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PSEUDO SYMMETRIC AND PSEUDO RICCI SYMMETRIC WARPED PRODUCT MANIFOLDS

Uday Chand De, Cengizhan Murathan, and Cihan Özgür

Abstract. We study pseudo symmetric (briefly $(PS)_n$) and pseudo Ricci symmetric (briefly $(PRS)_n$) warped product manifolds $M \times_F N$. If M is $(PS)_n$, then we give a condition on the warping function that M is a pseudosymmetric space and N is a space of constant curvature. If M is $(PRS)_n$, then we show that (i) N is Ricci symmetric and (ii) M is $(PRS)_n$ if and only if the tensor T defined by (2.6) satisfies a certain condition.

1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by E. Cartan [2], who, in particular, obtained a classification of these spaces.

Let (M, g), $n = \dim M$ be a Riemannian manifold, i.e., a manifold M with the metric tensor g with arbitrary signature and let ∇ be the Levi-Civita connection of (M, g). A Riemannian manifold is called *locally symmetric* [2] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). This condition of local symmetry is equivalent to the fact that at every point $p \in M$, the local geodesic symmetry $F(p)$ is an isometry [11]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

During the last five decades, the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as conformally symmetric manifolds by M. C. Chaki and B. Gupta [5], recurrent manifolds introduced by A. G. Walker [15], conformally recurrent manifolds by T. Adati and T. Miyazawa [1], pseudo symmetric manifolds introduced by M. C. Chaki [3], weakly symmetric manifolds by L. Tamássy and T. Q. Binh [14], projective symmetric manifolds by G. Soos [13], pseudo conformally symmetric spaces by U. C. De and H. A. Biswas [7], almost pseudo conformally symmetric

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spaces by U. C. De and A. K. Gazi [8], weakly conformally symmetric spaces by U. C. De and S. Bandyopadhyay [6], pseudosymmetric manifolds by R. Deszcz [9], etc.

In 1967, R. N. Sen and M. C. Chaki [12] studied certain curvature restrictions on a certain kind of conformally flat space of class one and they obtained the following expression of the covariant derivative of the curvature tensor:

$$R^h_{ijk,l} = 2\lambda_l R^h_{ijk} + \lambda_i R^h_{ljk} + \lambda_j R^h_{ilk} + \lambda_k R^h_{ijl} + \lambda^h R_{lijk}, \tag{1.1}$$

where R^h_{ijk} are the components of the curvature tensor R, $R_{lijk} = g_{hl} R^h_{ijk}$, λ_i is a non-zero covariant vector and ',' denotes covariant differentiation with respect to the metric tensor g_{ij}.

Later in 1987, M. C. Chaki [3] called a manifold whose curvature tensor satisfies (1.1), as a *pseudo symmetric manifold*. If all $\lambda_i = 0$, then the manifold reduces to a symmetric manifold in the sense of E. Cartan. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$.

This is to be noted that the notion of pseudo symmetric manifold studied in particular by R. Deszcz [9] is different from that of M. C. Chaki [3].

In 1988, M. C. Chaki introduced the notion of a pseudo Ricci symmetric manifold. A non-flat Riemannian manifold (M^n, g), $n \geq 2$, is called *pseudo Ricci symmetric* [4] if its Ricci tensor S satisfies the condition

$$S_{ij,k} = 2A_k S_{ij} + A_i S_{kj} + A_j S_{ik}, \tag{1.2}$$

where A is a non zero 1-form. Such an n-dimensional manifold is denoted by $(PRS)_n$. If $S_{ij,k} = 0$, then the manifold is called *Ricci-symmetric*.

Let $(M, \overline{g})$ and $(N, \overset{*}{g})$ (dim$M = q$, dim$N = n-q$, $1 \leq q < n$) be Riemannian manifolds covered by systems of charts $\{U, x^\alpha\}$ and $\{V, y^\alpha\}$, respectively. Let F be a positive C^∞ function on M. The warped product $M \times_F N$ of $(M, \overline{g})$ and $(N, \overset{*}{g})$ is the manifold $M \times N$ with the metric $g = \overline{g} \times_F \overset{*}{g}$. More precisely

$$\overline{g} \times_F \overset{*}{g} = \pi_1^* \overline{g} + (F \circ \pi_1)^2 \, \pi_2^* \overset{*}{g},$$

where π_i $(1 \leq i \leq 2)$ are natural projections from $M \times N \to M$ and $M \times N \to N$, respectively. The manifold $(M, \overline{g})$ is called the base manifold [11].

In the present paper, we study pseudosymmetric and pseudo Ricci symmetric warped product manifolds. The paper is organized as follows: In Section 2, we give some properties of warped product manifolds. In Section 3, we consider pseudosymmetric warped product manifolds. In Section 4, we study pseudo Ricci-symmetric warped product manifolds.

2. Warped product manifolds

Let $\left\{U \times V; x^1, \ldots, x^q, x^{q+1} = y^1, \ldots, x^n = y^{n-q}\right\}$ be a product chart for $M \times N$. The local components of the metric $g = \overline{g} \times_F \overset{*}{g}$ with respect to this chart are $g_{ij} = \overline{g}_{ab}$ if $i = a$ and $j = b$, $g_{ij} = F \overset{*}{g}_{\alpha\beta}$ if $i = \alpha$ and $j = \beta$ and $g_{ij} = 0$ otherwise, where $a, b, c, \ldots \in \{1, \ldots, q\}$, $\alpha, \beta, \gamma, \ldots \in \{q+1, \ldots, n\}$ and

$h, i, j, \ldots \in \{1, 2, \ldots, n\}$. We will mark by bars (resp., by stars) tensors formed from $\overline{g}$ (resp., $\overset{*}{g}$). The point and semicolon denote covariant differentiation in M and N, respectively. The local components Γ^h_{ij} of the Levi-Civita connection on $M \times_F N$ are the following [10]:

$$\Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, \quad \Gamma^a_{\alpha\beta} = -\tfrac{1}{2}\overline{g}^{ab} F_b \overset{*}{g}_{\alpha\beta}, \quad \Gamma^\alpha_{\beta\gamma} = \overset{*}{\Gamma}{}^\alpha_{\beta\gamma},$$
$$\Gamma^\alpha_{a\beta} = \tfrac{1}{2F} F_a \delta^\alpha_\beta, \quad \Gamma^a_{\alpha b} = \Gamma^\alpha_{ab} = 0, \quad F_a = \partial_a F = \tfrac{\partial F}{\partial x^a}.$$

The local components of the curvature tensor

$$R^l_{ijk} = \partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^r_{ij}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rj},$$

$\partial_j = \frac{\partial}{\partial x^j}$ and the local components $R_{rstu} = g_{rw} R^w_{stu}$ of the Riemannian-Christoffel curvature tensor and the local components S_{ts} of the Ricci tensor S of the warped product $M \times_F N$, which may not vanish are the following [10]:

$$R_{abcd} = \overline{R}_{abcd}, \tag{2.1}$$

$$R_{\alpha\beta\gamma\delta} = F \overset{*}{R}_{\alpha\beta\gamma\delta} - \frac{1}{4}(\Delta_1 F)\, \overset{*}{G}_{\alpha\beta\gamma\delta}, \tag{2.2}$$

and

$$\overset{*}{G}_{\alpha\beta\gamma\delta} = \overset{*}{g}_{\alpha\gamma}\overset{*}{g}_{\beta\delta} - \overset{*}{g}_{\alpha\delta}\overset{*}{g}_{\beta\gamma},$$

where $\Delta_1 F = \overline{g}^{ab} F_a F_b$. Differentiating the local components of the curvature tensor covariantly we get

$$R_{abcd,e} = \overline{R}_{abcd\cdot e}, \tag{2.3}$$

and

$$R_{\alpha\beta\gamma\delta,a} = -F_a \overset{*}{R}_{\alpha\beta\gamma\delta} + \frac{1}{2}\left[\frac{F_a}{F}\Delta_1 F - \frac{1}{2}\partial_a(\Delta_1 F)\right] \overset{*}{G}_{\alpha\beta\gamma\delta}. \tag{2.4}$$

For the local components S_{ij} of the Ricci tensor of $M \times_F N$ we have

$$S_{ab} = \overline{S}_{ab} - \frac{n-q}{2F} T_{ab}, \tag{2.5}$$

where T is a $(0,2)$-tensor with local components T_{ij} defined by

$$T_{\alpha\beta} = T_{\alpha a} = 0,$$

$$T_{ab} = F_{a.b} - \left(\frac{1}{2F}\right) F_a F_b. \tag{2.6}$$

Moreover, by covariant differentiation from last equations we have

$$S_{ab,c} = \overline{S}_{ab.c} - (n-q)\left(\frac{1}{2F} T_{ab}\right)_{.c}, \tag{2.7}$$

$$S_{ab,\alpha} = S_{a\alpha,b} = 0 \tag{2.8}$$

and

$$S_{\alpha\beta,\delta} = \overset{*}{S}_{\alpha\beta;\delta} \tag{2.9}$$

(for more details see [10]).

3. Pseudo symmetric warped product manifolds

In the present section, we consider pseudo symmetric warped product manifolds.

Now assume that $M \times_F N$ is a $(PS)_n$. From (2.4) we can write

$$R_{\alpha\beta\gamma\delta,e} = -F_e \overset{*}{R}_{\alpha\beta\gamma\delta} - \frac{1}{4}\left(\partial_e\left(\Delta_1 F\right) - \frac{2}{F}F_e \Delta_1 F\right)\overset{*}{G}_{\alpha\beta\gamma\delta}. \tag{3.1}$$

By (1.1), we have

$$R_{abcd,e} = 2A_e R_{abcd} + A_a R_{ebcd} + A_b R_{aecd} + A_c R_{abed} + A_d R_{abce}.$$

In virtue of (2.1) and (2.3) the above equation turns into

$$\overline{R}_{abcd\cdot e} = 2A_e \overline{R}_{abcd} + A_a \overline{R}_{ebcd} + A_b \overline{R}_{aecd} + A_c \overline{R}_{abed} + A_d \overline{R}_{abce}.$$

Hence we conclude that M is a $(PS)_n$. We know

$$R_{a\beta\gamma\delta} = 0. \tag{3.2}$$

So from (3.2) and (1.1) we have

$$R_{\alpha\beta\gamma\delta,e} = 2A_e R_{\alpha\beta\gamma\delta}.$$

In view of (3.1) and (2.2) the last equation can be written as

$$\begin{aligned} &- F_e \overset{*}{R}_{\alpha\beta\gamma\delta} - \frac{1}{4}\left(\partial_e\left(\Delta_1 F\right) - \frac{2}{F}F_e\left(\Delta_1 F\right)\right)\overset{*}{G}_{\alpha\beta\gamma\delta} \\ &= 2A_e\left(F R^*_{\alpha\beta\gamma\delta} - \frac{1}{4}\left(\Delta_1 F\right)\overset{*}{G}_{\alpha\beta\gamma\delta}\right), \end{aligned}$$

which gives us

$$\overset{*}{R}_{\alpha\beta\gamma\delta}\left(-F_e - 2A_e F\right) = m_e \overset{*}{G}_{\alpha\beta\gamma\delta}, \tag{3.3}$$

where

$$m_e = \frac{1}{4}\left(\partial_e\left(\Delta_1 F\right) - \frac{2}{F}F_e\left(\Delta_1 F\right) - 2A_e\left(\Delta_1 F\right)\right).$$

Now from (3.3) by a contraction we have

$$\overset{*}{S}_{\beta\gamma}\left(-F_e - 2A_e F\right) = \left(1 - n + q\right) m_e \overset{*}{g}_{\beta\gamma}. \tag{3.4}$$

Contracting (3.4) we obtain

$$\left(-F_e - 2A_e F\right) = \frac{\left(1 - n + q\right)\left(n - q\right)}{\overset{*}{r}} m_e. \tag{3.5}$$

If $m_e \neq 0$, putting (3.5) into (3.3) we find

$$\overset{*}{R}_{\alpha\beta\gamma\delta} = \frac{\overset{*}{r}}{\left(1 - n + q\right)\left(n - q\right)}\overset{*}{G}_{\alpha\beta\gamma\delta},$$

which means that N is of constant curvature. So we have the following theorem:

Theorem 3.1. *If a warped product manifold $M \times_F N$ with non-constant function F such that $\partial_e(\Delta_1 F) - \frac{2}{F} F_e(\Delta_1 F) - 2A_e(\Delta_1 F) \neq 0$ is $(PS)_n$, then M is $(PS)_n$ and N is a space of constant curvature.*

When the manifold N is a 3-dimensional space of constant curvature and M is 1-dimensional then the warped product is said to be a *Robertson-Walker space-time* [11].

Hence we can state the following corollary:

Corollary 3.2. *Under the conditions of Theorem 3.1, if* $\dim M = 1$ *and* $\dim N = 3$*, then the spacetime $M \times_F N$ is a Robertson-Walker space-time.*

4. Pseudo Ricci symmetric warped product manifolds

In this section, we consider pseudo Ricci symmetric warped product manifolds.

Now assume that $M \times_F N$ is a $(PRS)_n$. Since $S_{a\alpha} = 0$ we have $S_{a\alpha,b} = 0$. Hence we get from (1.2)

$$S_{a\alpha,b} = 2A_b S_{a\alpha} + A_a S_{b\alpha} + A_\alpha S_{ab}$$

which implies

$$A_\alpha S_{ab} = 0. \tag{4.1}$$

Since S_{ij} is nonzero in a $(PRS)_n$, hence

$$A_\alpha = 0. \tag{4.2}$$

Now from (1.2) we get

$$S_{\alpha\beta,\delta} = 2A_\delta S_{\alpha\beta} + A_\alpha S_{\delta\beta} + A_\beta S_{\delta\alpha}.$$

In virtue of (4.2) we obtain from above $S_{\alpha\beta,\delta} = 0$.

But from (2.9), we know that $S_{\alpha\beta,\delta} = \overset{*}{S}_{\alpha\beta;\delta}$. Hence it follows that $\overset{*}{S}_{\alpha\beta;\delta} = 0$, which means that N is Ricci symmetric.

Again from (1.2) we can write

$$S_{bc,e} = 2\lambda_e S_{bc} + \lambda_b S_{ec} + \lambda_c S_{be}.$$

By the use of (2.7), we obtain

$$\begin{aligned}
&\overline{S}_{bc.e} - \frac{n-q}{2F} T_{bc.e} + \frac{n-q}{2F^2} F_e T_{bc} \\
&= 2A_e\left(\overline{S}_{bc} - \frac{n-q}{2F} T_{bc}\right) + A_b\left(\overline{S}_{ec} - \frac{n-q}{2F} T_{ec}\right) + A_c\left(\overline{S}_{be} - \frac{n-q}{2F} T_{be}\right)
\end{aligned}$$

or,

$$\begin{aligned}
&\overline{S}_{bc.e} - 2A_e\overline{S}_{bc} - A_b\overline{S}_{ec} - A_c\overline{S}_{be} \\
&= \frac{n-q}{2F} T_{bc.e} - \left[\frac{n-q}{2F^2} F_e + 2A_e\frac{n-q}{2F}\right] T_{bc} - A_b\frac{n-q}{2F} T_{ec} - A_c\frac{n-q}{2F} T_{be}.
\end{aligned}$$

Thus M is pseudo Ricci symmetric if and only if

$$m_{bce} = 0,$$

where

$$m_{bce} = -\frac{n-q}{2\sigma}T_{bc.e} - \left[\frac{n-q}{2F^2}F_e + 2A_e\frac{n-q}{2F}\right]T_{bc} - A_b\frac{n-q}{2F}T_{ec} - A_c\frac{n-q}{2F}T_{be}.$$

Hence we can state the following theorem:

Theorem 4.1. *Let $M \times_F N$ be a pseudo Ricci symmetric warped product manifold. Then*

i) *N is Ricci symmetric and*

ii) *M is a pseudo Ricci symmetric manifold if and only if $m_{bce} = 0$.*

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Uday Chand De
Department of Pure Mathematics
University of Calcutta
35-Ballygunj Circular Road
Kolkata-700 019, West Bengal, India
E-mail address: uc_de@yahoo.com

Cengizhan Murathan
Department of Mathematics
Uludağ University
16059, Görükle, Bursa, Turkey
E-mail address: cengiz@uludag.edu.tr

Cihan Özgür
Department of Mathematics
Balikesir University
10145, Çağış, Balikesir, Turkey
E-mail address: cozgur@balikesir.edu.tr