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Pseudo-symmetry curvature conditions on hypersurfaces of Euclidean spaces and on Kahlerian manifolds

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RÉSUMÉ. — Nous étudions des variétés Riemanniennes pseudo-symétriques, qui sont des généralisations des espaces symétriques et semi-symétriques. Nous classifions les hypersurfaces pseudo-symétriques d'un espace Euclidien. Nous prouvons qu'il n'y a pas de variété Kaehlerienne pseudo-symétrique et non semi-symétrique.

ABSTRACT. — We study pseudo-symmetric Riemannian manifolds, which are generalizations of symmetric and semi-symmetric spaces. We classify the pseudo-symmetric hypersurfaces of a Euclidian space. We prove that there are no pseudo-symmetric Kaehlerian manifolds that are not semi-symmetric.

I - Introduction

In this paper we study Riemannian manifolds satisfying the curvature condition $R \cdot R = fQ(R)$ (this type of condition will be called a pseudo-symmetry curvature condition and will be explained in the next section). This condition arose during the study of umbilical hypersurfaces (see [AD], [DEP]) and is a generalization of the conditions $\nabla R = 0$ and $R \cdot R = 0$ (symmetric and semi-symmetric spaces [DDV]).

First, we study one simple case, namely isometric immersions into an $(N+1)$-dimensional Euclidean space of N-dimensional Riemannian manifolds satisfying this curvature condition or one of the related conditions $R \cdot C = fQ(C)$ or $R \cdot S = fQ(S)$ for the Weyl conformal curvature tensor C and the Ricci tensor S. We obtain a full classification of the

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hypersurfaces satisfying one of these conditions. We show that there are many non-conformally flat Riemannian manifolds satisfying $R \cdot R = fQ(R)$ (in this respect, see [DDV, Theorem 5.1]). Furthermore, we obtain that each conformally flat hypersurface of a Euclidean space satisfies $R \cdot R = fQ(R)$. Theorems 1 and 3 show that each hypersurface of a Euclidean space satisfying $R{\cdot}C = fQ(C)$ satisfies $R{\cdot}R = fQ(R)$. This is related to a theorem of Deszcz and Grycak which states that each analytic Riemannian manifold satisfying $R \cdot C = fQ(C)$ also satisfies $R \cdot R = fQ(R)$ or $C = 0$ in case $N \geq 5$ (for a precise formulation, see [DG]; see also [G]). Concerning Kähler manifolds we obtained a stronger result : there are no Kähler manifolds that satisfy $R \cdot R = fQ(R)$ and for which $R \cdot R \neq 0$.

More precisely, we will prove the following theorems.

THEOREM 1.— *Let $F : (M^n, g) \hookrightarrow E^{N+1}$ be an isometric immersion of a Riemannian manifold in a Euclidean space. Then (M^N, g) satisfies $R \cdot R = fQ(R)$ if and only if for each point p in M, F has at most two distinct principal curvatures in p or $R \cdot R = 0$ in p.*

THEOREM 2.— *Let $F : (M^N, g) \hookrightarrow E^{N+1}$ be an isometric immersion of a Riemannian manifold in a Euclidean space. Then (M^N, g) satisfies $R \cdot S = fQ(S)$ if and only if for each point p in M, F has at most two distinct principal curvatures in p or $R \cdot S = 0$ in p.*

THEOREM 3.— *Let $F : (M^N, g) \hookrightarrow E^{N+1}$ be an isometric immersion of a Riemannian manifold in a Euclidean space. Then (M^N, g) satisfies $R \cdot C = fQ(C)$ if and only if for each point p in M, F has at most two distinct principal curvatures in p or $R \cdot C = 0$ in p.*

THEOREM 4.— *Let (M^N, J, g) be a Kähler manifold satisfying $R \cdot R = fQ(R)$. Then (M^N, g) satisfies $R \cdot R = 0$.*

2 - Preliminaries

Let (M^N, g) be a (connected) n-dimensional Riemannian manifold ($N \geq 2$). In the following X, Y, Z denote vector fields that are tangent to M^N. ∇ is the Levi Civita connection of (M^N, g) and R is the Riemann-Christoffel curvature tensor of (M^N, g). $\widetilde{S}$ is the (1,1)-tensor related to the Ricci tensor S of (M^N, g) by $g(\widetilde{S}X, Y) = S(X, Y)$ for all X and Y. $\tau = tr\ \widetilde{S}$ is the scalar curvature of (M^N, g). $X \wedge Y$ is the (1,1)-tensor field defined by

$(X\Lambda Y)(Z) := g(Z,Y)X - g(Z,X)Y$. The *Weyl conformal curvature tensor* of (M^N, g) (for $N \geq 3$) is defined by

$$C(X,Y) := R(X,Y) - \frac{1}{N-2}(\widetilde{S}X \wedge Y + X \wedge \widetilde{S}Y) + \frac{\tau}{(N-1)(N-2)} X \wedge Y. \tag{2.1}$$

Let $F : (M^N, g) \hookrightarrow E^{N+1}$ be an isometric immersion of (M^N, g) in an $(N+1)$-dimensional Euclidean space. Let ξ be a local normal section on F. Then the *second fundamental form* h and the *second fundamental tensor* A of F are defined by the formulas of Gauss and Weingarten : $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi$ and $\widetilde{\nabla}_X \xi = -AX$ ($\widetilde{\nabla}$ is the standard connection of E^{N+1}). A is related to h by $h(X,Y) = g(AX, Y)$. We will not distinguish between A_p and its matrix ($p \in M$). The equation of Gauss is given by

$$R(X,Y) = AX \wedge AY. \tag{2.2}$$

Let $p \in M$. In the following x, y, z denote vectors in T_pM. Let $x\Lambda y$ denote the endomorphism $T_pM \to T_pM : z \mapsto g(z,y)x - g(z,x)y$. Since A_p is symmetric, there exists an orthonormal basis $\{e_1, \ldots, e_N\}$ of (T_pM, g_p) consisting of eigenvectors of A_p, i.e. such that

$$Ae_i = \lambda_i e_i, \tag{2.3}$$

where $\lambda_i \in \mathbf{R}$ for each $i \in \{1, \ldots, N\}$. $\lambda_1, \ldots, \lambda_N$ are called the *principal curvatures* of F in p. (2.1), (2.2) and (2.3) imply that

$$\begin{aligned} R(e_i, e_j) &= \lambda_i \lambda_j e_i \wedge e_j, \\ \widetilde{S}e_i &= \mu_i e_i, \\ C(e_i, e_j) &= a_{ij} e_i \wedge e_j, \end{aligned}$$

where (2.4)

$$\begin{aligned} \mu_i &= \lambda_i (tr\ A - \lambda_i), \\ a_{ij} &= \lambda_i \lambda_j - \frac{1}{N-2}(\mu_i + \mu_j) + \frac{(tr\ A)^2 - tr\ A^2}{(N-1)(N-2)} \end{aligned}$$

for all i, j and k in $\{1, \ldots, N\}$.

Let $\overline{\lambda}_1, \ldots, \overline{\lambda}_r$ denote the mutually distinct eigenvalues of A_p with multiplicities $s_1, \ldots, s_r$ respectively. Denote by V_α the space of eigenvectors with eigenvalue $\overline{\lambda}_\alpha (\alpha \in \{1, \ldots, r\})$. If $e_i, e_k \in V_\alpha$ and $e_j, e_\ell \in V_\beta$, then

$\mu_i = \mu_k$ and $a_{ij} = a_{k\ell}$, $\big(i,j,k,\ell \in \{1,\ldots,N\}$ and $\alpha,\beta \in \{1,\ldots,r\}\big)$. We define numbers $\overline{\mu}_\alpha := \mu_i$ and $\overline{a}_{\alpha\beta} := a_{ij}$ where $e_i \in V_\alpha$ and $e_j \in V_\beta$, $(i,j \in \{1,\ldots,N\}$ and $\alpha,\beta \in \{1,\ldots,r\})$.

Let (M,J,g) be a Kähler manifold and let $p \in M$. Then the following properties are well known :

$$R(JX,JY) = R(X,Y) \tag{2.5}$$

and

$$R(X,Y)J = JR(X,Y) \tag{2.6}$$

for all X and Y tangent to M.

(M^N,g) is called (locally) *conformally flat* if (M^N,g) is (locally) conformally equivalent to E^N. It is well known that (M^N,g) is conformally flat if and only if $C = 0$ for $N \geq 4$. We recall that every surface is conformally flat and that $C = 0$ for every 3-dimensional Riemannian manifold. F is called *quasi-umbilical* if for each point p in M A_p has an eigenvalue with multiplicity at least $N-1$. For $N \geq 4$, E.Cartan proved that F is quasi-umbilical if and only if (M^N,g) is conformally flat. We remark that $C = 0$ in p if and only if A_p has an eigenvalue with multiplicity at least $N-1$ if $N \geq 4$ (i.e. also the "pointwise" version of Cartan's result holds).

Concerning the notations $R\cdot C, R\cdot S, \ldots$ we say for example that (M^N,g) satisfies $R\cdot C = 0$ if and only if $R(X,Y)\cdot C = 0$ for all vectorfields X and Y tangent to M, where $R(X,Y)$ acts as a derivation on the algebra of tensor fields on M, i.e.

$$\begin{aligned}(R(X,Y)\cdot C)(Z,U;V,W) &= -C\big(R(X,Y)Z,U;V,W\big)\\ &- C\big(Z,R(X,Y)U;V,W\big) - C\big(Z,U;R(X,Y)V,W\big)\\ &- C\big(Z,U;V,R(X,Y)W\big)\end{aligned}$$

for X,Y,Z,U,V,W tangent to M^N. The derivation $R(X,Y)\cdot$ is the derivation $\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$.

For every $(0,s)$-tensor T on M a $(0,s+2)$-tensor $Q(T)$ is defined by

$$Q(T)(X_1,\ldots,X_s;Y,Z) = \big((Y\wedge Z)\cdot T\big)(X_1,\ldots,X_s)$$

(see, e.g. [T]). We say that a Riemannian manifold (M^N,g) satisfies $R\cdot T = fQ(T)$ if there exists a function $f : M \to \mathbf{R}$ such that

$$\big(R(Y,Z)\cdot T\big)(X_1,\ldots,X_s)(p) = f(p)Q(T)(X_1,\ldots,X_s;Y,Z)(p)$$

for every p in M and all $X_1,\dots,X_s,Y,Z$ tangent to M.

3 - Proof of theorem 1

Suppose that $F:(M^N,g)\hookrightarrow E^{N+1}$ is an isometric immersion of a Riemannian manifold. Let p be a point in M and let $\{e_1,\dots,e_N\}$ be a basis for T_pM satisfying (2.3). From (2.4) it is easy to obtain that

$$\begin{aligned}
&\big(R(e_i,e_j)\cdot R\big)(e_k,e_\ell;e_m,e_n)-f(p)Q(R)(e_k,e_\ell;e_m,e_n;e_i,e_j) =\\
&=\big(f(p)-\lambda_i\lambda_j\big)\big\{\delta_{jk}\lambda_i\lambda_\ell(\delta_{in}\delta_{\ell m}-\delta_{im}\delta_{\ell n})\\
&\qquad-\delta_{ik}\lambda_j\lambda_\ell(\delta_{jn}\delta_{\ell m}-\delta_{jm}\delta_{\ell n})\\
&\qquad+\delta_{j\ell}\lambda_i\lambda_k(\delta_{im}\delta_{kn}-\delta_{in}\delta_{km})\\
&\qquad-\delta_{i\ell}\lambda_j\lambda_k(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})\\
&\qquad+\delta_{jm}\lambda_k\lambda_\ell(\delta_{i\ell}\delta_{kn}-\delta_{ik}\delta_{\ell n})\\
&\qquad-\delta_{im}\lambda_k\lambda_\ell(\delta_{j\ell}\delta_{kn}-\delta_{jk}\delta_{\ell n})\\
&\qquad+\delta_{jn}\lambda_k\lambda_\ell(\delta_{ik}\delta_{\ell m}-\delta_{i\ell}\delta_{km})\\
&\qquad-\delta_{in}\lambda_k\lambda_\ell(\delta_{jk}\delta_{\ell m}-\delta_{j\ell}\delta_{km})\big\}
\end{aligned}$$

for all i,j,k,ℓ,m and n in $\{1,\dots,N\}$. Using this it can be verified that $R\cdot R=fQ(R)$ in p if and only if $\big(R(e_i,e_j)\cdot R\big)(e_i,e_k;e_j,e_k)=f(p)Q(R)(e_i,e_k;e_j,e_k;e_i,e_j)$ for all mutually distinct i,j and k in $\{1,\dots,N\}$, i.e. if and only if

$$\big(f(p)-\lambda_i\lambda_j\big)(\lambda_i-\lambda_j)\lambda_k=0 \tag{3.1}$$

for all mutually distinct i,j and k in $\{1,\dots,N\}$.

Let $\overline{\lambda}_1,\dots,\overline{\lambda}_r$ be the mutually distinct eigenvalues of $A(p)$ and denote their respective multiplicities by $s_1,\dots,s_r$.

If $r=1$, it is clear from (3.1) that $R\cdot R=fQ(R)$ in p.

If $r=2$, it is easy to see from (3.1) that $R\cdot R=fQ(R)$ for $f(p)=\overline{\lambda}_1\overline{\lambda}_2$.

Now suppose that $r\geq 3$ and choose mutually distinct indices α,β and γ in $\{1,\dots,r\}$. Assume that (M,g) satisfies $R\cdot R=fQ(R)$ in p. (3.1) implies that

$$\overline{\lambda}_\beta\big(f(p)-\overline{\lambda}_\alpha\overline{\lambda}_\gamma\big)=0 \tag{3.2}$$

and

$$\overline{\lambda}_\gamma\big(f(p)-\overline{\lambda}_\alpha\overline{\lambda}_\beta\big)=0. \tag{3.3}$$

Subtraction of (3.2) and (3.3) yields that $(\overline{\lambda}_\beta - \overline{\lambda}_\gamma)f(p) = 0$ from which we conclude that $f(p) = 0$ and hence that $R \cdot R = 0$ in p. The converse is trivial (take $f(p) = 0$). This proves Theorem 1.

From Theorem 1 and the fact that a hypersurface of a Euclidean space is conformally flat if and only if it is quasi-umbilical it easily follows that each conformally flat hypersurface of a Euclidean space satisfies $R \cdot R = fQ(R)$. Moreover it is now easy to give examples of non-conformally flat Riemannian manifolds satisfying $R \cdot R = fQ(R)$: in a Euclidean space all hypersurfaces with exactly two principal curvatures with multiplicities at least two provide examples of such manifolds.

4 - Proof of theorem 2

Suppose that $F : (M^N, g) \hookrightarrow E^{N+1}$ is an isometric immersion of a Riemannian manifold. Let p be a point in M and let $\{e_1, \dots, e_N\}$ be a basis for T_pM satisfying (2.3). From (2.4) it is easy to find that

$$\begin{aligned}&\big(R(e_i, e_j) \cdot S\big)(e_k, e_\ell) - f(p)Q(S)(e_k, e_\ell; e_i, e_j) = \\ &= \big(f(p) - \lambda_i\lambda_j\big)(\mu_i - \mu_j)(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk})\end{aligned}$$

for all i, j, k and ℓ in $\{1, \dots, N\}$. It can be verified that $R \cdot S = fQ(S)$ in p if and only if $\big(R(e_i, e_j) \cdot S\big)(e_i, e_j) = f(p)Q(S)(e_i, e_j; e_i, e_j)$ for all distinct i and j in $\{1, \dots, N\}$, i.e. if and only if

$$\big(f(p) - \lambda_i\lambda_j\big)(\lambda_i - \lambda_j)(tr\ A - \lambda_i - \lambda_j) = 0 \tag{4.1}$$

for all distinct i and j in $\{1, \dots, N\}$.

Denote by $\overline{\lambda}_1, \dots, \overline{\lambda}_r$ the mutually distinct eigenvalues of $A(p)$ and let $s_1, \dots, s_r$ be their respective multiplicities. Then $R \cdot S = fQ(S)$ in p if and only if

$$\big(f(p) - \overline{\lambda}_\alpha\overline{\lambda}_\beta\big)(tr\ A - \overline{\lambda}_\alpha - \overline{\lambda}_\beta) = 0 \tag{4.2}$$

for all distinct α and β in $\{1, \dots, r\}$.

If $r = 1$, then $R \cdot S = fQ(S)$ in p.

If $r = 2$, then $R \cdot S = fQ(S)$ in p for $f(p) = \overline{\lambda}_1\overline{\lambda}_2$.

Now assume that $r \geq 3$. Choose mutually distinct indices α, β and γ in $\{1, \dots, r\}$. Suppose that (M, g) satisfies $R \cdot S = fQ(S)$ in p. Since $\overline{\lambda}_\alpha, \overline{\lambda}_\beta$ and $\overline{\lambda}_\gamma$ are mutually distinct we may assume that tr $A - \overline{\lambda}_\alpha - \overline{\lambda}_\beta \neq 0$ and tr $A - \overline{\lambda}_\alpha - \overline{\lambda}_\gamma \neq 0$. (4.2) now implies that $f(p) - \overline{\lambda}_\alpha\overline{\lambda}_\beta = 0$ and $f(p) - \overline{\lambda}_\alpha\overline{\lambda}_\gamma = 0$.

Subtraction yields that $\overline{\lambda}_\alpha = 0$ and hence that $f(p) = 0$, which means that $R \cdot S = 0$. The converse is trivial.

5 - Proof of theorem 3

Suppose that $F : (M^N, g) \hookrightarrow E^{N+1}$ is an isometric immersion of a Riemannian manifold. Let p be a point in M and let $\{e_1, \ldots, e_N\}$ be a basis for T_pM satisfying (2.3). From (2.4) it is easy to obtain that

$$\begin{aligned}
&\big(R(e_i,e_j)\cdot C\big)(e_k,e_\ell;e_m,e_n) - f(p)Q(C)(e_k,e_\ell;e_m,e_n;e_i,e_j) = \\
&= \big(f(p)-\lambda_i\lambda_j\big)\{\delta_{jk}a_{i\ell}(\delta_{in}\delta_{\ell m}-\delta_{im}\delta_{\ell n}) \\
&\qquad - \delta_{ik}a_{j\ell}(\delta_{jn}\delta_{\ell m}-\delta_{jm}\delta_{\ell n}) \\
&\qquad + \delta_{j\ell}a_{ik}(\delta_{im}\delta_{kn}-\delta_{in}\delta_{km}) \\
&\qquad - \delta_{i\ell}a_{jk}(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km}) \\
&\qquad + \delta_{jm}a_{k\ell}(\delta_{i\ell}\delta_{kn}-\delta_{ik}\delta_{\ell n}) \\
&\qquad - \delta_{im}a_{k\ell}(\delta_{j\ell}\delta_{kn}-\delta_{jk}\delta_{\ell n}) \\
&\qquad + \delta_{jn}a_{k\ell}(\delta_{ik}\delta_{\ell m}-\delta_{i\ell}\delta_{km}) \\
&\qquad - \delta_{in}a_{k\ell}(\delta_{jk}\delta_{\ell m}-\delta_{j\ell}\delta_{km})\}
\end{aligned}$$

for all i, j, k, ℓ, m and n in $\{1, \ldots, N\}$. Using this it can be verified that $R \cdot C = fQ(C)$ in p if and only if $\big(R(e_i,e_j)\cdot C\big)(e_i,e_k;e_j,e_k) = f(p)Q(C)(e_i,e_k;e_j,e_k;e_i,e_j)$ for all mutually distinct i, j and k in $\{1, \ldots, N\}$, i.e. if and only if

$$\big(f(p)-\lambda_i\lambda_j\big)(\lambda_i-\lambda_j)\big(tr\ A-\lambda_i-\lambda_j-(N-2)\lambda_k\big) = 0 \tag{5.1}$$

for all mutually distinct i, j and k in $\{1, \ldots, N\}$. Let $\overline{\lambda}_1, \ldots, \overline{\lambda}_r$ be the mutually distinct eigenvalues of A in p and denote their respective multiplicities by $s_1, \ldots, s_r$.

If $r = 1$, it is clear from (5.1) that $R \cdot C = fQ(C)$ in p.

If $r = 2$, it is easy to see from (5.1) that $R \cdot C = fQ(C)$ in p for $f(p) = \overline{\lambda}_1\overline{\lambda}_2$.

Now suppose that $r \geq 3$ and assume that (M, g) satisfies $R \cdot C = fQ(C)$ in p. Choose mutually distinct indices α, β and γ in $\{1, \ldots, r\}$. Since $\overline{\lambda}_\alpha, \overline{\lambda}_\beta$ and $\overline{\lambda}_\gamma$ are mutually distinct we may suppose that tr $A - \overline{\lambda}_\alpha - \overline{\lambda}_\gamma - (N-2)\overline{\lambda}_\beta \neq 0$ and tr $A - \overline{\lambda}_\beta - \overline{\lambda}_\gamma - (N-2)\overline{\lambda}_\alpha \neq 0$. By (5.1) then, we obtain that $f(p) - \overline{\lambda}_\alpha\overline{\lambda}_\gamma = 0$ and $f(p) - \overline{\lambda}_\beta\overline{\lambda}_\gamma = 0$. It follows that $\overline{\lambda}_\gamma = 0$ and also that $f(p) = 0$ and hence $R \cdot C = 0$ in p. The converse is trivial.

Theorems 1 and 3 imply the following.

COROLLARY . — *Let $F : (M^N, g) \hookrightarrow E^{N+1}$ be an isometric immersion of a Riemannian manifold in a Euclidean space. The following conditions are equivalent :*

(i) (M^N, g) satisfies $R \cdot R = fQ(R)$,

(ii) (M^N, g) satisfies $R \cdot C = fQ(C)$.

Proof. — If (M^N, g) satisfies $R \cdot R = fQ(R)$, then (M^n, g) also satisfies $R \cdot S = fQ(S)$ since the derivations $R(X, Y)\cdot$ and $(X \wedge Y)\cdot$ commute with contractions (see Lemma 2.1 from [DDVV]). It is easy to see then that (M^N, g) also satisfies $R \cdot C = fQ(C)$ (use a reasoning similar to the one in part (iii) of Lemma 2.1 in [DDVV]).

Suppose that (M^N, g) satisfies $R \cdot C = fQ(C)$ and let p be a point in M. There are two possibilities : (i) $A(p)$ has at most two distinct eigenvalues, or (ii) $A(p)$ has more than two distinct eigenvalues and $R \cdot C = 0$ in p. In the first case it is clear that $R \cdot R = fQ(R)$ in p by Theorem 1. For the second case, it follows from Proposition 2 from [BVV] that $R \cdot R(p) = 0$ (use formula (3.1) with $f(p) = 0$).

6 - Proof of theorem 4

Suppose that (M^N, J, g) is a Kähler manifold satisfying $R \cdot R = fQ(R)$. Suppose that p is a point in M for which $R \cdot R(p) \neq 0$. We will derive a contradiction.

It is clear that $f(p) \neq 0$. First, observe that

$$Q(R)(u, v; Jz, Jw; x, y) = Q(R)(u, v; z, w; x, y) \tag{6.1}$$

for all $x, y, u, v, z, w \in T_pM$. Indeed, using (2.5) and (2.6),

$$\begin{aligned} Q(R)(u, v; Jz, Jw; x, y) &= \frac{1}{f(p)} \big(R(x, y) \cdot R\big)(u, v; Jz, Jw) \\ &= \frac{1}{f(p)} \big(R(x, y) \cdot R\big)(u, v; z, w) \\ &= Q(R)(u, v; z, w; x, y). \end{aligned}$$

(6.1) and (2.5) imply that

$$\begin{aligned} &R\big(u, v; (x \wedge y)Jz, Jw\big) + R\big(u, v; Jz, (x \wedge y)Jw\big) \\ &\quad - R\big(u, v; (x \wedge y)z, w\big) - R\big(u, v; z, (x \wedge y)w\big) = 0. \end{aligned} \tag{6.2}$$

Let $\{e_1, e_2, \dots, e_N\}$ be an orthonormal basis for T_pM. (6.2) yields that

$$
\begin{aligned}
0 = \sum_{i=1}^{N} \{ & R(u,v;(e_i \wedge y)Jz, Je_i) + R(u,v; Jz(e_i \wedge y)Je_i) \\
& - R(u,v;(e_i \wedge y)z, e_i) - R(u,v; z, (e_i \wedge y)e_i)\} \\
= & \left(\sum_{i=1}^{N} R(u,v;e_i,Je_i) \right) g(Jz,y) - (N-2)R(u,v;z,y)
\end{aligned} \tag{6.3}
$$

for all $u, v, z, y \in T_pM$.

Let $x \in T_pM \setminus \{0\}$. By (6.3)

$$
\begin{aligned}
\left(\sum_{i=1}^{N} R(u,v;e_i,Je_i) \right) g(Jx,Jx) &= (N-2)R(u,v;x,Jx) \\
&= (N-2)R(x,Jx;u,v) \\
&= \left(\sum_{i=1}^{N} R(x,Jx;e_i,Je_i) \right) g(Ju,v)
\end{aligned}
$$

for all $u, v \in T_pM$, which implies that

$$
\sum_{i=1}^{N} R(u,v;e_i,Je_i) = rg(Ju,v), \tag{6.4}
$$

for all $u, v \in T_pM$, where

$$
r = \frac{\sum_{i=1}^{N} R(x,Jx;e_i,Je_i)}{g(Jx,Jx)}
$$

Combination of (6.3) and (6.4) gives that

$$
R(u,v;z,w) = \frac{r}{N-2} g(Ju,v)g(Jz,w) \tag{6.5}
$$

for all $u, v, z, w \in T_pM$. From (6.5) and (2.6) it is easy to see now that $R \cdot R(p) = 0$, which contradicts our initial assumption.

This proves that $R \cdot R = 0$ on M.

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