then ξ is a critical point to E and $E(\xi) = E(\xi_t)$ for any variations of type $\xi_t(p) = \xi(p) + t\eta(p)$, where $\eta \in \chi(M)$.

Now, we shall study the relation between the tensor S_ξ and the harmonicity of ξ.

If $\nabla_k T_{ij} = 0$, or $T_{ij} = -T_{ji}$, or $T_{ij} = T_{ji}$ we have

$$((divS_\xi)^\sharp)^V \circ \xi = -\tau(\xi).$$

Consequently we obtain

Proposition 13 *The map* $\xi : (M, g) \longrightarrow (TM, G)$ *is harmonic if and only if* $divS_\xi = 0$.

We note that, in general, we have only: if ξ is harmonic then $divS_\xi = 0$.

4 The pseudo-Riemannian metric g^c

Let us define on TM the complete lift g^c of the Riemannian metric g by

$$g^c(X^V, Y^V) = 0, \quad g^c(X^H, Y^V) = g(X, Y), \quad g^c(X^H, Y^H) = 0. \tag{17}$$

The metric g^c is a pseudo-Riemannian metric with the signature (n, n). The vertical and horizontal distributions are maximally isotropic.

Proposition 14 *The Levi-Civita connection* ${}^{g^c}\nabla$ *of* g^c *is given locally by*

$$\begin{cases} {}^{g^c}\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = 0, \\ {}^{g^c}\nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial N_k^l}{\partial y^i}g_{lj} - \frac{\partial N_j^l}{\partial y^i}g_{lk}\right)g^{kh}\frac{\partial}{\partial y^h}, \\ {}^{g^c}\nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial N_k^l}{\partial y^j}g_{li} + \frac{\partial N_i^l}{\partial y^j}g_{lk}\right)g^{kh}\frac{\partial}{\partial y^h}, \\ {}^{g^c}\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial N_j^l}{\partial y^k}g_{li} - \frac{\partial N_i^l}{\partial y^k}g_{lj}\right)g^{kh}\frac{\delta}{\delta x^h} + \\ \qquad + \frac{1}{2}\left(R_{ji}^l g_{lk} + R_{ik}^l g_{lj} - R_{kj}^l g_{li}\right)g^{kh}\frac{\partial}{\partial y^h}. \end{cases} \tag{18}$$

Taking into account the relations (7), (11), (18) we obtain

Theorem 7 *a) The map $\pi : (TM, g^c) \longrightarrow (M, g)$ is a totally geodesic map if and only if the map $\pi : (TM, G) \longrightarrow (M, g)$ is a totally geodesic map.*

b) The map $\pi : (TM, g^c) \longrightarrow (M, g)$ is a harmonic map.

Proposition 15 *a) The identity map $1 : (TM, G) \longrightarrow (TM, g^c)$ is totally geodesic if and only if $\pi : (TM, G) \longrightarrow (M, g)$ is totally geodesic.*

b) g^c is biharmonic with respect to G.

From now on we shall consider only the case $N^i_j(x, y) = \Gamma^i_{jl} y^l - T^i_j(x)$. We have $\pi : (TM, g^c) \longrightarrow (M, g)$ is totally geodesic, ${}^G\nabla = {}^{g^c}\nabla$ and, consequently, we get

Theorem 8 *The map $\xi : (M, g) \longrightarrow (TM, g^c)$ is a harmonic map if and only if the map $\xi : (M, g) \longrightarrow (TM, G)$ is harmonic.*

We consider the Berwald connection on TM given by

$$(19) \qquad \begin{cases} B_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0, \quad B_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = 0, \\ \\ B_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^h_{ij} \frac{\delta}{\delta x^h}, \quad B_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma^h_{ij} \frac{\partial}{\partial y^h}. \end{cases}$$

B is a linear connection with torsion, and we shall consider its mean connection $\overset{m}{B}$.

The tension vector field $\tau(\xi)$ of the map $\xi : (M, g) \longrightarrow (TM, \overset{m}{B})$ is given by

$$(20) \qquad \tau(\xi) = g^{ij}\{\nabla_i \nabla_j \xi^h - \nabla_i T^h_j\} \frac{\partial}{\partial y^h}.$$

We know that a tensor field J of type $(1,1)$ on M is a harmonic endomorphism field if and only if

$$g^{ij}(\nabla_i J^h_j) = 0$$

(see [4]). Consequently we obtain

Proposition 16 *The following statements are equivalent:*

a) The tensor field J is a harmonic endomorphism field,

b) $J : (TM, G) \longrightarrow (TM, G)$ is a harmonic map,

c) $J : (TM, g^c) \longrightarrow (TM, g^c)$ is a harmonic map.

If (M, J, g) is a nearly-Kähler manifold, then J is a harmonic endomorphism field. As example of nearly-Kähler manifold, we can consider $M = S^6$, the tensor J defined by $J_x y = x \times y$, where $\times$ is the vectorial product of $\mathbb{R}^7$, and g is the usually Euclidean metric of $\mathbb{R}^7$.

From the relation (20) we obtain

Theorem 9 *a) If $T^j_i = \nabla_i \xi^j$ then $\xi : (M, g) \longrightarrow (TM, \overset{m}{B})$ is a harmonic map.*

b) If T is a harmonic endomorphism field then $\xi : (M, g) \longrightarrow (TM, \overset{m}{B})$ is a harmonic map if and only if $\nabla\xi$ is a harmonic endomorphism field.

Remarks:

1) If M is compact and orientable and T is a harmonic endomorphism field, then $\nabla\xi$ is harmonic if and only if $\nabla\xi = 0$; if (M, g) has the constant sectional curvature $c \neq 0$ then there is no $\xi \neq 0$ such that $\nabla\xi = 0$.

2) If (M, g) is Ricci-flat and T is a harmonic endomorphism field, then $\nabla\xi$ is a harmonic endomorphism field if and only if ξ is geodesic.

3) For any vector field ξ there is a Berwald connection of type (19) such that $\xi : (M, g) \longrightarrow (TM, \overset{m}{B})$ is a harmonic map.

If we consider on T_pM the induced connection of ${}^G\nabla$ or $\overset{m}{B}$ then $J_p : T_pM \longrightarrow T_pM$ is a totally geodesic map, because it carries the straight lines, which are the geodesics of T_pM, into straight lines.

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