International Journal of Geometric Methods in Modern Physics
Vol. 9, No. 1 (2012) 1250004 (21 pages)

DOI: 10.1142/S0219887812500041

PSEUDO Z SYMMETRIC RIEMANNIAN MANIFOLDS WITH HARMONIC CURVATURE TENSORS

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Received 23 February 2011
Accepted 27 July 2011
Published

In this paper we introduce a new notion of Z-tensor and a new kind of Riemannian manifold that generalize the concept of both *pseudo* Ricci symmetric manifold and *pseudo* projective Ricci symmetric manifold. Here the Z-tensor is a general notion of the *Einstein gravitational tensor* in General Relativity. Such a new class of manifolds with Z-tensor is named *pseudo Z* symmetric manifold and denoted by $(PZS)_n$. Various properties of such an n-dimensional manifold are studied, especially focusing the cases with *harmonic* curvature tensors giving the conditions of *closeness* of the associated one-form. We study $(PZS)_n$ manifolds with *harmonic* conformal and quasi-conformal curvature tensor. We also show the closeness of the associated one-form when the $(PZS)_n$ manifold becomes *pseudo* Ricci symmetric in *the sense of Deszcz* (see [A. Derdzinsky and C. L. Shen, Codazzi tensor fields, curvature and Pontryagin forms, *Proc. London Math. Soc.* **47**(3) (1983) 15–26; R. Deszcz, On pseudo symmetric spaces, *Bull. Soc. Math. Belg. Ser. A* **44** (1992) 1–34]). Finally, we study some properties of $(PZS)_4$ spacetime manifolds.

Keywords: Pseudo Ricci symmetric manifolds; pseudo projective Ricci symmetric; conformal curvature tensor; quasi-conformal curvature tensor; conformally symmetric; conformally recurrent; Riemannian manifolds.

Mathematics Subject Classification 2010: 53C15, 53C25

1. Introduction

In 1988 Chaki [4] introduced and studied a type of non-flat Riemannian manifold whose Ricci tensor is not identically zero and satisfies the following equation:

$$\nabla_k R_{j\ell} = 2A_k R_{j\ell} + A_j R_{k\ell} + A_\ell R_{kj}. \tag{1.1}$$

Such a manifold is called *pseudo* Ricci symmetric, A_k is a non-null covector called associated 1-form, ∇ is the operator of covariant differentiation with respect to the metric $g_{k\ell}$ and the manifold is denoted by $(PRS)_n$. Here we have defined the Ricci tensor to be $R_{k\ell} = -R^m_{mk\ell}$ [30] and the scalar curvature $R = g^{ij}R_{ij}$. This notion of *pseudo* Ricci symmetric is different from that of Deszcz [14, 15]. In [5] the authors considered conformally flat *pseudo* Ricci symmetric manifolds, where the conformal curvature tensor

$$\begin{aligned} C^m_{jk\ell} = R^m_{jk\ell} &+ \frac{1}{n-2}(\delta^m_j R_{k\ell} - \delta^m_k R_{j\ell} + R^m_j g_{k\ell} - R^m_k g_{j\ell}) \\ &- \frac{R}{(n-1)(n-2)}(\delta^m_j g_{k\ell} - \delta^m_k g_{j\ell}) \end{aligned} \tag{1.2}$$

vanishes, that is, $C^m_{jk\ell} = 0$. It may be scrutinized that the conformal curvature tensor vanishes identically for $n = 3$ [21]. In [2], a $(PRS)_n$ with *harmonic* curvature tensor, that is, $\nabla_m R^m_{jk\ell} = 0$ and with *harmonic* conformal curvature tensor, that is, $\nabla_m C^m_{jk\ell} = 0$ was studied (see [3]).

Such a notion of *harmonic* is related to the co-closeness of the curvature tensor. From this, together with the notion of closeness of the associated 1-form in (1.1), it gives us some important geometric meanings in the theory of *Yang–Mill's Connections*, *Harmonic Mappings* and *Mathematical Physics*, in particular, in *Einstein's Relativity*. From such a point of view, in this paper, we mainly consider the *closeness* of associated 1-forms for some generalized curvature tensors.

On the other hand, Suh, Kwon and Yang [27] introduced the notion of conformal curvature-like tensor on a semi-Riemannian manifold and have given a complete classification of conformally symmetric semi-Riemannian manifold with generalized non-null stress–energy tensor. More generally, Suh and Kwon [25] considered the notion of conformally recurrent semi-Riemannian manifolds with *harmonic* conformal curvature tensor and gave another generalization of conformal symmetric Riemannian manifolds. Moreover, in [26] due to Suh, Kwon and Pyo the importance of the *closeness* for the associated curvature-like 2-form corresponding to each concircular, projective and conformal curvature-like tensor defined on semi-Riemannian manifolds was remarked respectively.

Now let us consider a generalization of condition (1.1) introduced in a paper by Chaki and Saha [9]. They considered the so-called projective Ricci tensor $P_{k\ell}$ obtained by a suitable contraction of the projective curvature tensor $P_{jk\ell m}$ [16]. More precisely, one obtains

$$P_{j\ell} = \frac{n}{n-1}\left(R_{j\ell} - \frac{R}{n}g_{j\ell}\right), \tag{1.3}$$

where $R = g^{ij}R_{ij}$ denotes the scalar curvature.

Obviously $g^{j\ell}P_{j\ell} = 0$. The generalization defined in [9] is thus written as

$$\nabla_k P_{j\ell} = 2A_k P_{j\ell} + A_j P_{k\ell} + A_l P_{kj}. \tag{1.4}$$

This kind of manifold is called *pseudo* projective Ricci symmetric and denoted by $(PPRS)_n$. Recently in [7, 10] a further generalization of condition (1.1) was considered. More precisely a manifold whose Ricci tensor satisfies the condition

$$\nabla_k R_{j\ell} = (A_k + B_k)R_{j\ell} + A_j R_{k\ell} + A_\ell R_{kj} \tag{1.5}$$

is defined. Such a manifold is called almost *pseudo* Ricci symmetric and denoted by $(APRS)_n$. Here A_k and B_k are non-null covectors.

In [10] the properties of conformally flat $(APRS)_n$ are studied and the authors pointed out the importance of *pseudo* Ricci symmetric manifolds in the theory of General Relativity. It is therefore worthwhile to undertake the study of an n-dimensional manifold that generalizes both the $(PRS)_n$ and the $(PPRS)_n$ manifolds. In this paper we define a generalized $(0, 2)$ symmetric Z-tensor given by

$$Z_{k\ell} = R_{k\ell} + \phi g_{k\ell}, \tag{1.6}$$

where ϕ denotes an arbitrary scalar function. It is worth to notice that the Z-tensor allows us to reinterpret many well-known structures on Riemannian manifolds. In fact one can check that a Z flat Riemannian manifold is simply an Einstein space.

If a Z recurrent Riemannian manifold is considered i.e. a space satisfying the condition

$$\nabla_i Z_{k\ell} = \lambda_i Z_{k\ell}$$

one can easily find that this condition is equivalent to

$$\nabla_i R_{k\ell} = \lambda_i R_{k\ell} + (n-1)\mu_i g_{k\ell}$$

with the choice $(n-1)\mu_i = \lambda_i\phi - \nabla_i\phi$. So the manifold reduces to a *generalized Ricci recurrent manifold* [12] and if $\lambda_i\phi - \nabla_i\phi = 0$ a Ricci recurrent manifold is recovered (see also [19]).

If we consider a Riemannian manifold with Z-tensor of *Codazzi type* (see [13]), that is, with the property

$$\nabla_k Z_{j\ell} = \nabla_j Z_{k\ell}$$

one can easily find that this condition is equivalent to

$$\nabla_k R_{j\ell} - \nabla_j R_{k\ell} = (\nabla_j\phi)g_{k\ell} - (\nabla_k\phi)g_{j\ell}.$$

Transvecting the previous relation with $g^{j\ell}$ we get $\nabla_k(R + 2(n-1)\phi) = 0$ and finally

$$\nabla_k R_{j\ell} - \nabla_j R_{k\ell} = \frac{1}{2(n-1)}[(\nabla_k R)g_{j\ell} - (\nabla_j R)g_{k\ell}].$$

A manifold with Z-tensor of *Codazzi type* is thus a nearly conformal symmetric manifold $(NCS)_n$: this condition was introduced and studied by Roter [22] and Suh, Kwon and Yang [27]. Conversely, an $(NCS)_n$ manifold has a Z-tensor of *Codazzi type* if the condition $\nabla_k(R + 2(n-1)\phi) = 0$ is satisfied.

One can observe that the n-dimensional Einstein equations with *cosmological constant* Λ may be written in the same form $Z_{k\ell} = kT_{k\ell}$ with the choice

$\phi = -\frac{R}{2} + \Lambda$. This choice comes naturally from the condition $\nabla^\ell T_{k\ell} = 0$ for the stress–energy tensor and the second Bianchi identity that gives $\nabla_k(\frac{R}{2} + \phi) = 0$. In this case the vacuum solution $Z = 0$ implies $\Lambda = R\frac{n-2}{2n}$ and thus an Einstein space. So the generalized Z-tensor may be thought as a *generalized Einstein gravitational tensor* with arbitrary scalar function ϕ.

Finally one can also notice that Einstein's equations in General Relativity $(n = 4)$ can be written in the suggestive form

$$Z_{k\ell} = kT_{k\ell}$$

with the choice $\phi = -\frac{R}{2}$ without *cosmological constant*, that is, $\Lambda = 0$. In this case the tensor Z is said to be the *Einstein gravitational tensor*, T the *stress–energy tensor* and k a certain *gravitational constant* of a spacetime M, respectively. So for example the condition $\nabla_i Z_{k\ell} = 0$ describes a spacetime in which the stress–energy tensor is constant. When $T = 0$, then the tensor $Z = 0$, i.e. the spacetime M could be Ricci flat and M is said to be a vacuum (or empty) (see [20]).

In this paper we introduce a generalization of the condition $\nabla_i Z_{kl} = 0$ mentioned above. In this way we will extend the limit of validity of the properties of *pseudo* Ricci symmetric manifolds using this *generalized Einstein gravitational tensor*. More precisely, we introduce a new kind of Riemannian manifold whose non-null generalized Z tensor satisfies the following condition:

$$\nabla_k Z_{j\ell} = 2A_k Z_{j\ell} + A_j Z_{k\ell} + A_\ell Z_{kj}. \tag{1.7}$$

Such a manifold is called *pseudo* Z symmetric and denoted by $(PZS)_n$. It is worth to notice that if $\phi = 0$, we recover a *pseudo* Ricci symmetric manifold, that is, a $(PRS)_n$ manifold, while if $Z = g^{k\ell} Z_{k\ell} = R + n\phi = 0$, one has $\phi = -\frac{R}{n}$ and so we recover the classical Z tensor with $Z_{j\ell} = R_{j\ell} - \frac{R}{n} g_{j\ell} = \frac{n-1}{n} P_{j\ell}$. Thus the space reduces to a *pseudo* projective Ricci symmetric manifold, that is, a $(PPRS)_n$ manifold. Hereafter we call the generalized Z tensor simply Z tensor.

It is well known that in *pseudo* Ricci symmetric Riemannian manifold the condition $A_k R = \frac{1}{2}\nabla_k R$ is true giving a closed 1-form A_k in (1.1). From such a view point and the motivations mentioned above, in our paper we study in more detail the properties of *pseudo* Z symmetric manifold focusing our attention to peculiar conditions that give rise to the closeness of the associated 1-form A_k in (1.7).

In particular, we will note that these conditions naturally arise from a $(PZS)_n$ manifold endowed with *harmonic* curvature tensors: the case with *harmonic* conformal curvature tensor will be studied in a special way. Moreover, we will point out how these conditions depend on the choice of the scalar function ϕ in (1.6).

2. Elementary Properties of a $(PZS)_n$ Manifold

In this section elementary properties of a $(PZS)_n$ are shown. Let M be a non-flat n-dimensional $(n \geq 4)$ $(PZS)_n$ Riemannian manifold with metric g_{ij} and Riemannian connection ∇. We can state the following simple theorem.

Theorem 2.1. *Let M be an n-dimensional Riemannian $(PZS)_n$ manifold. If the scalar function ϕ satisfies the following differential equation*:

$$(\nabla_k\phi)g_{j\ell} = \phi(2A_k g_{j\ell} + A_j g_{k\ell} + A_\ell g_{kj}), \tag{2.1}$$

then a $(PZS)_n$ manifold reduces to a $(PRS)_n$ one.

Proof. The proof follows from a straightforward calculation by inserting the definition of the generalized Z-tensor in Eq. (1.7). If $R_{jl} = 0$, this condition is verified and the manifold is reduced to a trivial pseudo Ricci symmetric one. □

Now we point out some useful formulas concerning $(PZS)_n$ manifolds. Transvecting Eq. (1.7) with $g^{j\ell}$ gives immediately:

$$\nabla_k Z = 2A_k Z + 2A^\ell Z_{k\ell}. \tag{2.2}$$

In the same manner transvecting Eq. (1.7) with $g^{k\ell}$ and using the relation $\nabla^\ell Z_{j\ell} = \frac{1}{2}\nabla_j R + \nabla_j\phi$ coming from the contracted second Bianchi identity one obtains

$$\nabla_k R + 2\nabla_k\phi = 2A_k Z + 6A^\ell Z_{k\ell}. \tag{2.3}$$

By using both Eqs. (2.2) and (2.3) after a straightforward calculation, one obtains the following results:

$$A^\ell Z_{k\ell} = \frac{2-n}{4}\nabla_k\phi, \tag{2.4}$$

and

$$A_k Z = \frac{n-2}{4}\nabla_k\phi + \frac{1}{2}\nabla_k Z. \tag{2.5}$$

The last equations are the generalization of the correspondent results given in [4, 9]. In fact we can state the following simple remarks.

Remark 2.1. If $\nabla_k\phi = 0$ with $Z \neq 0$ and $\nabla_k R \neq 0$, one has $A^\ell Z_{k\ell} = 0$. That is $A^\ell R_{k\ell} = -\phi A_k$ and $A_k Z = \frac{1}{2}\nabla_k Z$. So A_k is a closed 1-form and it is an eigenvector of the Ricci tensor with eigenvalue $-\phi$. In particular if $\phi = 0$, the results given in [4] are recovered. We have shown that similar results are valid in more general conditions.

Remark 2.2. If $Z = 0$, then $\phi = -\frac{R}{n}$. And by a simple calculation we have $\nabla_k R = \nabla_k\phi = 0$. Furthermore, we have $A^\ell R_{k\ell} = \frac{R}{n}A_k$. So we have obtained that the scalar curvature is a covariant constant and that A_k is an eigenvector of the Ricci tensor with eigenvalue $\frac{R}{n}$. These are the results given in [9].

Remark 2.3. If the scalar curvature is constant, that is, $\nabla_k R = 0$ with $Z \neq 0$ and $\nabla_k\phi \neq 0$, one has $\nabla_k Z = n\nabla_k\phi$. Then from Eq. (2.5) it follows immediately that $A_k Z = \frac{3n-2}{4n}\nabla_k Z$. This means that A_k is a closed 1-form.

Remark 2.4. Let M be an n-dimensional Riemannian $(PZS)_n$ manifold with $Z \neq 0$ and $\nabla_k \phi \neq 0$ having harmonic curvature tensor which satisfies the property $\nabla_m R^m_{jk\ell} = 0$. Then A_k becomes a closed 1-form.

Proof. The contracted second Bianchi identity is invoked:

$$\nabla_m R^m_{jk\ell} = \nabla_k R_{j\ell} - \nabla_j R_{k\ell}.$$

Transvecting this with $g^{k\ell}$, one obtains $\nabla_j R = 0$. So we are in the hypothesis of Remark 2.3. It is well known that if the space is locally symmetric $\nabla_i R^m_{jk\ell} = 0$ [19] or Ricci symmetric $\nabla_i R_{k\ell} = 0$ [24] and if the Ricci tensor is of Codazzi type, i.e. $\nabla_k R_{j\ell} = \nabla_j R_{k\ell}$, then one has also $\nabla_m R^m_{jk\ell} = 0$. □

Remark 2.5. Let M be an n-dimensional Riemannian $(PZS)_n$ manifold whose Z-tensor is of Codazzi type, that is, $\nabla_k Z_{j\ell} = \nabla_j Z_{k\ell}$. Then: (1) $Z_{j\ell}$ must be a singular tensor, and (2) if $\nabla \phi = 0$ we have $Z_{j\ell} = 0$ (a trivial $(PZS)_n$ manifold) that is an Einstein space.

Proof. (1) From (1.7) and the condition $\nabla_k Z_{j\ell} = \nabla_j Z_{k\ell}$ we can easily find $A_k Z_{j\ell} = A_j Z_{k\ell}$. If the Z-tensor is non-singular, i.e. if $\det(Z_{j\ell}) \neq 0$, there exists a tensor $(Z^{-1})^{s\ell}$ of type $(2,0)$ with the property $(Z_{j\ell})(Z^{-1})^{s\ell} = \delta^s_j$. Thus we have

$$A_k Z_{j\ell}(Z^{-1})^{s\ell} = A_j Z_{k\ell}(Z^{-1})^{s\ell}$$

and so $A_k \delta^s_j = A_j \delta^s_k$. This gives finally $A_k = 0$. But the 1-form A_k is supposed to be non-null: thus we must have a singular Z-tensor.

(2) From (2.4) and being $\nabla_k \phi = 0$ we have $A^l Z_{k\ell} = 0$ and thus $A^k A_k Z_{j\ell} = A_j A^k Z_{k\ell} = 0$ from which $Z_{j\ell} = 0$. This contradicts the definition of a $(PZS)_n$ manifold and so such a kind of manifold can never exist. □

Now we consider other curvature tensors $K^m_{jk\ell}$ with the usual symmetries of the Riemann tensor satisfying the first Bianchi identity. We can thus state the following theorem.

Theorem 2.2. *Let M be an n-dimensional Riemannian manifold having a generalized curvature tensor $K^m_{jk\ell}$ with the property*:

$$\nabla_m K^m_{jk\ell} = a\nabla_m R^m_{jk\ell} + b[(\nabla_j R)g_{k\ell} - (\nabla_k R)g_{j\ell}], \tag{2.6}$$

where a and b are constants. If $\nabla_m K^m_{jk\ell} = 0$ and the condition $b \neq \frac{a}{2(n-1)}$ is satisfied, then the scalar curvature R is a covariant constant, that is, $\nabla_k R = 0$.

Proof. Transvecting Eq. (2.6) with $g^{k\ell}$ and using the second contracted Bianchi identity, one easily obtains $(\nabla_j R)[\frac{1}{2}a - (n-1)b] = 0$ from which one concludes immediately. □

Corollary 2.1. *Let M be an n-dimensional Riemannian $(PZS)_n$ manifold with $Z \neq 0$ and $\nabla_k \phi \neq 0$ having a generalized curvature tensor which satisfies the property* (2.6). *If $\nabla_m K^m_{jk\ell} = 0$, then A_k is a closed 1-form.*

Proof. It follows immediately from Remark 2.3 and Theorem 2.2. □

Some curvature tensors $K^m_{jk\ell}$ with the property (2.6) are well known. We give its examples as follows: the projective curvature tensor $P_{jk\ell}{}^m$ [16], the conformal curvature tensor $C^m_{jk\ell}$ [26, 27], the concircular curvature tensor $\tilde{C}^m_{jk\ell}$ [28], the conharmonic curvature tensor $N^m_{jk\ell}$ [24] and the quasi-conformal curvature tensor $W^m_{jk\ell}$ [29]. So we can state the following corollaries whose proofs are very similar.

Corollary 2.2. *Let M be an n-dimensional Riemannian $(PZS)_n$ manifold with $Z \neq 0$ and $\nabla_k \phi \neq 0$ having harmonic projective curvature tensor, that is, the property $\nabla_m P_{jk\ell}{}^m = 0$. Then A_k becomes a closed 1-form.*

Proof. The components of the projective curvature tensor are defined as (see [16, 24]):

$$P^m_{jk\ell} = R^m_{jk\ell} + \frac{1}{n-1}(\delta^m_j R_{k\ell} - \delta^m_k R_{j\ell}). \tag{2.7}$$

Applying the operator of covariant derivative to the previous equation and recalling the second contracted Bianchi identity, one obtains

$$\nabla_m P^m_{jk\ell} = \frac{n-2}{n-1}\nabla_m R^m_{jk\ell}. \tag{2.8}$$

Thus we are in the condition of Theorem 2.2 and Corollary 2.1. □

Corollary 2.3. *Let M be an n-dimensional Riemannian $(PZS)_n$ manifold with $Z \neq 0$ and $\nabla_k \phi \neq 0$ having harmonic concircular curvature tensor, that is, satisfying the property $\nabla_m \tilde{C}^m_{jk\ell} = 0$. Then A_k is a closed 1-form.*

Proof. The components of the concircular curvature tensor are defined as (see [26, 28]):

$$\tilde{C}^m_{jk\ell} = R^m_{jk\ell} + \frac{R}{n(n-1)}(\delta^m_j g_{k\ell} - \delta^m_k g_{j\ell}). \tag{2.9}$$

Applying the operator of covariant derivative to the previous equation and considering the second contracted Bianchi identity, one obtains

$$\nabla_m \tilde{C}^m_{jk\ell} = \nabla_m R^m_{jk\ell} + \frac{1}{n(n-1)}[(\nabla_j R) g_{k\ell} - (\nabla_k R) g_{j\ell}]. \tag{2.10}$$

Thus we are in the condition of Theorem 2.2 and Corollary 2.1. □

Corollary 2.4. *Let M be an n-dimensional Riemannian $(PZS)_n$ manifold with $Z \neq 0$ and $\nabla_k \phi \neq 0$ having harmonic conharmonic curvature tensor, that is, the property $\nabla_m N^m_{jk\ell} = 0$. Then A_k becomes a closed 1-form.*

Proof. The components of the conharmonic curvature tensor are defined as (see [24]):

$$N^m_{jk\ell} = R^m_{jk\ell} + \frac{1}{n-2}(\delta^m_j R_{k\ell} - \delta^m_k R_{j\ell} + R^m_j g_{k\ell} - R^m_k g_{j\ell}). \tag{2.11}$$

Applying the operator of covariant derivative to the previous equation and considering the second contracted Bianchi identity, one obtains

$$\nabla_m N^m_{jk\ell} = \frac{n-3}{n-2}\nabla_m R^m_{jk\ell} + \frac{1}{2(n-2)}[(\nabla_j R)g_{k\ell} - (\nabla_k R)g_{j\ell}]. \tag{2.12}$$

Thus we are in the condition of Theorem 2.2 and Corollary 2.1. □

Now we focus on Eq. (2.5). The operation of covariant derivation is applied on it and the following relation is obtained:

$$(\nabla_j A_k)Z + A_k(\nabla_j Z) = \frac{n-2}{4}\nabla_j\nabla_k\phi + \frac{1}{2}\nabla_j\nabla_k Z. \tag{2.13}$$

Now a similar equation with indices k and j exchanged is written and then subtracted from (2.13) to obtain

$$(\nabla_j A_k - \nabla_k A_j)Z + A_k(\nabla_j Z) - A_j(\nabla_k Z) = 0. \tag{2.14}$$

This result will have a discrete consequence in the properties of a $(PZS)_n$ manifold. Equation (2.5) is substituted in the previous relation and one has immediately:

$$(\nabla_j A_k - \nabla_k A_j)Z + \frac{2-n}{2}[A_k(\nabla_j\phi) - A_j(\nabla_k\phi)] = 0. \tag{2.15}$$

We can then state the following theorem.

Theorem 2.3. *Let M be an n-dimensional Riemannian $(PZS)_n$ manifold with $Z \neq 0$ and $\nabla_k\phi \neq 0$. Then A_k is a closed 1-form if and only if A_k and $\nabla_k\phi \neq 0$ are co-directional.*

3. The Manifold $(PZS)_n$ with Cyclic Ricci and Z-Tensors

In this section we consider the properties of a $(PZS)_n$ manifold having cyclic Ricci and Z-tensors. An n-dimensional Riemannian manifold is said to be cyclic Ricci tensor if the condition:

$$\nabla_k R_{j\ell} + \nabla_j R_{k\ell} + \nabla_\ell R_{kj} = 0 \tag{3.1}$$

holds. According to [6], this implies $\nabla_k R = 0$. So the following theorem also holds.

Theorem 3.1. *Let M be an n-dimensional Riemannian $(PZS)_n$ manifold with $Z \neq 0$ and $\nabla_k\phi \neq 0$ having cyclic Ricci tensor. Then A_k is a closed 1-form.*

Now an analogous definition of cyclic Z-tensor is introduced. An n-dimensional manifold is said to be cyclic Z-tensor if the following condition holds:

$$\nabla_k Z_{j\ell} + \nabla_j Z_{k\ell} + \nabla_\ell Z_{kj} = 0. \tag{3.2}$$

The previous equation is now transvected with $g^{j\ell}$ to give

$$\nabla_k Z + \nabla_k R + 2\nabla_k \phi = 0. \tag{3.3}$$

Recalling the relation $Z = R + n\phi$ the previous equation can be transformed in the following ones:

$$\nabla_k R = -\frac{n+2}{2}\nabla_k \phi, \tag{3.4}$$

and

$$\nabla_k Z = \frac{n-2}{2}\nabla_k \phi. \tag{3.5}$$

This result is then substituted in Eq. (2.5) to obtain

$$A_k Z = \frac{n-2}{4}\frac{2}{n-2}\nabla_k Z + \frac{1}{2}\nabla_k Z = \nabla_k Z. \tag{3.6}$$

We can thus state the following theorem.

Theorem 3.2. *Let M be an n-dimensional Riemannian $(PZS)_n$ manifold with $Z \neq 0$ and $\nabla_k \phi \neq 0$ having cyclic Z-tensor. Then A_k is a closed 1-form.*

4. Pseudo *Z* Symmetric Manifolds with Harmonic Conformal and Quasi-Conformal Curvature Tensors

In this section an n-dimensional $(n > 3)$ $(PZS)_n$ Riemannian manifold with the property $\nabla_m C^m_{jk\ell} = 0$ and $\nabla_m W^m_{jk\ell} = 0$ is considered. In other words, we study about a $(PZS)_n$ with *harmonic* conformal curvature tensor and *harmonic* quasi-conformal curvature tensor [3]. It is well known that the divergence of the conformal tensor satisfies the relation:

$$\nabla_m C^m_{jk\ell} = \frac{n-3}{n-2}\left[\nabla_m R^m_{jk\ell} + \frac{1}{2(n-1)}\{(\nabla_j R)g_{k\ell} - (\nabla_k R)g_{j\ell}\}\right]. \tag{4.1}$$

So if we consider $\nabla_m C_{jkl}{}^m = 0$, one immediately obtains

$$\nabla_m R^m_{jk\ell} = \nabla_k R_{j\ell} - \nabla_j R_{k\ell} = \frac{1}{2(n-1)}[(\nabla_k R)g_{j\ell} - (\nabla_j R)g_{k\ell}]. \tag{4.2}$$

This equation does not match with the hypothesis of Theorem 2.2. So we cannot conclude that A_k is a closed form in this way. From the contracted second Bianchi identity and from the definition of the Z-tensor $Z_{k\ell} = R_{k\ell} + \phi g_{k\ell}$ the following equation can be written as

$$\nabla_m R^m_{jk\ell} = \nabla_k Z_{j\ell} - \nabla_j Z_{k\ell} + [(\nabla_j \phi)g_{k\ell} - (\nabla_k \phi)g_{j\ell}]. \tag{4.3}$$

On the other hand, from the definition of a $(PZS)_n$ manifold one easily finds that

$$\nabla_k Z_{j\ell} - \nabla_j Z_{k\ell} = A_k Z_{j\ell} - A_j Z_{k\ell}. \tag{4.4}$$

In this way considering Eqs. (4.2)–(4.4) one can see that the following relation holds for a $(PZS)_n$ manifold with *harmonic* conformal curvature tensor:

$$A_k Z_{j\ell} - A_j Z_{k\ell} = \frac{1}{2(n-1)}[\nabla_k(R+2(n-1)\phi)g_{j\ell} - \nabla_j(R+2(n-1)\phi)g_{k\ell}]. \tag{4.5}$$

This is the starting point for the proofs of the most important properties of a $(PZS)_n$ manifold having *harmonic* conformal curvature tensor, that is, $\nabla_m C^m_{jkl} = 0$.

We note that the condition $\nabla_m C^m_{jkl} = 0$ implies that the manifold is a $(NCS)_n$ one: so if the condition $\nabla_k(R + 2(n-1)\phi) = 0$ is satisfied, the Z-tensor becomes a Codazzi tensor from (4.4) and (4.5). In order to avoid Z_{kl} to be singular we will suppose $\nabla_k(R + 2(n-1)\phi) \neq 0$ when we consider Remark 2.5 in the case $\nabla_m C^m_{jkl} = 0$. Moreover, we can state the following remark.

Remark 4.1. A nearly conformal symmetric $(PZS)_n$ Riemannian manifold with $R = h\phi$ being h and ϕ constants can never exist.

Proof. In fact the condition $R = h\phi$ implies $\nabla_k(R + 2(n-1)\phi) = 0$. From this, together with (4.5) it follows that

$$A_k Z_{j\ell} = A_j Z_{k\ell}$$

so the Z-tensor is of Codazzi type and by Remark 2.5 being $\nabla\phi = 0$ we achieve $Z_{j\ell} = 0$. This contradicts the definition of a $(PZS)_n$ manifold. This result generalizes the previous one obtained in [4]. □

Now we can state the following fundamental result.

Theorem 4.1. *Let M be an n-dimensional $(n > 3)$ $(PZS)_n$ Riemannian manifold with the property $\nabla_m C^m_{jk\ell} = 0$. If the tensor $Z_{k\ell}$ is non-singular, then A_k is a closed 1-form.*

Proof. By performing the covariant derivative of Eq. (4.5) one easily obtains

$$(\nabla_i A_k)Z_{j\ell} + A_k(\nabla_i Z_{j\ell}) - (\nabla_i A_j)Z_{k\ell} - A_j(\nabla_i Z_{k\ell}) = \frac{1}{2(n-1)}[(\nabla_i\nabla_k\rho)g_{j\ell} - (\nabla_i\nabla_j\rho)g_{k\ell}] \tag{4.6}$$

where $\rho = R + 2(n-1)\phi$ denotes a scalar function. Now a cyclic permutation of the indices i, j, k is performed and the resulting three equations are added to obtain

$$(\nabla_i A_k - \nabla_k A_i)Z_{j\ell} + (\nabla_j A_i - \nabla_i A_j)Z_{k\ell} + (\nabla_k A_j - \nabla_j A_k)Z_{i\ell} + A_j(\nabla_k Z_{i\ell} - \nabla_i Z_{k\ell}) + A_k(\nabla_i Z_{j\ell} - \nabla_j Z_{i\ell}) + A_i(\nabla_j Z_{k\ell} - \nabla_k Z_{j\ell}) = 0. \tag{4.7}$$

Inserting Eq. (4.4) in the previous result one easily writes

$$(\nabla_i A_k - \nabla_k A_i)Z_{j\ell} + (\nabla_j A_i - \nabla_i A_j)Z_{k\ell} + (\nabla_k A_j - \nabla_j A_k)Z_{i\ell} = 0. \tag{4.8}$$

Now if the Z-tensor is non-singular, i.e. if $\det(Z_{k\ell}) \neq 0$, then there exists a $(2,0)$ tensor $(Z^{-1})^{km}$ with the property $Z_{k\ell}(Z^{-1})^{km} = \delta^m_\ell$. Thus the previous equation is multiplied by $(Z^{-1})^{h\ell}$ to obtain

$$(\nabla_i A_k - \nabla_k A_i)\delta^h_j + (\nabla_j A_i - \nabla_i A_j)\delta^h_k + (\nabla_k A_j - \nabla_j A_k)\delta^h_i = 0. \tag{4.9}$$

Now we put $h = j$ and sum to obtain

$$(n-2)(\nabla_i A_k - \nabla_k A_i) = 0. \tag{4.10}$$

We can thus conclude that if $n > 2$, then A_k is a closed 1-form. □

Now we consider the case of harmonic quasi-conformal curvature tensor. In 1968 Yano and Sawaki [29] defined and studied a tensor $W_{jk\ell}^m$ on a Riemannian manifold of dimension n, which includes both the conformal curvature tensor $C_{jk\ell}^m$ and the concircular curvature tensor $\tilde{C}_{jk\ell}^m$ as particular cases. This tensor is known as *quasi-conformal curvature tensor* and its components are given by

$$W_{jk\ell}^m = -(n-2)bC_{jk\ell}^m + [a + (n-2)b]\tilde{C}_{jk\ell}^m. \tag{4.11}$$

In the previous equation $a \neq 0$, $b \neq 0$ are constants and $n > 3$ since the conformal curvature tensor vanishes identically for $n = 3$. A non-flat manifold has the harmonic quasi-conformal curvature tensor if $\nabla_m W_{jk\ell}^m = 0$. If the equations for $\nabla_m C_{jk\ell}^m = 0$ and $\nabla_m \tilde{C}_{jk\ell}^m = 0$ are employed and the covariant derivative with respect to the index m is applied on the definition of quasi-conformal curvature tensor, one obtains straightforwardly:

$$\begin{aligned}\nabla_m W_{jk\ell}^m &= (a+b)\nabla_m R_{jk\ell}^m \\ &\quad + \frac{2a - b(n-1)(n-4)}{2n(n-1)}[(\nabla_j R)g_{k\ell} - (\nabla_k R)g_{j\ell}].\end{aligned} \tag{4.12}$$

Now if $\nabla_m W_{jk\ell}^m = 0$, transvecting the previous equation with $g^{k\ell}$ after some calculations it follows that

$$(n-2)\frac{a + b(n-2)}{n}(\nabla_j R) = 0. \tag{4.13}$$

This means that $\nabla_j R = 0$ or $a + b(n-2) = 0$. Inserting this last result in (4.12), we recover easily Eq. (4.2). If $\nabla_j R = 0$, Remark 2.4 is invoked. Then it follows that A_k is a closed 1-form. On the other hand, if Eq. (4.2) is valid, Theorem 4.1 is used. Thus we can state the following theorem.

Theorem 4.2. *Let M be an n-dimensional $(n > 3)$ $(PZS)_n$ Riemannian manifold with the property $\nabla_m W_{jk\ell}^m = 0$. If the tensor $Z_{k\ell}$ is non-singular, then A_k is a closed 1-form.*

Now we follow the procedure explained in [10] to point out other properties of a $(PZS)_n$ manifold. Transvecting Eq. (4.5) with $g^{k\ell}$ gives

$$A_j Z - A^m Z_{jm} = \frac{1}{2}\nabla_j (R + 2(n-1)\phi). \tag{4.14}$$

Inserting this result in (4.5), one can write the following relation:

$$A_k Z_{j\ell} - A_j Z_{k\ell} = \frac{1}{(n-1)}[(A_k Z - A^m Z_{km})g_{j\ell} - (A_j Z - A^m Z_{jm})g_{k\ell}]. \tag{4.15}$$

Transvecting the previous equation with A^ℓ, one straightforwardly obtains

$$A_k A^\ell Z_{j\ell} = A_j A^\ell Z_{k\ell}. \tag{4.16}$$

Again we multiply the previous equation by A^j to obtain

$$A_k A^j A^\ell Z_{j\ell} = A_j A^j A^\ell Z_{k\ell}. \tag{4.17}$$

This last can be rewritten as

$$A^\ell Z_{k\ell} = \frac{A_k A^j A^\ell Z_{j\ell}}{A_j A^j} = t A_k, \tag{4.18}$$

where $t = \frac{A^j A^\ell Z_{j\ell}}{A_j A^j}$ is a scalar function. We have just proved the following theorem that generalizes a similar result in [10] for an $(APRS)_n$ Riemannian manifold.

Theorem 4.3. *Let M be an n-dimensional $(n > 3)$ $(PZS)_n$ Riemannian manifold with the property $\nabla_m C^m_{jk\ell} = 0$. Then the vector A^ℓ is an eigenvector of the $Z_{k\ell}$ tensor with eigenvalue t.*

Inserting (4.18) in Eq. (4.14), one easily obtains

$$A_k(t - Z) = -\frac{1}{2}\nabla_k (R + 2(n-1)\phi). \tag{4.19}$$

This result is again a natural generalization of a similar equation given in [10] for an $(APRS)_n$ Riemannian manifold.

Now transvecting Eq. (4.5) with A^j and using the result (4.18), one straightforwardly shows that the following equation holds:

$$R_{k\ell} = \frac{A_k A_\ell}{A_j A^j}\left[\frac{nt - Z}{n-1}\right] + g_{k\ell}\left[\frac{Z - t}{n-1} - \phi\right]. \tag{4.20}$$

In such a case a Riemannian manifold is said to be quasi-Einstein (see [8]). This result can be written in the more compact form:

$$R_{k\ell} = \alpha g_{k\ell} + \beta T_k T_\ell, \tag{4.21}$$

where $\alpha = \frac{Z-t}{n-1} - \phi$, $\beta = \frac{nt-Z}{n-1}$ are the associated scalars and $T_k = \frac{A_k}{\sqrt{A_j A^j}}$ is naturally a unit covector. We have just proved the following theorem.

Theorem 4.4. *Let M be an n-dimensional $(n > 3)$ $(PZS)_n$ Riemannian manifold with the property $\nabla_m C^m_{jk\ell} = 0$. Then M is quasi-Einstein.*

Now from Eqs. (4.18) and (2.4) one immediately writes

$$A_k t = \frac{2-n}{4}\nabla_k \phi. \tag{4.22}$$

The operation of covariant derivation is applied on the previous result and the following relation is obtained:

$$(\nabla_j A_k)t + A_k \nabla_j t = \frac{2-n}{4}\nabla_j \nabla_k \phi. \tag{4.23}$$

Now a similar equation with indices k and j exchanged is written and then subtracted from (4.23) to obtain finally:

$$(\nabla_j A_k - \nabla_k A_j)t = A_j \nabla_k t - A_k \nabla_j t. \tag{4.24}$$

We can thus state the following theorem.

Theorem 4.5. *Let M be an n-dimensional $(n > 3)$ $(PZS)_n$ Riemannian manifold with the property $\nabla_m C^m_{jk\ell} = 0$. Then the relation (4.24) holds.*

At this point it is worth to note the following geometric remark.

Remark 4.2. (1) If $\phi = 0$, we recover the $(PRS)_n$ manifold. Thus from (4.22) it follows that $t = 0$ and being $Z = R$, Eq. (4.20) takes the form:

$$R_{k\ell} = -\frac{A_k A_\ell}{A_j A^j}\left[\frac{R}{n-1}\right] + g_{k\ell}\left[\frac{R}{n-1}\right]. \tag{4.25}$$

So the manifold is quasi-Einstein with opposite associated scalars.

(2) If $\phi = -\frac{R}{n}$, then $Z = 0$. Thus we recover the $(PPRS)_n$ manifold. And from Remark 2.2 we easily obtain that $\nabla_k R = \nabla_k \phi = 0$. So again we have $t = 0$ and thus finally $R_{k\ell} = \frac{R}{n} g_{k\ell}$, that is, the manifold is Einstein.

According to Theorem 4.3, a $(PZS)_n$ Riemannian manifold with the property $\nabla_m C^m_{jk\ell} = 0$ is quasi-Einstein, that is, the Ricci tensor satisfies $R_{k\ell} = \alpha g_{k\ell} + \beta T_k T_\ell$ (see [8]). If $Z_{k\ell}$ is non-singular, the covector A_k is closed and by Eq. (4.24) we have $A_j(\nabla_k t) = A_k(\nabla_j t)$. This is taken in conjunction with the equation $A_j(\nabla_k Z) = A_k(\nabla_j Z)$ coming in the same situation from Eq. (2.14). One easily obtains the following relation:

$$A_j\left(\nabla_k \frac{nt - Z}{n-1}\right) = A_k\left(\nabla_j \frac{nt - Z}{n-1}\right). \tag{4.26}$$

Thus multiplying the previous result by $\frac{1}{\sqrt{A_j A^j}}$ and considering Theorem 4.3 we can state the following corollary.

Corollary 4.1. *Let M be an n-dimensional $(n > 3)$ $(PZS)_n$ Riemannian manifold with the property $\nabla_m C^m_{jk\ell} = 0$. Then the manifold M is quasi-Einstein. Moreover, if the tensor $Z_{k\ell}$ is non-singular, the following holds*

$$T_j(\nabla_k \beta) = T_k(\nabla_j \beta). \tag{4.27}$$

5. Conformally Flat Pseudo Z Symmetric Manifolds: Local Form of the Metric Tensor

In this section we study in depth conformally flat $(PZS)_n$ manifold. In particular we point out the existence of a proper concircular vector in such a manifold and give the local form of the metric tensor. It is worth to notice that the proof in the present paper is based only on the request of a non-singular Z-tensor. First we recall the following theorem whose proof is different from that of [11].

Theorem 5.1. *Let M be an n-dimensional $(n > 3)$ manifold whose Ricci tensor is given by $R_{k\ell} = \alpha g_{k\ell} + \beta T_k T_\ell$ where T_k is a unit vector. If the manifold is conformally flat and the condition $T_j(\nabla_k\beta) = T_k(\nabla_j\beta)$ is satisfied, then T_k is a proper concircular vector.*

Proof. If the manifold is conformally flat, then the following naturally holds:

$$\nabla_k R_{j\ell} - \nabla_j R_{k\ell} = \frac{1}{2(n-1)}[(\nabla_k R)g_{j\ell} - (\nabla_j R)g_{k\ell}]. \tag{5.1}$$

Equation (4.21) is then substituted in previous relation and the operations of covariant differentiation are performed to give straightforwardly:

$$(\nabla_k\beta)T_jT_\ell + \beta(\nabla_kT_j)T_\ell + \beta T_j(\nabla_kT_\ell) - (\nabla_j\beta)T_kT_\ell - \beta(\nabla_jT_k)T_\ell - \beta T_k(\nabla_jT_\ell)$$
$$= \frac{1}{2(n-1)}[(\nabla_k\tilde{R})g_{j\ell} - (\nabla_j\tilde{R})g_{k\ell}], \tag{5.2}$$

where $\tilde{R} = R - 2(n-1)\alpha$. Recalling that T_k is a unit vector and so $(\nabla_kT_\ell)T^\ell = 0$, Eq. (5.2) is then transvected with $g^{j\ell}$ to obtain

$$\nabla_k\beta - (\nabla^\ell\beta)T_kT_\ell - \beta(\nabla^\ell T_k)T_\ell - \beta T_k(\nabla^\ell T_\ell) = \frac{1}{2}\nabla_k\tilde{R}. \tag{5.3}$$

Transvecting again Eq. (5.2) with T^jT^ℓ gives

$$\nabla_k\beta - (\nabla_\ell\beta)T_kT^\ell - \beta T^\ell(\nabla_\ell T_k) = \frac{1}{2(n-1)}\nabla_k\tilde{R} - \frac{1}{2(n-1)}(\nabla_\ell\tilde{R})T_kT^\ell. \tag{5.4}$$

Comparing the last two equations gives immediately:

$$\beta T_k(\nabla^\ell T_\ell) = \frac{2-n}{2(n-1)}\nabla_k\tilde{R} - \frac{1}{2(n-1)}(\nabla_\ell\tilde{R})T_kT^\ell. \tag{5.5}$$

The last result is then transvected with T^k so that the following holds:

$$\beta(\nabla^\ell T_\ell) = -\frac{1}{2}(\nabla_\ell\tilde{R})T^\ell. \tag{5.6}$$

Now Eq. (5.6) is substituted in (5.5) to give

$$(\nabla_\ell\tilde{R})T_kT^\ell = \nabla_k\tilde{R}. \tag{5.7}$$

If the last result is substituted in (5.4), one can easily obtain

$$\nabla_k\beta - (\nabla_\ell\beta)T_kT^\ell = \beta T^\ell(\nabla_\ell T_k). \tag{5.8}$$

It is worth to notice that, by (5.7) one easily has $(\nabla_k\tilde{R})T_j = (\nabla_j\tilde{R})T_k$. Thus transvecting Eq. (5.2) with T^ℓ gives immediately:

$$\beta[\nabla_kT_j - \nabla_jT_k] + T_j\nabla_k\beta - T_k\nabla_j\beta = 0. \tag{5.9}$$

Following the hypothesis of the theorem, we immediately conclude that T_k is a closed 1-form, that is,

$$\nabla_kT_j - \nabla_jT_k = 0. \tag{5.10}$$

Now using Eqs. (5.10), (5.9) and (5.7) we can transvect again Eq. (5.2) with T^j recalling that $(\nabla_jT_\ell)T^j = (\nabla_\ell T_j)T^j = 0$ being T_j closed to obtain the following relation:

$$\nabla_kT_\ell = \frac{T^m(\nabla_m\tilde{R})}{2\beta(n-1)}[T_\ell T_k - g_{k\ell}]. \tag{5.11}$$

So we conclude that T_k is a concircular vector. □

Now we can state the following remarks.

Remark 5.1. From $(\nabla_m\tilde{R})T^mT_k = \nabla_k\tilde{R}$ we easily obtain by a covariant derivative that the following is true:

$$\nabla_j\nabla_k\tilde{R} = (\nabla_jT_k)(\nabla_m\tilde{R})T^m + T_k\nabla_j((\nabla_m\tilde{R})T^m). \tag{5.12}$$

A similar relation is written with indices k and j exchanged and the resulting equations are then subtracted recalling that T_k is a closed 1-form to obtain finally:

$$\nabla_j((\nabla_m\tilde{R})T^m) = T_j(T^k\nabla_k(T^m\nabla_m\tilde{R})). \tag{5.13}$$

Remark 5.2. From $\nabla_k\beta - (\nabla_\ell\beta)T_kT^\ell = \beta T^\ell(\nabla_\ell T_k)$ recalling that T_k is a closed 1-form, one easily writes

$$\nabla_k\beta = (\nabla_\ell\beta)T_kT^\ell. \tag{5.14}$$

Now if the scalar function $f = \frac{T^m(\nabla_m\tilde{R})}{2\beta(n-1)}$ is considered by the previous remarks, one can write $\nabla_jf = \mu T_j$ where μ is another scalar function. Thus the 1-form $\omega_k = fT_k$ is closed and T_k is a *proper* concircular vector.

Now taking account of Corollary 4.1 and Theorem 5.1 one can state the following theorem.

Theorem 5.2. *Let M be an n-dimensional $(n > 3)$ conformally flat $(PZS)_n$. If the tensor $Z_{k\ell}$ is non-singular, then the manifold admits a proper concircular vector.*

In [1] it is well known that if a conformally flat space admits a proper concircular vector, then this space is subprojective in the sense of Kagan. In this way the following holds.

Theorem 5.3. *Let M be an n-dimensional $(n > 3)$ conformally flat $(PZS)_n$. If the tensor $Z_{k\ell}$ is non-singular, then the manifold is a subprojective space.*

In [28] Yano proved that a necessary and sufficient condition for a Riemannian manifold admits a concircular vector that there is a coordinate system in which the first fundamental form may be written as

$$ds^2 = (dx^1)^2 + e^{q(x^1)} g^*_{\alpha\beta} dx^\alpha dx^\beta, \tag{5.15}$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are functions of x^γ only $(\alpha, \beta, \gamma = 2, 3, \ldots, n)$ and q is a function x^1 only. Since a conformally flat $(PZS)_n$ manifold with a non-singular $Z_{k\ell}$ tensor admits a proper concircular vector field, this space is the warped product $1 \times e^q M^*$ where (M^*, g^*) is a $(n-1)$-dimensional Riemannian manifold. Gebarosky in [18] proved that the warped product $1 \times e^q M^*$ satisfies the condition (5.1) if and only if M^* is Einstein. Thus the following theorem holds.

Theorem 5.4. *Let M be an n-dimensional $(n > 3)$ conformally flat $(PZS)_n$. If the tensor $Z_{k\ell}$ is non-singular, then the manifold M is the warped product $1 \times e^q M^*$ where M^* is Einstein.*

6. Sufficient Conditions for a $(PZS)_n$ Riemannian Manifold to be Ricci Pseudo Symmetric in the Sense of Deszcz

In this section we explore the connection between $(PZS)_n$ Riemannian manifolds and the notion of pseudo Ricci symmetric manifolds in the sense of Deszcz. We shall give sufficient conditions for a $(PZS)_n$ manifold to be Ricci pseudo symmetric in the sense of Deszcz [14, 15]. The starting point is the definition of a $(PZS)_n$ manifold, that is,

$$\nabla_s Z_{k\ell} = 2A_s Z_{k\ell} + A_k Z_{s\ell} + A_\ell Z_{sk}. \tag{6.1}$$

By performing a covariant derivative and employing (6.1) one obtains the following results:

$$\begin{aligned}\nabla_i \nabla_s Z_{k\ell} = {} & 2(\nabla_i A_s) Z_{k\ell} + 2A_s(2A_i Z_{k\ell} + A_k Z_{i\ell} + A_\ell Z_{ik}) \\ & + (\nabla_i A_k) Z_{s\ell} + A_k(2A_i Z_{s\ell} + A_s Z_{i\ell} + A_\ell Z_{is}) \\ & + (\nabla_i A_\ell) Z_{sk} + A_\ell(2A_i Z_{sk} + A_s Z_{ik} + A_k Z_{is}).\end{aligned} \tag{6.2}$$

Exchanging the indices s and i in the previous equation and subtracting after a long calculation one gets

$$\begin{aligned}(\nabla_s \nabla_i - \nabla_i \nabla_s) Z_{k\ell} = {} & 2(\nabla_s A_i - \nabla_i A_s) Z_{k\ell} + Z_{i\ell}(\nabla_s A_k - A_k A_s) \\ & - Z_{s\ell}(\nabla_i A_k - A_k A_i) + Z_{ki}(\nabla_s A_\ell - A_s A_\ell) \\ & - Z_{sk}(\nabla_i A_\ell - A_\ell A_i).\end{aligned} \tag{6.3}$$

Now if $\nabla_s A_k - A_k A_s = \gamma g_{ks}$ being γ an arbitrary scalar function, we obtain the following identities chain:

$$\begin{aligned}
&(\nabla_s\nabla_i - \nabla_i\nabla_s)Z_{k\ell}\\
&\quad = \gamma(Z_{i\ell}g_{sk} - Z_{s\ell}g_{ik} + Z_{ki}g_{s\ell} - Z_{sk}g_{i\ell})\\
&\quad = \gamma[(R_{i\ell} + \phi g_{i\ell})g_{sk} - (R_{s\ell} + \phi g_{s\ell})g_{ik} + (R_{ki} + \phi g_{ki})g_{s\ell} - (R_{sk} + \phi g_{sk})g_{i\ell}]\\
&\quad = \gamma[R_{i\ell}g_{sk} + \phi g_{i\ell}g_{sk} - R_{s\ell}g_{ik} - \phi g_{s\ell}g_{ik} + R_{ki}g_{s\ell} + \phi g_{ki}g_{s\ell} - R_{sk}g_{i\ell} - \phi g_{sk}g_{i\ell}]\\
&\quad = \gamma[R_{i\ell}g_{sk} - R_{s\ell}g_{ik} + R_{ki}g_{s\ell} - R_{sk}g_{i\ell}]. \qquad (6.4)
\end{aligned}$$

Now we observe that $(\nabla_s\nabla_i - \nabla_i\nabla_s)Z_{k\ell} = (\nabla_s\nabla_i - \nabla_i\nabla_s)[R_{k\ell} + \phi g_{k\ell}] = (\nabla_s\nabla_i - \nabla_i\nabla_s)R_{k\ell}$ being ϕ a scalar function. So we have obtained the final result:

$$(\nabla_s\nabla_i - \nabla_i\nabla_s)R_{k\ell} = \gamma[R_{i\ell}g_{sk} - R_{s\ell}g_{ik} + R_{ki}g_{s\ell} - R_{sk}g_{i\ell}]. \qquad (6.5)$$

In such a case we call the manifold *pseudo* Ricci symmetric in the sense of Deszcz [14, 15]. We can state the following theorem.

Theorem 6.1. *Let M be an n-dimensional $(PZS)_n$ manifold. If the associated form is concircular and satisfies the condition $\nabla_s A_k - A_k A_s = \gamma g_{ks}$, then the manifold is Ricci pseudo symmetric in the sense of Deszcz.*

On the other hand, if we consider a *pseudo Z* symmetric manifold, that is, also *pseudo* Ricci symmetric in the sense of Deszcz, we can achieve an interesting result. Let us consider Eqs. (4.3) and (4.4), by which we can write easily:

$$\nabla_m R^m_{jk\ell} = A_k Z_{j\ell} - A_j Z_{k\ell} + [(\nabla_j\phi)g_{k\ell} - (\nabla_k\phi)g_{j\ell}]. \qquad (6.6)$$

Now the covariant derivative of the previous equation is performed and thus one obtains

$$\begin{aligned}
\nabla_i\nabla_m R^m_{jk\ell} &= (\nabla_i A_k)Z_{j\ell} + A_k(\nabla_i Z_{j\ell}) - (\nabla_i A_j)Z_{k\ell} - A_j(\nabla_i Z_{k\ell})\\
&\quad + [(\nabla_i\nabla_j\phi)g_{k\ell} - (\nabla_i\nabla_k\phi)g_{j\ell}]. \qquad (6.7)
\end{aligned}$$

By performing a cyclic permutation of indices i, j, k, adding the resulting three equations and using the contracted Bianchi identity one writes

$$\begin{aligned}
&(\nabla_i\nabla_k - \nabla_k\nabla_i)R_{j\ell} + (\nabla_j\nabla_i - \nabla_i\nabla_j)R_{k\ell} + (\nabla_k\nabla_j - \nabla_j\nabla_k)R_{i\ell}\\
&\quad = (\nabla_i A_k - \nabla_k A_i)Z_{j\ell} + (\nabla_j A_i - \nabla_i A_j)Z_{k\ell} + (\nabla_k A_j - \nabla_j A_k)Z_{i\ell}. \qquad (6.8)
\end{aligned}$$

If we consider that the manifold is also *pseudo* Ricci symmetric in the sense of Deszcz, one can write

$$(\nabla_i\nabla_k - \nabla_k\nabla_i)R_{j\ell} = L_s[g_{j\ell}R_{k\ell} - g_{jk}R_{i\ell} + g_{li}R_{jk} - g_{lk}R_{j\ell}], \qquad (6.9)$$

where L_s is an arbitrary scalar function. Then it can be easily checked that the sum of the three commutators vanishes so the previous equation takes the form:

$$(\nabla_i A_k - \nabla_k A_i)Z_{j\ell} + (\nabla_j A_i - \nabla_i A_j)Z_{k\ell} + (\nabla_k A_j - \nabla_j A_k)Z_{i\ell} = 0. \qquad (6.10)$$

Now using the same procedure as in Sec. 4 we can finally state the following theorem.

Theorem 6.2. *Let M be an n-dimensional $(PZS)_n$ manifold. If the manifold is also pseudo Ricci symmetric in the sense of Deszcz and the Z-tensor is non-singular, then A_k is a closed 1-form.*

7. $(PZS)_n$ Spacetime Manifolds

In this section we study general properties of pseudo Z symmetric spacetimes; we will focus on the closeness of the 1-form A_k and will state some theorems concerning the perfect fluid spacetime.

If a $(PZS)_4$ spacetime is considered i.e. $Z_{kl} = kT_{kl}$, then the condition $\nabla^\ell Z_{k\ell} = 0$ coming from the stress–energy tensor must be satisfied and so from Eq. (2.3) it follows that $2A_k Z + 6A^\ell Z_{k\ell} = 0$, together with $\nabla_k(\frac{R}{2} + \phi) = 0$, and from Eq. (2.2) it follows that $\nabla_k Z = 2A_k Z + 2A^\ell Z_{k\ell}$. Combining this fact we obtain $\nabla_k Z = \frac{4}{3} A_k Z$ and $A^\ell Z_{k\ell} = -\frac{A_k}{3} Z$. Then we can state the following theorem.

Theorem 7.1. *For a $(PZS)_4$ spacetime manifold the 1-form A_k is closed and it is an eigenvector of the Z-tensor with eigenvalue $-\frac{Z}{3}$.*

On the other hand, from the condition $\nabla_k(\frac{R}{2} + \phi) = 0$ it follows that $\frac{R}{2} + \phi = \Lambda$, where Λ is a *cosmological constant* and thus Einstein's equation takes the form $R_{k\ell} - \frac{R}{2} g_{k\ell} + \Lambda g_{k\ell} = kT_{k\ell}$. An example of this situation is a perfect fluid spacetime with non-null stress–energy tensor given by the following equation:

$$T_{k\ell} = (\mu + p) u_k u_\ell + p g_{k\ell}, \tag{7.1}$$

where μ is the energy density, p is the isotropic pressure and u_i the fluid flow velocity with the condition $u_i u^i = -1$. In addition p and μ are related by an *equation of state* governing the particular sort of perfect fluid under consideration. In general this is an equation of the form $p = p(\mu, T)$ where T is the absolute temperature. However, we shall only be concerned with situations in which T is effectively constant so that the equation of state reduces to $p = p(\mu)$.

If the condition $A^\ell Z_{k\ell} = -\frac{A_k}{3} Z$ is applied on Einstein's equation $Z_{k\ell} = kT_{k\ell}$ one can obtain the following relation:

$$T_{k\ell} = (\mu + p) u_k u_\ell + p g_{k\ell}, \tag{7.2}$$

$$\left(k\mu - \frac{Z}{3}\right) A_k u^k = 0. \tag{7.3}$$

If the relation $A_k u^k \neq 0$ is fulfilled, then we have $k\mu = \frac{Z}{3}$. Now $Z_{k\ell} = kT_{k\ell}$ gives rise to $Z = kT$ and so by contracting (7.1) and using $n = 4$, we have $T = 3p - \mu$ and $Z = k(3p - \mu)$. It follows immediately that $kp = \frac{4}{9} Z$ and that $\mu = \frac{3}{4} p$: this is the *equation of state* of such a spacetime. Inserting $k\mu = \frac{Z}{3}$ in Eq. (7.2) one easily obtain $k(\mu + p) u_k A^\ell u_\ell = -k(\mu + p) A_k$ and thus $A_k = -u_k A^\ell u_\ell$. Now from $\frac{R}{2} + \phi = \Lambda$ and $Z = R + 4\phi$ it follows that $Z = 4\Lambda - R$. Inserting these relations in

Einstein's equation it follows that

$$R_{k\ell} = \frac{7}{9}(4\Lambda - R)u_k u_\ell + \frac{1}{9}\left(7\Lambda + \frac{R}{2}\right) g_{kl}. \tag{7.4}$$

So we have obtained a quasi-Einstein manifold and so the following theorem is true.

Theorem 7.2. *A $(PZS)_4$ perfect fluid spacetime manifold is quasi-Einstein and the 1-form A_k is proportional to the fluid flow velocity.*

This theorem is a generalization of the result due to Ray-Guha [23] which says that a perfect fluid pseudo Ricci symmetric spacetime is a quasi-Einstein manifold with each of its associated scalars equal to $\frac{R}{3}$.

If a $(PZS)_n$ spacetime with further condition $\nabla_k Z_{j\ell} = 0$ (a spacetime with the conservation of the stress–energy density) is considered, we get $\nabla_k Z = 0$ and from the previous results we may write $2A_k Z + 6A^\ell Z_{k\ell} = 0$ and $2A_k Z + 2A^\ell Z_{k\ell} = 0$. From this it follows that $A^\ell Z_{k\ell} = 0$ and $Z = g^{k\ell} Z_{k\ell} = 0$. So we obtain $R = -n\phi$ and from $\nabla_k(\frac{R}{2} + \phi) = 0$ we know that $\nabla_k R = 0 = \nabla_k \phi$ and $\Lambda = R\frac{n-2}{2n}$. We note that in this case a non-null *cosmological constant* is necessary. Thus Einstein's equations take the form $R_{k\ell} - \frac{R}{2} g_{k\ell} + \Lambda g_{k\ell} = kT_{k\ell}$, that is, $R_{k\ell} - \frac{R}{n} g_{k\ell} = kT_{k\ell}$. In four dimension this implies $R_{k\ell} - \frac{R}{4} g_{k\ell} = kT_{k\ell}$. The vacuum solution is still $R_{k\ell} - \frac{R}{4} g_{k\ell} = 0$. On the other hand, the 1-form A_k becomes an eigenvector of the stress–energy tensor with null eigenvalue. In other words, the 1-form A_k is an eigenvector of the Ricci tensor with eigenvalue $\frac{R}{n}$. Moreover, by the result of $\nabla_k R = 0 = \nabla_k \phi$ our space becomes also Ricci symmetric. The vacuum solution is thus an Einstein space.

If the conservation of the stress–energy tensor $k\nabla_k T_{j\ell} = \nabla_k Z_{j\ell} = 0$ is taken under consideration for the perfect fluid spacetime we get the previous results like $A^\ell Z_{k\ell} = 0$ and $Z = 0$. The condition $Z = 0$ implies $T = 0$ and consequently $\mu = 3p$: this is known as *radioactive perfect fluid spacetime* (see [17]). In fact $\mu = 3p$ is the *equation of state* for radiation and the corresponding spacetime is isotropic and homogeneous. The stress–energy tensor becomes $T_{k\ell} = 4pu_k u_\ell + pg_{k\ell}$. The condition $A^\ell Z_{k\ell} = 0$ implies that $A^\ell T_{k\ell} = 0$ and thus, if $p \neq 0$, then we have

$$A_k = -4u_k A^\ell u_\ell. \tag{7.5}$$

The previous result is thus transvected by u^k to obtain easily $A_k u^k = 0$ and finally $A_k = 0$. Thus we obtain $\nabla_k R_{j\ell} = 0$ and we state the following theorem.

Theorem 7.3. *A $(PZS)_4$ perfect fluid spacetime manifold with stress–energy conservation is a radioactive perfect fluid isotropic and homogeneous spacetime and it is Ricci symmetric.*

Acknowledgment

The second author was supported by Grant Proj. No. BSRP-2011-0025687 from National Research Foundation of Korea.

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