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PSEUDO-SYMMETRIC CONTACT 3-MANIFOLDS III

BY

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Abstract. A trans-Sasakian 3-manifold is pseudo-symmetric if and only if it is η-Einstein. In particular, a quasi-Sasakian 3-manifold is pseudo-symmetric if and only if it is a coKähler manifold or a homothetic Sasakian manifold. Some examples of non-Sasakian pseudo-symmetric contact 3-manifolds are exhibited.

Introduction. A Riemannian 3-manifold (M,g) is said to be a proper *pseudo-symmetric space* if its Ricci eigenvalues $\{\varrho_1,\varrho_2,\varrho_3\}$ satisfy the relation $\varrho_1=\varrho_2\neq\varrho_3$ ($\varrho_3\neq 0$) up to numbering [14]. In particular, a proper pseudo-symmetric 3-space (M,g) is said to be of *constant type* if ϱ_3 is a nonzero constant.

Such spaces have been studied from different motivations. For instance, in hypersurface geometry of nonflat 4-dimensional Riemannian space forms, it is shown that isometrically deformable hypersurfaces of type number two are pseudo-symmetric spaces of constant type [20].

O. Kowalski explained some other motivations of the study of pseudo-symmetric 3-spaces with *constant* principal Ricci curvatures in [28].

In our previous paper [11], we have investigated pseudo-symmetry of contact Riemannian 3-manifolds. In particular, we have shown that every Sasakian 3-manifold is constant type pseudo-symmetric. Moreover, in [12], we proved that tangent sphere bundles over Riemannian 2-manifolds are pseudo-symmetric if and only if the base manifolds are of constant curvature.

As is well known, odd-dimensional spheres are typical examples of Sasakian manifolds. On the other hand, odd-dimensional hyperbolic spaces cannot admit a Sasakian structure, but have a so-called *Kenmotsu structure*. K. Kenmotsu manifolds are normal (noncontact) almost contact Riemannian manifolds. Kenmotsu [25] investigated fundamental properties and local structure of such manifolds. Kenmotsu manifolds are locally isometric to warped product spaces with 1-dimensional base and Kähler fiber.

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As a generalization of both Sasakian manifolds and Kenmotsu manifolds, J. A. Oubiña [36] introduced the notion of trans-Sasakian manifold. An almost contact Riemannian manifold $(M; \varphi, \xi, \eta, g)$ is said to be a *trans-Sasakian manifold* if it satisfies

$$(\nabla_X \varphi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}$$

for some functions α and β. Here ∇ denotes the Levi-Civita connection.

J. C. Marrero [30] has proven that there are no proper trans-Sasakian manifolds in higher dimensions. Moreover, Marrero has shown the existence of proper trans-Sasakian 3-manifolds.

N. Hashimoto and M. Sekizawa [21] investigated conformally flat (irreducible) pseudo-symmetric 3-spaces of constant type. Their (local) classification says such spaces are warped products with 1-dimensional base and constant curvature fiber. One can see that every 3-dimensional warped product with 1-dimensional base and 2-dimensional fiber admits a trans-Sasakian structure with $\alpha = 0$.

In this paper, motivated by these observations, we study pseudo-symmetry of trans-Sasakian 3-manifolds.

As another generalization of Sasakian manifolds, generalized (κ, μ)-spaces have been extensively studied ([5], [6], [9], [16], [17], [24], [26], [27]).

A contact Riemannian manifold is said to be a *generalized* (κ, μ)*-space* if

$$R(X,Y)\xi = (\kappa I + \mu \mathrm{h})\{\eta(Y)X - \eta(X)Y\}, \qquad X, Y \in \mathfrak{X}(M),$$

for some functions κ and μ. Here h is an endomorphism field defined by $\mathrm{h} = \pounds_\xi \varphi/2$. If both κ and μ are constants, M is called a (κ, μ)*-space*. One can see that Sasakian manifolds are (κ, μ)-spaces with $\kappa = 1$ and $\mathrm{h} = 0$.

In the final section, we shall study pseudo-symmetry of 3-dimensional generalized (κ, μ)-spaces.

Throughout this paper we assume that all manifolds are connected.

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1. Preliminaries. Let (M, g) be a Riemannian manifold with its Levi-Civita connection ∇. Denote by R the Riemannian curvature of M:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \qquad X, Y \in \mathfrak{X}(M).$$

Here $\mathfrak{X}(M)$ is the Lie algebra of all vector fields on M. A tensor field F of type $(1,3)$,

$$F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

is said to be *curvature-like* provided that F has the symmetry properties of R. For example,

$$(X \wedge Y)Z = g(Y,Z)X - g(Z,X)Y, \qquad X, Y \in \mathfrak{X}(M), \tag{1.1}$$

defines a curvature-like tensor field on M. Note that the curvature R of a Riemannian manifold (M, g) of constant curvature c satisfies the formula $R(X, Y) = c(X \wedge Y)$.

As is well known, every curvature-like tensor field F acts on the algebra $\mathcal{T}_s^1(M)$ of all tensor fields on M of type $(1, s)$ as a derivation [35, p. 44]:

$$(F \cdot P)(X_1, \dots, X_s; Y, X) = F(X, Y)\{P(X_1, \dots, X_s)\} - \sum_{j=1}^{s} P(X_1, \dots, F(X, Y)X_j, \dots, X_s),$$
$$X_1, \dots, X_s \in \mathfrak{X}(M),\ P \in \mathcal{T}_s^1(M).$$

The derivative $F \cdot P$ of P with respect to F is a tensor field of type $(1, s+2)$.

For a tensor field P of type $(1, s)$, we denote by $\mathcal{Q}(g, P)$ the derivative of P with respect to the curvature-like tensor defined by (1.1):

$$\mathcal{Q}(g, P)(X_1, \dots, X_s; Y, X) = (X \wedge Y)P(X_1, \dots, X_s) - \sum_{j=1}^{s} P(X_1, \dots, (X \wedge Y)X_j, \dots, X_s).$$

A Riemannian manifold (M, g) is said to be *semi-symmetric* if $R \cdot R = 0$. Obviously, locally symmetric spaces ($\nabla R = 0$) are semi-symmetric.

More generally, a Riemannian manifold (M, g) is said to be *pseudo-symmetric* if

$$R \cdot R = L\mathcal{Q}(g, R)$$

for some function L. In particular, if L is constant, then M is called a *pseudo-symmetric space of constant type* [29]. A pseudo-symmetric space is said to be *proper* if it is not semi-symmetric.

For Riemannian 3-manifolds, the following characterizations of pseudo-symmetry are known (cf. [29]).

PROPOSITION 1.1. *A Riemannian* 3*-manifold* (M, g) *is pseudo-symmetric if and only if it is quasi-Einstein. This means that there exists a one-form* ω *such that the Ricci tensor field* ϱ *has the form*

$$\varrho = ag + b\omega \otimes \omega.$$

Here a *and* b *are functions.*

PROPOSITION 1.2. *Let* (M, g) *be a Riemannian* 3*-manifold. Then* (M, g) *is a pseudo-symmetric space of constant type if and only if there exists a one-form* ω *such that the Ricci tensor field* ϱ *is expressed as* $\varrho = ag + b\omega \otimes \omega$, *where* a *is a function and* $a + b|\omega|^2$ *is a constant* (*provided that* $\omega \neq 0$).

REMARK 1. The preceding proposition can be rephrased as follows (see [29, Proposition 0.1]):

A Riemannian 3-manifold is a pseudo-symmetric space of constant type with $R \cdot R = L\mathcal{Q}(g, R)$ if and only if the principal Ricci curvatures (eigenvalues of the Ricci tensor) locally satisfy the following relations (up to numbering):

$$\varrho_1 = \varrho_2, \quad \varrho_3 = 2L.$$

2. Almost contact Riemannian manifolds

2.1. Let M be an odd-dimensional manifold. An *almost contact structure* on M is a quadruple of tensor fields (φ, ξ, η, g), where φ is an endomorphism field, ξ is a vector field, η is a one-form and g is a Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.2}$$

A $(2n+1)$-dimensional manifold together with an almost contact structure is called an *almost contact Riemannian manifold* (or *almost contact manifold*). The *fundamental* 2*-form* Φ of M is defined by

$$\Phi(X, Y) := g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

If an almost contact Riemannian manifold $(M; \varphi, \xi, \eta, g)$ satisfies the condition

$$\varrho = ag + b\eta \otimes \eta$$

for some functions a and b, then M is said to be η*-Einstein*. Clearly, every η-Einstein almost contact 3-manifold is pseudo-symmetric.

2.2. Let $(M; \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold. A tangent plane at a point of M is said to be a *holomorphic plane* if it is invariant under φ. The sectional curvature of a holomorphic plane is called its *holomorphic sectional curvature*. If the sectional curvature function of M is constant on all holomorphic planes in TM, then M is said to be of *constant holomorphic sectional curvature*.

On the other hand, if the sectional curvature function is constant on all planes in TM which contain ξ, then M is said to be of *constant* ξ*-sectional curvature*.

2.3. An almost contact Riemannian manifold $(M; \varphi, \xi, \eta, g)$ is called a *contact Riemannian manifold* if

$$\Phi = d\eta. \tag{2.3}$$

The formula (2.3) implies that the one-form η is actually a *contact form*, namely η satisfies $(d\eta)^n \wedge \eta \neq 0$. On a contact Riemannian manifold M, the structure vector field ξ is traditionally called the *characteristic vector field* (or *Reeb vector field*).

2.4. An almost contact Riemannian manifold M is said to be of *rank* $r = 2s$ (> 0) if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$, and of *rank* $r = 2s + 1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. Thus contact Riemannian manifolds are of rank $2n + 1$.

An almost contact Riemannian manifold M is said to be *normal* if it satisfies $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ.

A normal almost contact Riemannian manifold is said to be a *quasi-Sasakian manifold* if its fundamental 2-form Φ is closed ($d\Phi = 0$) [1]. In particular, a contact Riemannian manifold is called a *Sasakian manifold* if it is normal. By definition, Sasakian manifolds are quasi-Sasakian manifolds of rank $2n + 1$.

2.5. According to Oubiña [36], an almost contact manifold $(M; \varphi, \xi, \eta, g)$ is said to be a *trans-Sasakian manifold* (of type (α, β)) if

$$(\nabla_X \varphi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\} \tag{2.4}$$

for some functions α and β.

In particular, a trans-Sasakian manifold is said to be a

- *Sasakian manifold* if $(\alpha, \beta) = (1, 0)$,
- *Kenmotsu manifold* if $(\alpha, \beta) = (0, 1)$,
- *coKähler manifold* if $(\alpha, \beta) = (0, 0)$.

More generally a trans-Sasakian manifold of type $(\alpha, 0)$ with nonzero constant α is homothetic to a Sasakian manifold and called a *homothetic Sasakian manifold* or α-Sasakian manifold. Analogously, a *homothetic Kenmotsu manifold* (or β-Kenmotsu manifold) is a trans-Sasakian manifold of type $(0, \beta)$ with nonzero constant β [23].

Remark 2. Trans-Sasakian manifolds are normal [36].

There are two typical subclasses of the class of trans-Sasakian manifolds.

A trans-Sasakian manifold of type (α, β) is said to be of *class* C_5 if $\alpha = 0$. This class C_5 contains the class of β-Kenmotsu manifolds. On the other hand, a trans-Sasakian manifold is said to be of *class* C_6 if $\beta = 0$. α-Sasakian manifolds and coKähler manifolds are of class C_6.

Let $(M; \varphi, \xi, \eta, g)$ be a trans-Sasakian manifold. Then from (2.1) and (2.4), we have

$$\nabla_X \xi = -\alpha\varphi X + \beta\{X - \eta(X)\xi\}, \qquad X, Y \in \mathfrak{X}(M). \tag{2.5}$$

In particular, we have $\nabla_\xi \xi = 0$. Hence on trans-Sasakian manifolds, integral curves (trajectories) of ξ are geodesics.

Moreover, trans-Sasakian manifolds satisfy the following formula ([7], [42, (4.9)]):

$$2\alpha\beta + \xi\alpha = 0. \tag{2.6}$$

The formula (2.6) implies the following characterization of α-Sasakian manifolds.

LEMMA 2.1 ([7]). *Let M be a trans-Sasakian manifold of type (α, β). If α is a nonzero constant, then $\beta = 0$ and hence M is α-Sasakian.*

Marrero proved the following fundamental result (see also [42, Theorem 4.8]).

PROPOSITION 2.1 ([30]). *Trans-Sasakian manifolds of dimension ≥ 5 are either of class C_5 or of class C_6 with constant α.*

From (2.5)–(2.6), one can deduce the following formulas:

$$\alpha = -(\nabla_X \Phi)(X, \xi), \qquad \beta = -\frac{1}{2n}\,\delta\eta, \qquad X \perp \xi,\ |X| = 1.$$

Here δ denotes the codifferential operator. The function $\delta\eta$ is defined by $\delta\eta = -\mathrm{trace}(\nabla\eta)$.

3. Pseudo-symmetric trans-Sasakian 3-manifolds

3.1. Let $(M; \varphi, \xi, \eta, g)$ be an almost contact Riemannian 3-manifold. Then the covariant derivative $\nabla\varphi$ of φ satisfies ([33])

$$(\nabla_X \varphi)Y = g(\varphi(\nabla_X \xi), Y)\xi - \eta(Y)\varphi\nabla_X \xi, \qquad X, Y \in \mathfrak{X}(M). \tag{3.1}$$

In dimension 3, there exist *proper* trans-Sasakian manifolds, namely, trans-Sasakian manifolds which are neither of class C_5 or of class C_6 (see Proposition 3.7).

On the other hand, Olszak obtained the following characterization of trans-Sasakian 3-manifolds.

PROPOSITION 3.1. *Let M be an almost contact Riemannian 3-manifold. Then the following three conditions are equivalent:*

- $\nabla\xi \circ \varphi = \varphi \circ \nabla\xi$.
- *M is normal.*
- *M is trans-Sasakian.*

In that case, M is a trans-Sasakian manifold of type (α, β) with

$$\alpha = \frac{1}{2}\,\mathrm{trace}(\varphi\nabla\xi), \qquad \beta = \frac{1}{2}\,\mathrm{div}\,\xi.$$

Moreover, Olszak gave the following characterization of quasi-Sasakian 3-manifolds.

PROPOSITION 3.2 ([33]). *Let M be an almost contact Riemannian 3-manifold. Then M is quasi-Sasakian if and only if M is a trans-Sasakian manifold of type $(\alpha, 0)$ with $d\alpha(\xi) = 0$.*

In particular, every quasi-Sasakian 3-manifold is of class C_6.

The Ricci operator of a trans-Sasakian 3-manifold is given by the following formula due to Olszak [33].

PROPOSITION 3.3. *Let M be a trans-Sasakian 3-manifold. Denote by Q the Ricci operator of M defined by*

$$\varrho(X,Y) = g(QX,Y), \qquad X,Y \in \mathfrak{X}(M).$$

Then Q is given by

$$QX = \{\mathrm{s}/2 + \xi\beta - (\alpha^2 - \beta^2)\}I + \{-\mathrm{s}/2 - \xi\beta + 3(\alpha^2 - \beta^2)\}\eta(X)\xi$$
$$-\eta(X)\{\operatorname{grad}\beta - \varphi\operatorname{grad}\alpha\} - \{d\alpha(\varphi X) + d\beta(X)\}\xi,$$

where $\mathrm{s} = \mathrm{tr}\ \varrho$ *is the scalar curvature of M.*

Now let M be a pseudo-symmetric trans-Sasakian 3-manifold. Let us take a local orthonormal frame field $\{e_1, e_2, e_3\}$ such that $\eta(e_1) = 0,\ e_2 = \varphi e_1,\ e_3 = \xi$. Denote by ϱ_{ij} the components of the Ricci tensor field ϱ with respect to this frame;

$$\varrho_{11} = \varrho_{22} = \mathrm{s}/2 - \alpha^2 + \beta^2 + d\beta(\xi), \qquad \varrho_{33} = 2\alpha^2 - 2\beta^2 - 2d\beta(\xi),$$
$$\varrho_{12} = 0, \qquad \varrho_{13} = d\alpha(\varphi e_1) + d\beta(e_1), \qquad \varrho_{23} = d\alpha(\varphi e_2) + d\beta(e_2).$$

Then the characteristic polynomial $\Psi(\lambda) = \det(\lambda\delta_{ij} - \varrho_{ij})$ for ϱ is given by

$$\Psi(\lambda) = (\lambda - \varrho_{11})F(\lambda),$$
$$F(\lambda) = \lambda^2 - (\varrho_{11} + \varrho_{33})\lambda + \varrho_{11}\varrho_{33} - 4\sum_{i=1}^{2}\{d\alpha(\varphi e_i) + d\beta(e_i)\}^2.$$

Hence $\varrho_0 := \varrho_{11} = \varrho_{22}$ is a Ricci eigenvalue. The solutions $\varrho_\pm$ to $F(\lambda) = 0$ are given by

$$\varrho_\pm := \frac{1}{2}\left[(\varrho_0 + \varrho_{33}) \pm \sqrt{(\varrho_0 - \varrho_{33})^2 + 4\Big\{\sum_{i=1}^{2}\{d\alpha(\varphi e_i) + d\beta(e_i)\Big\}^2}\,\right].$$

CASE 1: ϱ_0 solves $F(\lambda) = 0$. In this case, $F(\varrho_0) = 0$ is equivalent to

$$d\alpha(\varphi e_i) + d\beta(e_i) = 0, \qquad i = 1, 2.$$

In other words, $F(\varrho_0) = 0$ if and only if

$$g(\operatorname{grad}\beta - \varphi\operatorname{grad}\alpha, X) = 0 \tag{3.2}$$

for all $X \in \mathfrak{X}(M)$ orthogonal to ξ. In this case, the Ricci eigenvalues are ϱ_0, ϱ_0 and ϱ_{33}.

CASE 2: $\varrho_+ = \varrho_-$. The trans-Sasakian manifold M satisfies $\varrho_+ = \varrho_-$ if and only if M satisfies (3.2) and $\varrho_{33} = \varrho_0$. In this case, all the Ricci eigenvalues are the same function. Hence M is of constant curvature.

Hence we obtain the following result.

Lemma 3.1. *Every pseudo-symmetric trans-Sasakian* 3*-manifold satisfies* (3.2)*.*

Here we give an interpretation of the condition (3.2).

Lemma 3.2. *On a trans-Sasakian* 3*-manifold M, ξ is an eigenvector field of the Ricci operator Q if and only if M satisfies* (3.2)*.*

Proof. Direct computations using Proposition 3.3 show that

$$Q\xi = 2(\alpha^2 - \beta^2 - d\beta(\xi))\xi - (\operatorname{grad}\beta - \varphi \operatorname{grad}\alpha).$$

Hence ξ is an eigenvector field of Q if and only if (3.2) holds. In that case, the following formulas hold:

$$\operatorname{grad}\beta - \varphi \operatorname{grad}\alpha = d\beta(\xi)\xi, \quad Q\xi = (2(\alpha^2 - \beta^2) - 3d\beta(\xi))\xi. \ \blacksquare$$

Lemma 3.3. *Let M be a trans-Sasakian* 3*-manifold. Then M is pseudo-symmetric if and only if M is η-Einstein.*

Proof. ($\Leftarrow$) If M is η-Einstein, then M is pseudo-symmetric by Proposition 1.1.

($\Rightarrow$) Assume that M is pseudo-symmetric. Then M satisfies (3.2). Hence the Ricci tensor field is given by

$$\varrho = \{\mathrm{s}/2 + \xi\beta - (\alpha^2 - \beta^2)\}g + \{-\mathrm{s}/2 - 3\xi\beta + 3(\alpha^2 - \beta^2)\}\eta \otimes \eta.$$

This formula says M is η-Einstein. $\blacksquare$

E. Vergara-Diaz and C. M. Wood gave the following characterization of (3.2).

Lemma 3.4 ([42]). *A trans-Sasakian* 3*-manifold M satisfies* (3.2) *if and only if ξ is a harmonic section of the unit tangent sphere bundle T_1M of M.*

Hence we obtain the following result.

Theorem 3.1. *Let M be a trans-Sasakian* 3*-manifold. Then the following conditions are equivalent*:

(1) *M is pseudo-symmetric.*
(2) *M is η-Einstein.*
(3) *ξ is an eigenvector field of Q.*
(4) *ξ is a harmonic section of the unit tangent sphere bundle T_1M,*
(5) *M satisfies* (3.2).

In this case, the Ricci tensor field of M is given by

$$\varrho = \{\mathrm{s}/2 + \xi\beta - (\alpha^2 - \beta^2)\}g + \{-\mathrm{s}/2 - 3\xi\beta + 3(\alpha^2 - \beta^2)\}\eta \otimes \eta. \tag{3.3}$$

Example 3.1 (CoKähler 3-manifolds). *Let M be a coKähler* 3*-manifold. Then its Ricci operator is given by*

$$Q = \frac{s}{2}I - \frac{s}{2}\eta \otimes \xi.$$

Thus the principal Ricci curvatures are

$$\varrho_1 = \varrho_2 = \mathrm{s}/2, \quad \varrho_3 = 0.$$

Hence M is semi-symmetric.

Example 3.2 (Homothetic Kenmotsu manifolds). Let M be a 3-dimensional almost contact Riemannian manifold of class C_5. Then its principal Ricci curvatures are

$$\varrho_1 = \varrho_2 = \mathrm{s}/2 + \beta^2 + d\beta(\xi), \quad \varrho_3 = -2\beta^2 - 2d\beta(\xi).$$

Thus M is pseudo-symmetric if and only if $d\beta(X) = 0$ for all $X \perp \xi$. In particular, every homothetic Kenmotsu 3-manifold is a pseudo-symmetric space of constant type.

Example 3.3 (Homothetic Sasakian manifolds). The principal Ricci curvatures of α-Sasakian manifold M are

$$\varrho_1 = \varrho_2 = \mathrm{s}/2 - \alpha^2, \quad \varrho_3 = 2\alpha^2 > 0.$$

Thus every α-Sasakian 3-manifold is a pseudo-symmetric space of constant type.

Remark 3. Let (M^3, g) be a locally symmetric Riemannian 3-manifold. Then M is (locally) isometric to one of the following spaces:

- Euclidean 3-space $\mathbb{E}^3$ (coKähler),
- the 3-sphere $\mathbb{S}^3(c^2)$ of curvature c^2 (homothetic Sasakian) or hyperbolic 3-space $\mathbb{H}^3(-c^2)$ of curvature $-c^2$ (homothetic Kenmotsu),
- Riemannian products $\mathbb{S}^2(c^2) \times \mathbb{E}^1$ or $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$ (coKähler).

It is known that semi-symmetric Kenmotsu manifolds are locally symmetric and hence of constant curvature -1 [25]. On the other hand, semi-symmetric Sasakian manifolds are locally symmetric and hence of constant curvature 1. Thus we obtain

Corollary 3.1.

(1) *β-Kenmotsu 3-manifolds other than hyperbolic space forms are proper pseudo-symmetric spaces of constant type.*
(2) *α-Sasakian 3-manifolds other than spherical space forms are proper pseudo-symmetric spaces of constant type.*

Here we give a classification of pseudo-symmetric quasi-Sasakian 3-manifolds.

Corollary 3.2. *A quasi-Sasakian 3-manifold is pseudo-symmetric if and only if it is a coKähler manifold or a homothetic Sasakian manifold.*

Proof. For a quasi-Sasakian 3-manifold M, (3.2) reduces to

$$g(\varphi \operatorname{grad} \alpha, e_1) = g(\varphi \operatorname{grad} \alpha, e_2) = 0.$$

Since $e_2 = \varphi e_1$ and $e_1 = -\varphi e_2$, (3.2) is equivalent to the equation

$$e_1\alpha = e_2\alpha = 0.$$

Thus M is pseudo-symmetric if and only if α is constant, because $\xi\alpha = 0$ by Proposition 3.2. ∎

Every Sasakian 3-manifold satisfies the condition $Q\varphi = \varphi Q$. We consider here the commutator $[Q, \varphi]$. Direct computation shows that

$$(Q\varphi - \varphi Q)X = g(X, \operatorname{grad}\alpha + \varphi \operatorname{grad}\beta)\xi - \eta(X)(\operatorname{grad}\alpha + \varphi \operatorname{grad}\beta).$$

From this formula, we get the following result.

Proposition 3.4. *On a trans-Sasakian 3-manifold M, the following three conditions are equivalent.*

- $\eta(Q\varphi - \varphi Q) = 0$.
- $Q\varphi = \varphi Q$.
- $\operatorname{grad}\alpha + \varphi \operatorname{grad}\beta = 0$.

In this case, $\xi\alpha = -2\alpha\beta = 0$ and M is η-Einstein with Ricci tensor field (3.3).

Proof. It is clear that $\eta([Q, \varphi]) = 0$ if and only if $Z := \operatorname{grad}\alpha + \varphi \operatorname{grad}\beta = 0$. By (2.6), we have $\eta(Z) = \xi\alpha = -2\alpha\beta$. ∎

Example 3.4 (Warped products). Let (N, h, J) be a Riemannian 2-manifold together with the compatible orthogonal complex structure J. Take a direct product $M = \mathbb{E}^1(t) \times N$ and denote by π and σ the natural projections onto the first and second factors, respectively.

Take the warped product $M = \mathbb{E}^1 \times_f N$ and define $\xi = \partial/\partial t$. Then the Levi-Civita connection ∇ of M is given by (cf. [35])

$$\begin{aligned}
\nabla_{\overline{X}^{\mathrm{v}}} \overline{Y}^{\mathrm{v}} &= (\overline{\nabla}_{\overline{X}} \overline{Y})^{\mathrm{v}} - \frac{1}{f}\, g(\overline{X}^{\mathrm{v}}, \overline{Y}^{\mathrm{v}}) f'\xi, \\
\nabla_\xi \overline{X}^{\mathrm{v}} &= \nabla_{\overline{X}^{\mathrm{v}}}\xi = \frac{f'}{f}\, \overline{X}^{\mathrm{v}}, \\
\nabla_\xi \xi &= 0.
\end{aligned}$$

Here the superscript v means the vertical lift operation of vector fields from N to M. Define φ by $\varphi X = \{J(\sigma_* X)\}^{\mathrm{v}}$. Then we get

$$\begin{aligned}
\nabla_X \xi &= \beta(X - \eta(X)\xi), \\
(\nabla_X \varphi)Y &= \beta\{g(\varphi X, Y) - \eta(Y)\varphi X\}, \qquad \beta = f'/f.
\end{aligned}$$

Hence $M = \mathbb{E}^1 \times_f N$ is of class C_5.

Take a local orthonormal frame field $\{\bar{e}_1, \bar{e}_2\}$ of (N, h) such that $\bar{e}_2 = J\bar{e}_1$. Then we obtain a local orthonormal frame field $\{e_1, e_2, e_3\}$ by

$$e_1 = \frac{1}{f}\, \bar{e}_1^{\mathrm{v}}, \qquad e_2 = \frac{1}{f}\, \bar{e}_2^{\mathrm{v}} = \varphi e_1, \qquad e_3 = \xi.$$

Then the holomorphic sectional curvature of M is given by

$$H = K(e_1 \wedge e_2) = \frac{1}{f^2}\{K_N - (f')^2\}.$$

On the other hand, the sectional curvature of a plane containing ξ is

$$K(e_1 \wedge e_3) = K(e_2 \wedge e_3) = -\frac{f''}{f}.$$

The Ricci tensor components $\varrho_{ij} = \varrho(e_i, e_j)$ are given by

$$\varrho_{11} = \varrho_{22} = \frac{K}{f^2} - \frac{f''}{f} - \left(\frac{f'}{f}\right)^2, \quad \varrho_{33} = -\frac{2f''}{f}$$

Hence M is a pseudo-symmetric space. In particular, M is of constant type if and only if f is a solution to $f'' = -Lf$ for some constant L.

The local structure of Kenmotsu manifolds is described as follows.

Proposition 3.5 ([25]).

- *Kenmotsu manifolds of constant holomorphic sectional curvature are hyperbolic space forms of curvature* -1*.*
- *A Kenmotsu manifold M is locally isomorphic to a warped product $I \times_f N$ whose base $I \subset \mathbb{E}^1(t)$ is an open interval and N is a Kähler manifold with warping function $f(t) = e^{ct}$, $c \neq 0$. The structure vector field is $\xi = \partial/\partial t$.*

As we saw before, warped products of the form $M = \mathbb{E}^1 \times_f N$ with 2-dimensional standard fiber are pseudo-symmetric trans-Sasakian 3-manifolds. In particular M is of constant type if and only if the warping function f satisfies the ODE $f'' = -Lf$ for some constant L. In particular, if we assume that, in addition, N is of constant Gaussian curvature, the warped product is conformally flat. Conversely, 3-dimensional conformally flat irreducible pseudo-symmetric space of constant type are locally isometric to warped products as above. More precisely, Hashimoto and Sekizawa obtained the following result.

Theorem 3.2 ([21]). *Let (M, g) be a 3-dimensional conformally flat irreducible pseudo-symmetric space of constant type. Then M is locally isometric to the warped product space $\mathbb{E}^1 \times_f N^2(k)$, whose base is the real line $\mathbb{E}^1$ and standard fiber $N^2(k)$ is a 2-dimensional space form of curvature k, respectively. The warping function f is one of the following*:

$$f(t) = \begin{cases} t, & L = 0, \\ \sinh(\lambda t) \ \textit{or} \ \cosh(\lambda t), & L = -\lambda^2 < 0, \\ \sin(\lambda t) & L = \lambda^2 > 0. \end{cases}$$

The principal Ricci curvatures are given by

$$\varrho_1 = \varrho_2 = \pm\frac{a^2}{f(t)^2} + 2L, \quad \varrho_3 = 2L,$$

where a is a positive constant. The curvature constant k is determined as follows:

- *If (M, g) is semi-symmetric, then $k = 1 \pm a^2$.*
- *If $L = -\lambda^2 < 0$, then $k = \lambda^2 \pm a^2$ when $f(t) = \sinh(\lambda t)$, and $k = -\lambda^2 \pm a^2$ when $f(t) = \cosh(\lambda t)$, respectively.*
- *If $L = \lambda^2 > 0$, then $k = \lambda^2 \pm a^2$.*

REMARK 4. M. S. Goto [15] studied global structures of compact conformally flat semi-symmetric spaces of dimension 3. Olszak [34] gave an example of a conformally flat quasi-Sasakian 3-manifold which is not pseudo-symmetric.

CoKähler manifolds are characterized as follows.

PROPOSITION 3.6 ([8, Lemma 2]). *Let $(M; \varphi, \xi, \eta, g)$ be an almost contact manifold such that ξ is Killing and $d\eta = 0$. Then M is locally isometric to a Riemannian product $N \times I$, where I is an open interval and N is an almost Hermitian manifold.*

In particular, a coKähler manifold is locally isometric to a Riemannian product $N \times I$, where I is an open interval and N is a Kähler manifold.

Marrero [30] showed the nonexistence of proper trans-Sasakian manifolds of dimension greater than 3. On the other hand, he showed the following method of constructing proper trans-Sasakian 3-manifolds (see also [32]).

PROPOSITION 3.7 ([30], [32]). *Let M be a Sasakian 3-manifold and σ a nonconstant positive function on M. Then the pseudo-conformal deformation*

$$g \mapsto g^\sigma := \sigma g + (1 - \sigma)\eta \otimes \eta$$

induces a trans-Sasakian manifold $(M; \varphi, \xi, \eta, g^\sigma)$ of type $(\alpha^\sigma, \beta^\sigma)$, where

$$\alpha^\sigma = \frac{1}{\sigma}, \qquad \beta^\sigma = \frac{1}{2\sigma} d\sigma(\xi).$$

- *If $d\sigma(\xi) \neq 0$, then $(M; \varphi, \xi, \eta, g^\sigma)$ is a proper trans-Sasakian manifold. Moreover, $(M; \varphi, \xi, \eta, g^\sigma)$ is neither of class C_5 nor of class C_6.*
- *If $d\sigma(\xi) = 0$, then M is quasi-Sasakian. Conversely, every quasi-Sasakian 3-manifold can be obtained in this way ([32]).*

Let $\mathbb{R}^3(-3)$ be the Heisenberg group

$$\left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \middle| (x,y,z) \in \mathbb{R}^3 \right\}$$

with the canonical Sasakian structure (φ, ξ, η, g) of constant holomorphic sectional curvature -3:

$$g = \frac{1}{4}(dx^2 + dy^2) + \eta \otimes \eta, \qquad \eta = \frac{1}{2}(dz - xdy), \qquad \xi = 2\frac{\partial}{\partial z},$$

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}.$$

We take a global orthonormal frame field:

$$e_1 = 2\frac{\partial}{\partial x}, \qquad e_2 = 2\left(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right), \qquad e_3 = 2\frac{\partial}{\partial z} = \xi.$$

Then the endomorphism field φ satisfies $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$ and $\varphi\xi = 0$.

Now let us take a positive function σ on $\mathbb{R}^3(-3)$ such that $d\sigma(\xi) \neq 0$ and consider the pseudo-conformal deformation $g \mapsto \widetilde{g} := g^\sigma$. The resulting proper trans-Sasakian 3-manifold is of type

$$\widetilde{\alpha} = \frac{1}{\sigma}, \qquad \widetilde{\beta} = \frac{\sigma_z}{2\sigma}.$$

We can take a global orthonormal frame field

$$\widetilde{e}_1 = \frac{1}{\sqrt{\sigma}}e_1, \qquad \widetilde{e}_2 = \frac{1}{\sqrt{\sigma}}e_2, \qquad \widetilde{e}_3 = \xi.$$

Let us consider the pseudo-symmetry condition:

$$\widetilde{g}(\widetilde{e}_i, \operatorname{grad}_{\widetilde{g}}\widetilde{\beta} - \varphi \operatorname{grad}_{\widetilde{g}}\widetilde{\alpha}) = 0, \qquad i = 1, 2,$$

for the deformed manifold. Direct computation shows that the deformed manifold is pseudo-symmetric if and only if

$$\left(\frac{\sigma_z}{2\sigma}\right)_x + \left(\frac{1}{\sigma}\right)_y + x\left(\frac{1}{\sigma}\right)_z = 0, \tag{3.4}$$

$$-\left(\frac{1}{\sigma}\right)_x + \left(\frac{\sigma_z}{2\sigma}\right)_y + x\left(\frac{\sigma_z}{2\sigma}\right)_z = 0. \tag{3.5}$$

Proposition 3.8. *Let $\sigma(x,y,z)$ be a positive solution to the system* (3.4)–(3.5) *such that $\sigma_z \neq 0$. Then the pseudo-conformal deformation of $\mathbb{R}^3(-3)$ by σ is a pseudo-symmetric proper trans-Sasakian* 3*-manifold.*

For simplicity, we assume that σ depends only on z. Then the pseudo-symmetry condition reduces to

$$\left(\frac{1}{\sigma}\right)_z = \left(\frac{\sigma_z}{\sigma}\right)_z = 0.$$

Hence σ is a constant. Thus the example due to Marrero (pseudo-conformal deformation of $\mathbb{R}^3(-3)$ with $\sigma = e^z$) is not pseudo-symmetric.

4. Pseudo-symmetric homogeneous contact Riemannian 3-manifolds. A contact Riemannian manifold $(M; \varphi, \xi, \eta, g)$ is said to be a *homogeneous contact Riemannian manifold* if there exists a connected Lie group G acting transitively on M as a group of isometries which leave the contact form η invariant.

Assume that M is simply connected. Then by a theorem due to Sekigawa [40], M is a Riemannian symmetric space or a Lie group with a left invariant metric. By using the classification of 3-dimensional Lie groups with left invariant metric due to J. Milnor [31], D. Perrone classified all simply connected homogeneous contact Riemannian 3-manifolds.

PROPOSITION 4.1 ([37]). *Let $(M; \varphi, \xi, \eta, g)$ be a simply connected homogeneous contact Riemannian 3-manifold. Then M is a Lie group G together with a left invariant contact Riemannian structure (η, g) and Webster scalar curvature $\mathcal{W} = (s - \varrho(\xi, \xi) + 4)/8$ and torsion invariant $\tau = \pounds_\xi g$. Here $\pounds_\xi$ denotes the Lie differentiation with respect to ξ.*

- *If G is unimodular, then G is one of the following*:
  (1) *the Heisenberg group $\mathbb{H}_3$ if $\mathcal{W} = |\tau| = 0$;*
  (2) $\mathrm{SU}(2)$ *if* $4\sqrt{2}\mathcal{W} > |\tau|$;
  (3) $\widetilde{E}(2)$ *if* $4\sqrt{2}\mathcal{W} = |\tau| > 0$;
  (4) $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ *if* $-|\tau| \neq 4\sqrt{2}\,\mathcal{W} < |\tau|$;
  (5) $E(1,1)$ *if* $4\sqrt{2}\,\mathcal{W} = -|\tau| < 0$.

  The Lie algebra $\mathfrak{g}$ of G is generated by an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ with commutation relations

  $$[e_1, e_2] = 2e_3, \qquad [e_2, e_3] = c_2 e_1, \qquad [e_3, e_1] = c_3 e_2.$$

- *If G is nonunimodular, then the Lie algebra $\mathfrak{g}$ of G satisfies the commutation relations*

  $$[e_1, e_2] = \alpha e_2 + 2e_3, \qquad [e_2, e_3] = 0, \qquad [e_3, e_1] = \gamma e_2,$$

  where $e_3 = \xi$, $e_1, e_2 \in \operatorname{Ker}\eta$, $e_2 = \varphi e_1$, $\alpha \neq 0$ and $4\sqrt{2}\,\mathcal{W} < |\tau|$. If $\gamma = 0$ then the structure is Sasakian $(\tau = 0)$ and $\mathcal{W} = -\alpha^2/4$.

In our previous work [11], we obtained the following result.

Proposition 4.2. *Every* 3*-dimensional unimodular Lie groups with special left invariant contact Riemannian structure is a pseudo-symmetric space of constant type.*

On the other hand, unfortunately, our result on nonunimodular groups in [11] is not correct. We take this opportunity to give a correct classification of pseudo-symmetric *nonunimodular* Lie groups with left invariant contact Riemannian structure (cf. [22]).

Let G be a 3-dimensional nonunimodular Lie group with a left invariant contact Riemannian structure. Then there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = \gamma e_2,$$

where $\alpha \neq 0$. In particular, $\gamma = 0$ if and only if G is a Sasakian manifold of constant holomorphic sectional curvature $-3 - \alpha^2$. In this case G is isomorphic to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ with left invariant Sasakian structure of some constant holomorphic curvature as a contact Riemannian manifold, but not isomorphic as a homogeneous contact manifold. The Ricci curvatures of G are given in terms of $\{e_1, e_2, e_3\}$ as follows:

$$\begin{aligned} \varrho_{11} &= -\alpha^2 - 2 + 2\gamma - \gamma^2/2, \\ \varrho_{22} &= -\alpha^2 - 2 + \gamma^2/2, \\ \varrho_{32} &= \varrho_{23} = \alpha\gamma, \quad \varrho_{33} = 2 - \gamma^2/2. \end{aligned}$$

The characteristic polynomial $\Psi(\lambda) = \det(\lambda\delta_{ij} - \varrho_{ij})$ for the Ricci tensor field is given by

$$\begin{aligned} \Psi(\lambda) &= (\lambda - \varrho_{11})F(\lambda), \\ F(\lambda) &= \lambda^2 + \alpha^2\lambda - \{\alpha^2(2 + \gamma^2/2) + (2 - \gamma^2/2)^2\}. \end{aligned}$$

The discriminant $\mathcal{D}$ of $F(\lambda) = 0$ is

$$\mathcal{D} = \alpha^4 + 4\{\alpha^2(2 + \gamma^2/2) + (2 - \gamma^2/2)^2\} > 0.$$

Thus the equation $F(\lambda) = 0$ has no double roots. On the other hand, we have $F(\varrho_{11}) = 2\gamma\{(\gamma + 2)^2 + \alpha^2\}$. Thus $F(\varrho_{11}) = 0$ if and only if $\gamma = 0$. In this case,

$$\varrho_{11} = \varrho_{22} = -\alpha^2 - 2, \quad \varrho_{33} = 2.$$

Thus we obtain the following result.

Theorem 4.1. *A* 3*-dimensional nonunimodular Lie group with a left invariant contact Riemannian structure is pseudo-symmetric if and only if it is a Sasakian space form of constant holomorphic sectional curvature* $-3 - \alpha^2 < -3$.

5. Pseudo-symmetric non-Sasakian contact Riemannian 3-manifolds. As we saw in the preceding section, there exist many pseudo-symmetric homogeneous Riemannian 3-manifolds. Moreover, the unit tangent sphere bundle of a Riemannian 2-manifold of constant curvature is locally homogeneous and pseudo-symmetric. In fact, in our previous paper [12], we have shown that for every Riemannian 2-manifold of constant curvature c, its unit tangent sphere bundle T_1M equipped with the standard contact Riemannian structure is a pseudo-symmetric space of constant type. In particular, if $c \neq 1$, the unit tangent sphere bundle is non-Sasakian. It was pointed out by D. E. Blair, Th. Koufogiorgos and B. J. Papantoniou [5] that the unit tangent sphere bundle of a surface with constant curvature c with standard contact Riemannian structure is a so-called (κ, μ)*-space* with $\kappa = c(2-c)$ and $\mu = -2c$.

Note that non-Sasakian 3-dimensional (κ, μ)-spaces are locally homogeneous and of constant holomorphic sectional curvature $H = -(\kappa + \mu)$

On the other hand, O. Kowalski [28] gave examples of nonhomogeneous pseudo-symmetric 3-spaces. Nonhomogeneous Sasakian 3-manifolds provide examples of nonhomogeneous pseudo-symmetric spaces.

In view of the results of our previous papers, one may raise the following question:

Are there examples of nonhomogeneous, non-Sasakian, pseudo-symmetric contact Riemannian 3*-manifolds?*

In this section we exhibit some examples of non-Sasakian pseudo-symmetric contact Riemannian 3-manifolds.

5.1. Let M be a contact Riemannian 3-manifold. Then the formula (3.1) reduces to ([41])

$$(\nabla_X \varphi)Y = g((I+\mathrm{h})X, Y)\xi - \eta(Y)(I+\mathrm{h})X, Y \qquad X \in \mathfrak{X}(M),$$

where I is the identity transformation and the endomorphism field h is defined by $\mathrm{h} = \pounds_\xi \varphi/2$.

Now let us define an endomorphism field ℓ by

$$\ell(X) = R(\xi, X)\xi, \qquad X \in \mathfrak{X}(M).$$

Then ℓ and h satisfy the following relations:

$$\mathrm{h}\xi = \ell(\xi) = 0, \qquad \eta \circ \mathrm{h} = 0, \qquad \mathrm{tr}\,\mathrm{h} = \mathrm{tr}(\mathrm{h}\varphi) = 0, \qquad \mathrm{h}\varphi + \varphi\mathrm{h} = 0,$$
$$\nabla_\xi \mathrm{h} = \varphi(I - \ell - \mathrm{h}^2), \qquad \mathrm{tr}\,\ell = 2 - \mathrm{tr}(\mathrm{h}^2).$$

Lemma 5.1 (cf. [10]). *Let M be a* 3*-dimensional contact Riemannian manifold. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ such that*

$$\mathrm{h}e_1 = \lambda e_1, \qquad e_2 = \varphi e_1, \qquad e_3 = \xi.$$

With respect to $\mathcal{E}$, *the Levi-Civita connection* ∇ *is given by*

$$\nabla_{e_1}e_1 = be_2, \quad \nabla_{e_1}e_2 = -be_1 + (1+\lambda)\xi, \quad \nabla_{e_1}\xi = -(1+\lambda)e_2,$$
$$\nabla_{e_2}e_1 = -ce_2 + (\lambda-1)e_3, \quad \nabla_{e_2}e_2 = ce_1, \quad \nabla_{e_2}\xi = (1-\lambda)e_1,$$
$$\nabla_{\xi}e_1 = \alpha e_2, \quad \nabla_{\xi}e_2 = -\alpha e_1, \quad \nabla_{\xi}\xi = 0.$$

The Ricci operator Q *is given by*

$$Qe_1 = \varrho_{11}e_1 + \xi(\lambda)e_2 + (2b\lambda - e_2(\lambda))\xi,$$
$$Qe_2 = \xi(\lambda)e_1 + \varrho_{22}e_2 + (2c\lambda - e_1(\lambda))\xi,$$
$$Q\xi = (2b\lambda - e_2(\lambda))e_1 + (2c\lambda - e_1(\lambda))e_2 + 2(1-\lambda^2)\xi,$$

where

$$\varrho_{11} = \mathrm{s}/2 + \lambda^2 - 2\alpha\lambda - 1, \qquad \varrho_{22} = \mathrm{s}/2 + \lambda^2 + 2\alpha\lambda - 1.$$

PROPOSITION 5.1 ([16]). *On a contact Riemannian* 3*-manifold with local orthonormal frame field* $\mathcal{E}$ *as in Lemma* 5.1, $Q\varphi = \varphi Q$ *if and only if* $b = c = 0$.

PROPOSITION 5.2. *Let* M *be a contact Riemannian* 3*-manifold with local orthonormal frame field* $\mathcal{E}$ *as in Lemma* 5.1*. Then* $\varrho_{11} = \varrho_{22}$ *if and only if* $\alpha = 0$ *or* M *is Sasakian.*

COROLLARY 5.1 (cf. [18, Proposition 2]). *If a contact Riemannian* 3*-manifold* M *has constant* ξ*-sectional curvature*, *then* $\alpha=0$ *or* M *is Sasakian.*

REMARK 5. A contact Riemannian 3-manifold is said to be a (3-τ)-*manifold* if $\nabla_\xi \tau = 0$ [3], [16]. Every contact Riemannian 3-manifold of constant ξ-sectional curvature is a (3-τ)-manifold with constant $\operatorname{tr}\ell$ [18, Proposition 2].

Now let M be a contact Riemannian 3-manifold with constant ξ-sectional curvature. Then the Ricci operator has the form:

$$Qe_1 = (\mathrm{s}/2 + \lambda^2 - 1)e_1 + 2b\lambda\xi,$$
$$Qe_2 = (\mathrm{s}/2 + \lambda^2 - 1)e_2 + 2c\lambda\xi,$$
$$Q\xi = 2b\lambda e_1 + 2c\lambda e_2 + 2(1-\lambda^2)\xi.$$

Hence the characteristic polynomial $\Psi(t) = \det(t\delta_{ij} - \varrho_{ij})$ for the Ricci tensor field ϱ is

$$\Psi(t) = (t - \varrho_{11})F(t),$$
$$F(t) = t^2 - (\varrho_{11} + 2 - 2\lambda^2)t + \{2(1-\lambda^2)\varrho_{11} - 4\lambda^2(b^2+c^2)\}.$$

CASE 1: ϱ_{11} solves $F(t) = 0$. Direct computation shows that $F(\varrho_{11}) = 0$ if and only if $\lambda = 0$ (i.e., M is Sasakian) or $b = c = 0$ (i.e., $Q\varphi = \varphi Q$).

CASE 2: $F(t) = 0$ has real double solutions. The discriminant $\mathcal{D}$ of the equation $F(t) = 0$ is

$$\mathcal{D} = (\varrho_{11} + 2\lambda^2 - 2)^2 + 16(b^2 + c^2).$$

Hence $F(t) = 0$ has two equal real solutions if and only if $\varrho_{11} + 2\lambda^2 - 2 = 0$ and $b = c = 0$.

5.2.

DEFINITION 5.1. A contact Riemannian manifold is said to be a *generalized* (κ, μ)*-space* if

$$R(X,Y)\xi = (\kappa I + \mu \mathrm{h})\{\eta(Y)X - \eta(X)Y\}, \qquad X, Y \in \mathfrak{X}(M),$$

for some functions κ and μ. If both κ and μ are constants, M is called a (κ, μ)*-space*. A generalized (κ, μ)-space is said to be *proper* if $(d\kappa)^2 + (d\mu)^2 \neq 0$.

Sasakian manifolds are (κ, μ)-spaces with $\kappa = 1$, $\mu = 0$ and $\mathrm{h} = 0$. Generalized (κ, μ)-spaces are of particular interest in dimension 3. In fact, the following results are known.

THEOREM 5.1 ([26]). *Let M be a non-Sasakian generalized (κ, μ)-space of dimension greater than 3. Then M is a (κ, μ)-space.*

PROPOSITION 5.3 ([27, Lemma 1]). *Let M be a 3-dimensional generalized (κ, μ)-space. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ such that*

$$\mathrm{h}e_1 = \lambda e_1, \qquad e_2 = \varphi e_1, \qquad e_3 = \xi,$$

where $\lambda = \sqrt{1-\kappa} > 0$. The Ricci operator Q is given by

$$QX = aX + b\eta(X)\xi + \mu \mathrm{h}X, \qquad X \in \mathfrak{X}(M).$$

with

$$a = \tfrac{1}{2}(\mathrm{s} - 2\kappa), \qquad b = \tfrac{1}{2}(6\kappa - \mathrm{s}).$$

Hence the principal Ricci curvatures of a 3-dimensional generalized (κ, μ)-space are given by

$$\begin{aligned} \varrho_1 &= \tfrac{1}{2}(\mathrm{s} - 2\kappa) + \mu\sqrt{1-\kappa}, \\ \varrho_2 &= \tfrac{1}{2}(\mathrm{s} - 2\kappa) - \mu\sqrt{1-\kappa}, \\ \varrho_3 &= 2\kappa. \end{aligned}$$

From these we can see that

$$\begin{aligned} \varrho_1 = \varrho_2 \ &\Leftrightarrow\ \mu = 0 \text{ or } \kappa = 1, \\ \varrho_1 = \varrho_3 \ &\Leftrightarrow\ \mu = \frac{1}{\sqrt{1-\kappa}}\,(3\kappa - \mathrm{s}/2), \\ \varrho_2 = \varrho_3 \ &\Leftrightarrow\ \mu = -\frac{1}{\sqrt{1-\kappa}}\,(3\kappa - \mathrm{s}/2). \end{aligned}$$

PROPOSITION 5.4. *A 3-dimensional proper generalized (κ, μ)-space is pseudo-symmetric if and only if $\mu = 0$ or $\mu = \pm\frac{1}{\sqrt{1-\kappa}}(3\kappa - \mathrm{s}/2)$.*

Perrone gave a characterization of "generalized (κ, μ)-property" as follows:

THEOREM 5.2 ([39]). *On a contact Riemannian 3-manifold M, its Reeb vector field $\xi : M \to T_1M$ is a harmonic map with respect to the Sasaki-lift metric if and only if M satisfies the generalized (κ, μ)-condition on an everywhere dense open subset of M.*

For 3-dimensional (κ, μ)-spaces, the following characterization is known.

THEOREM 5.3 ([6]). *Let M be a contact Riemannian 3-manifold. Then the following three conditions are equivalent*:

(1) *M is η-Einstein.*
(2) *$Q\varphi = \varphi Q$.*
(3) *M is a $(\kappa, 0)$-space with $\kappa \leq 1$.*

In the third case, M is of constant holomorphic sectional curvature $-\kappa$.

THEOREM 5.4 ([6]). *Let M be a contact Riemannian 3-manifold. Then M satisfies $Q\varphi = \varphi Q$ if and only if M is either*

(1) *a Sasakian 3-manifold*,
(2) *a flat contact Riemannian 3-manifold*, *or*
(3) *a non-Sasakian contact Riemannian space form of constant holomorphic sectional curvature $-\kappa$ and constant ξ-sectional curvature κ.*

In the third case, $\kappa < 1$.

These results imply that every $(\kappa, 0)$-space with $\kappa \leq 1$ is a pseudo-symmetric space.

To close this paper we exhibit two examples.

EXAMPLE 5.1. In [38], D. Perrone gave the following example of weakly φ-symmetric 3-space which is neither homogeneous nor strongly φ-symmetric. Let M be the open submanifold $\{(x, y, z) \in \mathbb{R}^3(x, y, z) \mid x \neq 0\}$ of Cartesian 3-space $\mathbb{R}^3$ together with a contact form $\eta = xydx + dz$. The Reeb vector field of this contact 3-manifold is $\xi = \partial/\partial z$. Take a global frame field

$$e_1 = -\frac{2}{x}\frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}, \quad e_3 = \xi$$

and define a Riemannian metric g by the condition that $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. Moreover, define an endomorphism field φ by $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$ and $\varphi\xi = 0$. Then (φ, ξ, η, g) is the associated almost contact structure of (M, η). The endomorphism field h satisfies $\mathrm{h}e_1 = e_1$, $\mathrm{h}e_2 = -e_2$. Hence M is non-Sasakian. Perrone showed that this contact Riemannian 3-manifold is nonhomogeneous. The Ricci operator of (M, g) is given by $Q = -8\omega^1 \otimes e_1$, where ω^1 is the dual 1-form of e_1. Hence

(M,g) is pseudo-symmetric. Thus Perrone's example is a nonhomogeneous and non-Sasakian contact Riemannian 3-manifold which is pseudo-symmetric.

Next we recall an example of a generalized (κ,μ)-space constructed by Koufogiorgos and Ch. Tsichlias [26] (see also [24, Section 4.3]).

EXAMPLE 5.2. Let $M = \{(x,y,z) \in \mathbb{R}^3 \mid z \neq 0\}$. Define a frame field $\mathcal{U} = \{u_1, u_2, u_3\}$ by

$$u_1 = \frac{\partial}{\partial x}, \qquad u_2 = -2yz\,\frac{\partial}{\partial x} + \frac{2x}{z^2}\,\frac{\partial}{\partial y} - \frac{1}{z^2}\,\frac{\partial}{\partial z}, \qquad u_3 = \frac{1}{z}\,\frac{\partial}{\partial y}.$$

Then we have

$$[u_1,u_2] = \frac{2}{z^2}\,u_3, \qquad [u_2,u_3] = 2u_1 + \frac{1}{z^3}\,u_3, \qquad [u_3,u_1] = 0.$$

Put $\xi = u_1$ and define a Riemannian metric g by $g(u_i,u_j) = \delta_{ij}$. Then we have a contact Riemannian manifold $M = (M;\varphi,\xi,\eta,g)$ with structure $\eta = g(\xi,\cdot)$ and

$$\varphi u_1 = 0, \qquad \varphi u_2 = u_3, \qquad \varphi u_3 = -u_2.$$

Then $\mathcal{E} = \{e_1,e_2,e_3\} = \{u_2,u_3,u_1\}$ satisfies the condition

$$\mathrm{h}e_1 = \lambda e_1, \qquad \mathrm{h}e_2 = -\lambda e_2, \qquad \mathrm{h}\xi = 0,$$

where $\lambda = 1/z^2$. Moreover this contact Riemannian 3-manifold is a generalized (κ,μ)-space with

$$\kappa = \frac{z^4-1}{z^4}, \qquad \mu = 2\left(1 - \frac{1}{z^2}\right).$$

The Ricci operator Q is given by

$$Qe_1 = \varrho_{11}e_1, \qquad Qe_2 = \varrho_{22}e_2, \qquad Q\xi = 2(1-\lambda^2)\xi,$$

where

$$\varrho_{11} = \mathrm{s}/2 + \lambda^2 - 2\alpha\lambda - 1, \qquad \varrho_{22} = \mathrm{s}/2 + \lambda^2 + 2\alpha\lambda - 1,$$
$$\alpha = -1 + 1/z^2, \qquad b = 1/z^3, \qquad c = 0.$$

The scalar curvature is

$$\mathrm{s} = \frac{6}{z^6} - \frac{2}{z^4} - \frac{2}{z^3} + \frac{4}{z^2} - 2.$$

Hence this space is not pseudo-symmetric.

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