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Pseudo-symmetric contact 3-manifolds II – When is the tangent sphere bundle over a surface pseudo-symmetric? –

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Abstract. The tangent sphere bundles over surfaces are pseudo-symmetric if and only if the base surfaces are of constant curvature. It is pointed out that semi-symmetry of the tangent sphere bundle of a surface of constant positive curvature depends on the radius.

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Dedicated to professor Oldřich Kowalski

Introduction

This is the second part of our series of study on pseudo-symmetric contact metric 3-manifolds.

There are two large classes of contact metric manifolds.

(1) The total spaces of the Boothby-Wang fibrations;

(2) unit tangent sphere bundles over Riemannian manifolds.

Sasakian space forms are typical examples of the former class. Here we point out that Sasakian space forms are homogeneous contact metric manifolds.

It was proved by Okumura [16] that locally symmetric Sasakian manifolds are of constant curvature 1. Tanno generalized Okumura's result. In fact he proved that every locally symmetric K-contact manifold is Sasakian [17].

Recently, Boeckx and the first author [6] showed that locally symmetric contact metric manifolds are either Sasakian manifolds of constant curvature 1 or locally isometric to the unit tangent sphere bundles over Euclidean space.

In the case of unit tangent sphere bundles, Blair [2] showed that unit tangent sphere bundles over surfaces equipped with the standard contact metric structure are locally symmetric if and only if the base surfaces are of constant curvature 1 or 0.

These results imply that for unit tangent sphere bundles, to be locally symmetric is a very strong condition.

Calvaruso and Perrone [8] generalized Blair's result. They showed that semi-symmetric unit tangent sphere bundles over surfaces are locally symmetric. Recently, Boeckx and Calvaruso [5] generalized this result to unit tangent sphere bundles over Riemannian manifolds of general dimension.

In our previous work [9], we have investigated more mild condition for contact metric 3-manifolds–*pseudo-symmetry*.

A Riemannian manifold (M, g) is said to be *pseudo-symmetric* if there exists a function L such that $R(X, Y) \cdot R = L\{ (X \wedge Y) \cdot R \}$ for all vector fields X and Y on M. Here R is the Riemannian curvature and $(X \wedge Y)$ is the endomorphism field defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y,$$

respectively. The tensor field $R\cdot R$ denotes the derivative of R by R. In particular, a pseudo-symmetric space is called a *pseudo-symmetric space of constant type* if L is a constant.

Note that a Riemannian manifold is said to be *semi-symmetric* if it is a pseudo-symmetric space with $L = 0$.

A pseudo-symmetric space (M, g) is said to be *proper* if $L \neq 0$.

In [9] we have shown that every Sasakian 3-manifold is a pseudo-symmetric space of constant type. More generally, it is shown that 3-dimensional unimodular Lie groups equipped with left invariant contact metric structure are constant type pseudo-symmetric.

Motivated by those results due to Blair, Boeckx, Calvaruso, Perrone and the authors of the present paper, we study pseudo-symmetry of tangent sphere bundles (with arbitrary radii) over surfaces in this paper.

Our main result is:

1 Theorem. *Let M be a Riemannian 2-manifold. Then its tangent sphere bundle $T^{(r)}M$ of radius r equipped with the Sasaki lift metric is pseudo-symmetric if and only if M is of constant curvature.*

From the proof of Theorem, we obtain the following corollary:

2 Corollary. *Let M be a Riemannian 2-manifold. Then its tangent sphere* *bundle* $T^{(r)}M$ *equipped with the metric induced by the Sasaki lift metric is semi-symmetric if and only if* M *is flat or of constant positive curvature* $1/r^2$.

This corollary implies that although pseudo-symmetry of tangent sphere bundles does not depend on the radius, semi-symmetry does depend on the radius. In fact, the tangent sphere bundle of a surface of constant positive curvature c can be semi-symmetric only when the radius is $1/\sqrt{c}$.

Notational convention: Throughout this paper, we denote by $\Gamma(\mathcal{E})$, the space of all smooth sections of a vector bundle $\mathcal{E}$.

1 Tangent sphere bundles

1.1

Let M be an n-manifold with tangent bundle TM. Every element of TM is a tangent vector of M and can be represented as a pair $(x;u)$, where x is a point of M and u is a tangent vector of M at x. The tangent space of M at x is denoted by T_xM. Denote by π the natural projection of TM. Take a local coordinate system $(x^1, x^2, \ldots, x^n)$ then this coordinate system induces a local coordinate system on TM:

$$\left(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n;\, u^1, u^2, \ldots, u^n\right),\ \bar{x}^i := x^i \circ \pi,\ u^i := dx^i.$$

One can easily check that

$$\boldsymbol{U} = u^i \frac{\partial}{\partial u^i}$$

is globally defined on TM and regarded as the position vector field of TM. This $\boldsymbol{U}$ is called the *canonical vertical vector field* of TM. The natural projection $\pi : TM \to M$ induces a foliation $\mathcal{V} = \mathrm{Ker}(d\pi)$. This foliation and its associated distribution are called the *vertical foliation* and *vertical subbundle* of TM, respectively.

The Levi-Civita connection ∇ defines a splitting of the tangent bundle $T(TM)$ of TM:

$$T(TM) = \mathcal{H} \oplus \mathcal{V}.$$

The complimentary subbundle $\mathcal{H}$ is called the *horizontal distribution* of TM determined by ∇. For a vector $X \in T_xM$, the *horizontal lift* of X to a point $(x;u) \in TM$ is a unique vector $X^{\mathrm{h}} \in \mathcal{H}_u$ such that $\pi_{*u}X^{\mathrm{h}}_u = X$. The *vertical lift* of X to u is a unique vector $X^{\mathrm{v}} \in \mathcal{V}_u$ such that $X^{\mathrm{v}}(df) = Xf$ for all smooth function f on M. Here we regard df naturally as a smooth function on TM. These two lifting operations are extended naturally to those for vector fields.

The *canonical almost complex structure* J of TM associated to ∇ is given by

$$JX^{\mathrm{h}} = X^{\mathrm{v}},\ JX^{\mathrm{v}} = -X^{\mathrm{h}},\ X \in \Gamma(TM).$$

The *Sasaki lift metric* g^{S} of TM is defined by

$$g^{\mathrm{S}}\left(X^{\mathrm{h}}, Y^{\mathrm{h}}\right) \circ \pi = g^{\mathrm{S}}\left(X^{\mathrm{v}}, Y^{\mathrm{v}}\right) \circ \pi = g\left(X, Y\right),\ g^{\mathrm{S}}\left(X^{\mathrm{h}}, Y^{\mathrm{v}}\right) \circ \pi = 0\,,$$

for all $X, Y \in \Gamma(TM)$.

One can see that g^{S} is Hermitian with respect to J. Moreover the resulting almost Hermitian manifold (TM, g^{S}, J) is an almost Kähler manifold.

For general theory of almost Kähler structure of (TM, g^{S}), we refer to [2] and [15].

1.2

The *tangent sphere bundle* of radius $r > 0$ is the hypersurface

$$T^{(r)}M := \left\{\, (x; u) \in TM \mid g_x(u, u) = r^2 \,\right\}.$$

One can see that $\boldsymbol{n} := \boldsymbol{U}/r$ is a global unit normal vector field to $T^{(r)}M$. We denote by $\bar{g}$ the Riemannian metric on $T^{(r)}M$ induced by g^{S}.

For any vector filed $X \in \Gamma(TM)$, its horizontal lift X^{h} is tangent to $T^{(r)}M$ at each point $u \in T^{(r)}M$. Yet, in general, the vertical lift X^{v} is not tangent to $T^{(r)}M$. Boeckx and Vanhecke [7] introduced the following new lifting operation. The *tangential lift* X^{t} of X is a vector field defined by

$$X^{\mathrm{t}} = X^{\mathrm{v}} - \bar{g}_x(X, \boldsymbol{n})\boldsymbol{n}.$$

1.3

An odd-dimensional Riemannian manifold (M, g) admits an *almost contact metric structure* compatible to g if it admits a vector field ξ, one-form η and an endomorphism field φ such that

$$\varphi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1\,,$$
$$g\left(\varphi X, \varphi Y\right) = g(X, Y) - \eta(X)\eta(Y)\,,$$

for all X, $Y \in \Gamma(TM)$.

An odd-dimensional Riemannian manifold together with an almost contact metric structure is called an *almost contact metric manifold*.

The almost Kähler structure (J, g^{S}) induces an almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$ on $T^{(r)}M$ in the following way.

$$JE = \varphi E + \eta(E)\,\boldsymbol{n},\ E \in \Gamma\left(T\left(T^{(r)}M\right)\right), \tag{1}$$

$$\xi = -J\boldsymbol{n}. \tag{2}$$

Direct computation show that the following formula;

$$\bar{g}(E, \varphi F) = 2r d\eta(E, F),\ E, F \in \Gamma\left(T\left(T^{(r)}M\right)\right).$$

This formula implies the following equation:

$$\bar{g}(E, \varphi F) = r\left\{\bar{g}\left(\bar{\nabla}_E\,\xi, F\right) - \bar{g}\left(\bar{\nabla}_F\,\xi, E\right)\right\}. \tag{3}$$

From (3), one can see that ξ is a Killing vector field if and only if

$$\overline{\nabla}_E\,\xi = -\frac{1}{2r}\varphi\,E,$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\left(T^{(r)}M, \bar{g}\right)$.

The Killing property of ξ is characterized by Tashiro as follows (See [1, p. 136]):

3 Proposition. *On the tangent sphere bundle $T^{(r)}M$, ξ is Killing if and only if M is of constant curvature* 1.

4 Remark. An almost contact metric manifold $(M, g; \varphi, \xi, \eta)$ is said to be a *contact metric manifold* if it satisfies

$$d\eta(X, Y) = g(X, \varphi Y),\ X, Y \in \Gamma(TM).$$

If one wish to normalize the structure tensors on $T^{(r)}M$ to adopt with contact Riemannian geometry [1], the following normalization is required:

$$\tilde{\eta} = \frac{1}{2r}\eta,\ \tilde{\xi} = 2r\xi,\ \tilde{\varphi} = \varphi,\ \tilde{g} = \frac{1}{4r^2}\,\bar{g}.$$

Then $(T^{(r)}M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a contact metric manifold (in the sense of [1]).

2 Pseudo-symmetry of tangent sphere bundles

2.1

Both the local symmetry and conformal flatness are a very strong restrictions for tangent sphere bundles. In fact, Blair and Koufogiorgos obtained the following results.

5 Theorem ([2]). *The unit tangent sphere bundle $T^{(1)}M$ of a Riemannian manifold M equipped with adjusted metric $\tilde{g} = \bar{g}/4$ is locally symmetric if and only if either M is flat or M is a surface of constant curvature* 1*.*

6 Theorem ([3]). *The unit tangent sphere bundle $T^{(1)}M$ of a Riemannian manifold M equipped with adjusted metric $\tilde{g}$ is conformally flat if and only if M is a surface of constant curvature* 0 *or* 1*.*

Recently, Boeckx and Calvaruso generalized Theorem 5 as follows:

7 Theorem ([5]). *Let M be a Riemannian n-manifold. If the unit tangent sphere bundle $(T^{(1)}M, \tilde{g})$ is semi-symmetric then it is locally symmetric.*

In the case $\dim M = 2$, the only possibility for M to have semi-symmetric or conformally flat unit tangent sphere bundle $T^{(1)}M$ with metric $\tilde{g}$ is to be of constant curvature 0 or 1. Note that, the classification of semi-symmetric $(T^{(1)}M, \tilde{g})$ with $\dim M = 2$ was obtained by Calvaruso and Perrone [8].

8 Remark.

(1) When M is the unit 2-sphere S^2 then $T^{(1)}M$ (with adjusted metric $\tilde{g}$) is the real projective 3-space P^3 and hence it is a (Sasakian) space form of constant curvature 1 (*cf.* [11]). In the case $M = \mathbb{R}^2$, $\left(T^{(1)}M, \tilde{g}\right) = \mathbb{R}^2 \times S^1$ is flat and identified with the Euclidean rigid motion group $SE(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ as a contact metric manifold.

(2) Blair and Sharma classified 3-dimensional locally symmetric contact metric manifolds [4]. Three-dimensional locally symmetric contact metric manifolds are either Sasakian manifolds of constant curvature 1 (*i.e.*, locally isometric to the unit 3-sphere S^3 or $P^3 = T^{(1)}S^2$) or locally isometric to $SE(2) = T^{(1)}\mathbb{R}^2$. This classification is generalized to general dimension in [6].

2.2

As we explained above, semi-symmetry is still a strong restriction for the unit tangent sphere bundles. Hereafter we shall investigate more mild condition–pseudo-symmetry for tangent sphere bundles (of arbitrary radii) over surfaces. For our use we recall the following useful lemma.

9 Lemma. ([12, Proposition 0.1]) *A Riemannian* 3*-manifold is a pseudo-symmetric space with $R(X,Y)\cdot R = L\{\,(X\wedge Y)\cdot R\,\}$ if and only if the eigenvalues of the Ricci tensor locally satisfy the following relations (up to numeration);*

$$\rho_1 = \rho_2, \ \rho_3 = 2L.$$

In particular, the Riemannian manifold is of constant type pseudo-symmetric space if and only if L is constant.

2.3

Now let $M = (M^2, g)$ be a Riemannian 2-manifold. In this subsection, we compute the Ricci curvatures of $T^{(r)}M$ with respect to the metric $\bar{g}$ induced by the Sasaki lift metric g^{S}. Take a point $u = (x; u) \in T^{(r)}M$. Then we can take an orthonormal basis $\{ e_1, e_2 \}$ of T_xM of the form $e_2 = u/r$. Then via the lifting operations, we get the following orthonormal basis

$$\bar{e}_1 := \left(e_1^{\mathrm{h}}\right)_u,\ \bar{e}_2 := \left(e_2^{\mathrm{h}}\right)_u,\ \bar{e}_3 := \left(e_1^{\mathrm{t}}\right)_u$$

of $T_u\left(T^{(r)}M\right)$. Using this orthonormal basis, we compute the Ricci tensor $\bar{\rho}$ of $\left(T^{(r)}M, \bar{g}\right)$. To this end, here we recall the following general formula.

10 Proposition ([13])**.** *Let (M, g) be a Riemannian n-manifold and $u = (x; u)$ a point of the tangent sphere bundle $T^{(r)}M$ of radius r. Take an orthonormal basis $\{ E_1, E_2, \ldots, E_{n-1}, E_n = u/r \}$. Then the Ricci tensor $\bar{\rho}$ of the tangent sphere bundle is given by*

$$\begin{aligned}
\bar{\rho}_u\left(X^{\mathrm{h}}, Y^{\mathrm{h}}\right) &= \rho_x(X, Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u, E_i)X, R(u, E_i)Y), \\
\bar{\rho}_u\left(X^{\mathrm{h}}, Y^{\mathrm{t}}\right) &= \frac{1}{2}\left\{(\nabla_u\rho)(Y, X) - (\nabla_Y\rho)(u, X)\right\}, \\
\bar{\rho}_u\left(X^{\mathrm{t}}, Y^{\mathrm{t}}\right) &= \frac{1}{4}\sum_{i=1}^{n} g(R(u, X)E_i, R(u, Y)E_i) + \frac{n-2}{r^2}g(X, Y),
\end{aligned}$$

for all $X, Y \in T_xM$. Here ρ denotes the Ricci tensor of (M, g).

By direct computations using Proposition 10, we obtain the following formulas for the components $\bar{\rho}_{ij} = \bar{\rho}(\bar{e}_i, \bar{e}_j)$:

11 Proposition. *The Ricci tensor $\bar{\rho}$ of $\left(T^{(r)}M, \bar{g}\right)$ is given by*

$$\begin{aligned}
&\bar{\rho}_{11} = \bar{\rho}_{22} = \tilde{\kappa} - \frac{\tilde{\kappa}^2 r^2}{2},\ \bar{\rho}_{33} = \frac{\tilde{\kappa}^2 r^2}{2}, \\
&\bar{\rho}_{12} = 0,\ \bar{\rho}_{13} = \frac{r}{2}\bar{e}_2(\tilde{\kappa}),\ \bar{\rho}_{23} = -\frac{r}{2}\bar{e}_1(\tilde{\kappa}),
\end{aligned}$$

where $\tilde{\kappa} = \kappa \circ \pi$ is the (vertical) lift of the Gaussian curvature κ of M to $T^{(r)}M$.

12 Remark. Yampol'skií obtained the Ricci curvatures $\bar{\rho}$ of the unit tangent sphere bundle $T^{(1)}M$ over a surface M of constant curvature c ([18, p. 115, (27)]).

From the list of Ricci curvatures given in Proposition 11, we obtain our main theorem.

Proof of Theorem. By using Proposition 11, the characteristic equation of the matrix valued function (ρ_{ij}) is computed as

$$(\lambda - \bar{\rho}_{11})\left\{\lambda^2 - (\bar{\rho}_{11} + \bar{\rho}_{33})\,\lambda + \bar{\rho}_{11}\bar{\rho}_{33} + (\bar{\rho}_{13})^2 + (\bar{\rho}_{23})^2\right\} = 0.$$

Thus $\bar{\rho}_0 := \bar{\rho}_{11} = \bar{\rho}_{22}$ is a Ricci eigenvalue of $T^{(r)}M$. The other Ricci eigenvalues of $T^{(r)}M$ are solutions to the equation:

$$F(\lambda) = \lambda^2 - (\bar{\rho}_{11} + \bar{\rho}_{33})\lambda + \bar{\rho}_{11}\bar{\rho}_{33} + (\bar{\rho}_{13})^2 + (\bar{\rho}_{23})^2 = 0. \tag{4}$$

The solutions to (4) are given explicitly by

$$\bar{\rho}_{\pm} = \frac{1}{2}\left\{\tilde{\kappa} \pm \sqrt{\tilde{\kappa}^2(1-\tilde{\kappa}r^2)^2 + r^2|\operatorname{grad}_M \kappa|^2 \circ \pi}\right\}.$$

Here $\operatorname{grad}_M \kappa$ is the gradient vector field of κ defined on M.
($\Leftarrow$) Assume that the base surface is of constant curvature, then the solutions to (4) are $\{\,\bar{\rho}_{11}, \bar{\rho}_{11}, \bar{\rho}_{33}\,\}$. Hence by Lemma 9, $T^{(r)}M$ is a pseudo-symmetric space. In particular, since all the Ricci eigenvalues are constant, $T^{(r)}M$ is of constant type.
($\Rightarrow$) Conversely, let us assume that $T^{(r)}M$ is pseudo-symmetric.

(1) If $\bar{\rho}_0 = \bar{\rho}_+$ or $\bar{\rho}_0 = \bar{\rho}_-$, then $0 = F(\bar{\rho}_0) = |\operatorname{grad}_M \kappa|^2 \circ \pi$. Hence κ is a constant. In this case, the Ricci eigenvalues are

$$\tilde{\kappa} - \frac{\tilde{\kappa}^2 r^2}{2},\ \tilde{\kappa} - \frac{\tilde{\kappa}^2 r^2}{2},\ \frac{\tilde{\kappa}^2 r^2}{2}. \tag{5}$$

All Ricci eigenvalues coincide if and only if $\kappa = 0$ or $\kappa = 1/r^2 > 0$.

(2) $\bar{\rho}_+ = \bar{\rho}_-$ if and only if $\kappa\left(1 - \kappa r^2\right) = 0$ and $\operatorname{grad}_M \kappa = 0$. Namely $\kappa = 0$ or $\kappa = 1/r^2 > 0$. In this case, the Ricci eigenvalues are

$$\tilde{\kappa} - \frac{\tilde{\kappa}^2 r^2}{2},\ \frac{1}{2}\tilde{\kappa},\ \frac{1}{2}\tilde{\kappa}. \tag{6}$$

QED

Proof of Corollary. ($\Leftarrow$) Let M be a Riemannian 2-manifold of constant curvature 0 or $1/r^2$. Then all the Ricci eigenvalues of $T^{(r)}M$ coincide. Hence $T^{(r)}M$ is locally symmetric, especially, semi-symmetric.
($\Rightarrow$) Assume that $T^{(r)}M$ is semi-symmetric. Then M is of constant curvature c by Theorem. From (5) and (6), semi-symmetric case can only occur when $c = 0$ or $c = 1/r^2$. QED

13 Remark. By using Proposition 11, the sectional curvature function of $\left(T^{(r)}M, \bar{g}\right)$ is computed as follows.

$$K(\bar{e}_1 \wedge \bar{e}_2) = \tilde{\kappa} - \frac{3}{4}r^2\tilde{\kappa}^2,\ K(\bar{e}_2 \wedge \bar{e}_3) = K(\bar{e}_1 \wedge \bar{e}_3) = \frac{r^2}{4}\tilde{\kappa}^2.$$

Note that $\xi_u = \bar{e}_2$. Hence if M is of constant curvature c, $T^{(r)}M$ has constant holomorphic sectional curvature $K(\bar{e}_1 \wedge \bar{e}_3) = r^2c^2/4$. Compare with higher dimensional case described in [10, Theorem 4.3].

14 Remark. A contact metric manifold is said to be a *strongly locally φ-symmetric space* if all reflections with respect to the flows of the Reeb vector field ξ are local isometries.

In [7], it is shown that $\left(T^{(1)}M, \tilde{g}\right)$ is strongly locally φ-symmetric if and only if the base manifold is of constant curvature.

Hence for unit tangent sphere bundles over surfaces equipped with the adjusted metric $\tilde{g} = \bar{g}/\left(4r^2\right)$, pseudo-symmetry is equivalent to strong local φ-symmetry.

2.4

The scalar curvature $\bar{s}$ of $(T^{(r)}M, \bar{g})$ is computed as

$$\bar{s} = \bar{\rho}_{11} + \bar{\rho}_{22} + \bar{\rho}_{33} = \frac{1}{2}\left(4\tilde{\kappa} - \tilde{\kappa}^2r^2\right). \tag{7}$$

The formula (7) implies that the tangent sphere bundle $T^{(r)}M$ of a surface $\left(M^2, g\right)$ has constant scalar curvature if and only if M is of constant curvature [14, Proposition 3.5].

3 Concluding remarks

The Riemannian geometry of tangent sphere bundles depends on the radius. For example, $(T^{(1/\sqrt{2})}M, \bar{g})$ is isometric to the unit tangent sphere bundle $T^{(1)}M$ with metric induced from the *Cheeger-Gromoll metric* on TM. Kowalski and Sekizawa obtained the following result:

15 Theorem. ([14, Corollary 3.4]) *Let M^2 be a surface of constant curvature c. Then the scalar curvature $\bar{s}$ of $\left(T^{(r)}M, \bar{g}\right)$ is a constant satisfying the following relations;*

(1) $c > 0$, then $\bar{s} > 0$ for $r < \frac{2}{\sqrt{c}}$, $\bar{s} = 0$ for $r = \frac{2}{\sqrt{c}}$, $\bar{s} < 0$ for $r > \frac{2}{\sqrt{|c|}}$.

(2) If $c < 0$, then $\bar{s} < 0$ for all r.

(3) If $c = 0$, then $\bar{s} = 0$ for all r.

Our discussions show that pseudo-symmetry of $T^{(r)}M$ does not depend on the radius. On the other hand, semi-symmetry of the tangent sphere bundles $T^{(r)}M$ over a surface M of constant curvature $\kappa \neq 0$ depends on the radius. In fact, $T^{(r)}M$ is semi-symmetric if and only if $r = 1/\sqrt{\kappa}$.

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