Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:3** (2011), 93–103 DOI: 10.2298/FIL1103093L

PSEUDOBOUNDED OR ω-PSEUDOBOUNDED PARATOPOLOGICAL GROUPS

Fucai Lin and Shou Lin

Abstract

We say that a paratopological group G is *pseudobounded* (ω*-pseudobounded*), if for every neighborhood V of the identity element e of G, there exists a natural number n such that $G = V^n$ ($G = \bigcup_{n=1}^{\infty} V^n$). In this paper, we mainly discuss the pseudobounded and ω-pseudobounded paratopological groups. First, we give an example to show that a theorem in [4] is not true. And then, we define the concept of premeager, and discuss when a pseudobounded paratopological group is a topological group. Moreover, we also discuss some properties of ω-pseudobounded topological groups, and show that the class of connected topological groups is contained in the class of ω-pseudobounded topological groups. Finally, some open problems concerning the paratopological groups are posed.

1 Introduction

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product maps of $G \times G$ into G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product maps of $G \times G$ into G is jointly continuous.

It is well known that paratopological groups is a good generalization of topological groups. The topic of paratopological groups is quite popular nowadays and one can see a lot of activities going on in what concerns of the study of these objects, see [1, 3, 7, 10, 11].

Recently, K.H. Azar defined the bounded topological groups [4]. However, in this paper, we call it pseudobounded instead of bounded since the boundedness has other meaning in topological algebra. In this paper, we define the pseudobounded and ω-pseudobounded paratopological groups.

2010 *Mathematics Subject Classifications.* 54A05; 54B05; 54C05; 54H11.

Key words and Phrases. Pseudobounded; ω-pseudobounded; paratopological groups; topological groups; quasi-metric; premeager spaces; Polish spaces.

Received: December 11, 2010; Revised: March 16, 2011

Communicated by Ljubiša D.R. Kočinac

Supported by the NSFC (No. 10971185) and the Educational Department of Fujian Province (No. JA09166) of China.

Definition 1. [4] Let G be a paratopological group and $A \subset G$. We say that A is a *pseudobounded subset* of G, if for every neighborhood V of the identity element e of G, there exists a natural number n such that $A \subset V^n$. If G is a pseudobounded subset, then we say that G is *pseudobounded*.

It is well known that the Sorgenfrey line [5, Example 1. 2. 2] is a first-countable and non-pseudobounded paratopological group, where as a set the Sorgenfrey line is the set of real numbers and its topology is generated by taking as a basis the half open intervals $[a, b), a < b$.

Moreover, there exists a pseudobounded paratopological group G such that G is not a topological group, see Example 2.

Definition 2. A *quasi-metric* d on a set X is a function from $X \times X$ into the set R^+ of positive real numbers such that for any $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) = 0 \Leftrightarrow x = y$;

2. $d(x, y) \leq d(x, z) + d(z, y)$.

If d also satisfies the additional condition (3) $d(x, y) = d(y, x)$, then d is called a *metric* on X.

In this paper, we mainly discuss the pseudobounded and ω-pseudobounded paratopological groups. In Section 2, we give an example to show that a theorem in [4] is not true. Moreover, we also discuss when a pseudobounded paratopological group G has a quasi-metric such that G is pseudobounded with respect to the quasi-metric. In Section 3, we define the concept of premeager, and discuss when a pseudobounded paratopological group is a topological group. In Section 4, we define the concept of ω-pseudobounded and discuss some properties of ω-pseudobounded topological groups, and show that the class of connected topological groups is contained in the class of ω-pseudobounded topological groups. Finally, some open problems concerning the paratopological groups are posed.

All spaces are Hausdorff unless stated otherwise, and all maps are onto. $\mathbb{N}$ denotes the set of all positive natural numbers. The letter e denotes the neutral element of a group. Readers may refer to [2, 5, 6] for notations and terminology not explicitly given here.

2 Pseudobounded topological groups and paratopological groups

In [4], K. H. Azar proved the following theorem:

Theorem 1. [4] *Let G be a topological group metrizable with respect to a left invariant metric d. Then G is pseudobounded with respect to the topology if and only if G is bounded with respect to the metric d.*

However, the proof of Theorem 1 has a gap. Indeed, the result of this theorem is not true, see the following counterexample.

Example 1. Let $\mathbb{R}$ be the real line endowed with the Euclidean topology $\mathcal{F}$. Then $\mathbb{R}$ has an Euclidean metric ρ on $\mathbb{R}$ by the equation $\rho(x,y) = |x-y|$ for any two points $x, y \in \mathbb{R}$. Put $d(x,y) = \min\{\rho(x,y), 1\}$ for any two points $x, y \in \mathbb{R}$. Then $d(x,y)$ is a standard bounded metric that induces the same topology as ρ. It is well known that $(\mathbb{R}, \mathcal{F}, +)$ is an Abelian topological group with respect to the usual addition. Also, d is an invariant metric. In fact, for any two points $x, y \in \mathbb{R}$, it is easy to see that $d(z+x, z+y) = d(x,y)$ for any $z \in \mathbb{R}$. However, the space $\mathbb{R}$ is not pseudobounded. Indeed, let $B_d(0, \frac{1}{2}) = \{x \in \mathbb{R} : d(0,x) < \frac{1}{2}\}$. Then it is easy to see that $nB_d(0, \frac{1}{2}) = (-\frac{n}{2}, \frac{n}{2}) \neq \mathbb{R}$ for each $n \in \mathbb{N}$.

However, we have the following theorem.

Theorem 2. *Let G be a paratopological group with a continuous and left invariant quasi-metric d. If G is pseudobounded, then G is bounded with respect to the quasi-metric d.*

Proof. Fix an $A > 0$. Since $d : G \times G \to R^+$ is continuous, there exist open neighborhoods U and V of e such that $d(U \times V) \subset [0, A)$. Because G is a paratopological group, there is an open neighborhood W of e such that $W \subset U \cap V$. Since G is pseudobounded, we can find an $n \in \mathbb{N}$ such that $W^n = G$. Therefore, we have $d(W \times W) \subset [0, A)$. Next, we shall show that $d(G \times G) \subset [0, (2n-1)A)$. Since $d(G \times G) = d(W^n \times W^n)$, it is equivalent to show that $d(W^n \times W^n) \subset [0, (2n-1)A)$.

First, we show that $d(W^2 \times W^2) \subset [0, 3A)$. For each $(x, y) \in W^2 \times W^2$, we have $x = x_1x_2$ and $y = y_1y_2$, where $x_1, x_2, y_1, y_2 \in W$. Then

$$\begin{aligned} d(x,y) &= d(x_1x_2, y_1y_2) \\ &\leq d(x_1x_2, x_1) + d(x_1, y_1y_2) \\ &= d(x_2, e) + d(x_1, y_1y_2) \\ &\leq d(x_2, e) + d(x_1, y_1) + d(y_1, y_1y_2) \\ &= d(x_2, e) + d(x_1, y_1) + d(e, y_2) \\ &< A + A + A \\ &= 3A. \end{aligned}$$

Suppose that $d(W^k \times W^k) \subset [0, (2k-1)A)$, where $2 \leq k < n$. We show that $d(W^{k+1} \times W^{k+1}) \subset [0, (2k+1)A)$. In fact, for each $(u, v) \in W^{k+1} \times W^{k+1}$, we have

$u = u_1u_2$ and $v = v_1v_2$, where $u_1, v_1 \in W^k$ and $u_2, v_2 \in W$. Then

$$\begin{aligned} d(u,v) &= d(u_1u_2, v_1v_2) \\ &\leq d(u_1u_2, u_1) + d(u_1, v_1v_2) \\ &= d(u_2, e) + d(u_1, v_1v_2) \\ &\leq d(u_2, e) + d(u_1, v_1) + d(v_1, v_1v_2) \\ &= d(u_2, e) + d(u_1, v_1) + d(e, v_2) \\ &< A + (2k-1)A + A \\ &= (2k+1)A. \end{aligned}$$

Therefore, we have $d(W^n \times W^n) \subset [0, (2n-1)A)$, and it follows that d is bounded. □

Let G be a paratopological group, and H be a closed subgroup. Denote by G/H the set of all left cosets aH of H in G, and endow it with the quotient topology with respect to the canonical mapping $\pi : G \to G/H$ defined by $\pi(a) = aH$, for every $a \in G$.

Theorem 3. *Let G be a paratopological group and let H be a normal subgroup of G. If H and G/H are pseudobounded, then G is pseudobounded.*

Proof. Let U be a neighborhood of e. Obviously, the set $V = U \cap H$ is an open neighborhood of e in H. Since G/H and H are pseudobounded, there exist $m, n \in \mathbb{N}$ such that $V^n = H$ and $(U/H)^m = G/H$. We claim that $U^{m+n} = G$. In fact, let $x \in G$.

Case 1: $x \in H$.

Obviously, we have $x \in H = V^n \subset U^n \subset U^{m+n}$.

Case 2: $x \notin H$.

Then we have $xH \in G/H = (U/H)^m$, and therefore, there exist points $x_1, \cdots, x_m \in U$ such that $xH = x_1 \cdots x_m H$. Hence, there exists an $h \in H$ such that $xh \in U^m$. It follows that $x \in U^m H = U^m V^n \subset U^m U^n = U^{m+n}$.

Therefore, we have $U^{m+n} = G$, that is, G is pseudobounded. □

The following proposition is easy, and we omit it.

Proposition 1. *Let each G_α be a pseudobounded paratopological group, where $\alpha \in I$. Then the product topology $\prod_{\alpha \in I} G_\alpha$ is also a pseudobounded paratopological group.*

Example 2. There exists a normal pseudobounded paratopological group G such that G is not a topological group.

Proof. Let $X = \{(x, 1) : 0 \leq x < 1\}$, and let the topology on X be generated by the base consisting of sets of the form

$$\{(x,1) \in X : x_0 < x < x_0 + \frac{1}{k}\} \cup \{(x_0, 1)\},$$

where $0 \le x_0 < 1$ and $k \in \mathbb{N}$.

Then the space X is the arrow space which is homeomorphic to the Sorgenfrey line. Moreover, there exists a natural structure of an Abelian group on X such that the multiplication $(u, v) \mapsto u \cdot v$ is continuous, that is, the space X admits a structure of a paratopological group. For example, if $u = (x, 1)$ and $v = (y, 1)$ are two points in X, then $u \cdot v = (x + y, 1)$ if $x + y < 1$, and $u \cdot v = (x + y - 1, 1)$ if $x + y \ge 1$. Obviously, X is pseudobounded. However, the Sorgenfrey line is not a topological group, and hence X is not a topological group. $\square$

Note 1. It follows from Example 2 that the pseudoboundedness is not a topological invariant.

3 Premeager paratopological groups

Definition 3. Let G be a paratopological group. G is called *premeager* if, for any its nowhere dense subset A of G, we have $A^n \neq G$ for each $n \in \mathbb{N}$.

A *Lusin space* is an uncountable space such that every its nowhere dense set is countable. A *Polish space* is a separable completely metrizable space. A *Polish group* is a topological group G regarded as a topological space which is itself a Polish space.

It is well known that a Polish space is a Lusin space. Therefore, we have the following proposition.

Proposition 2. *A paratopological Lusin group has the premeager property. In particular, a Polish group has the premeager property.*

The proof of the following Proposition 3 is due to M. Sakai.

Proposition 3. *The Sorgenfrey line X ($X = \mathbb{R}$) does not have the premeager property. In particular, the Euclidean line does not have the premeager property.*

Proof. Let C be the usual Cantor set in [0,1]. It is well known that C is nowhere dense in X. By [9, p.37, Lemma 1], we have $C + C = [0, 2]$, where '+' is the usual additive. Let $A = \bigcup_{n\in\mathbb{Z}}(2n + C)$, where $\mathbb{Z}$ is the integer. Then A is nowhere dense in X, but $A + A = X$ since $C + C = [0, 2]$. $\square$

A map $f : X \to Y$ is called *quasi-open* if we have $\operatorname{int}(f(U)) \neq \emptyset$ for each non-empty open subset U of X.

First, we discuss some properties of the premeager.

Proposition 4. *Let $f : G \to H$ be a continuous quasi-open homomorphism map, where G, H are paratopological groups. If G is premeager, then H is also premeager.*

Proof. Let A be any nowhere dense subset of H. Suppose that there exists some $n \in \mathbb{N}$ such that $A^n = H$. Therefore, $(f^{-1}(A))^n = f^{-1}(A^n) = f^{-1}(H) = G$. Since G is premeager, the set $f^{-1}(A)$ is a non-nowhere dense subset of X. Hence there is a non-empty open subset U of X such that $U \subset \overline{f^{-1}(A)}$. It follows from $\overline{f^{-1}(A)} \subset$

$f^{-1}(\overline{A})$ that $f(U) \subset \overline{A}$. Since f is quasi-open, we have $\emptyset \neq \mathrm{int}(f(U)) \subset f(U) \subset \overline{A}$, which is a contradiction. □

Since open maps are quasi-open maps, we have the following corollary.

Corollary 1. *Let $f : G \to H$ be an open and continuous homomorphism map, where G, H are paratopological groups. If G is premeager, then H is also premeager.*

Proposition 5. *Let G be a pseudobounded and premeager paratopological group. Then every open subgroup of G is premeager.*

Proof. Let H be an open subgroup of G. Suppose that H is non-premeager. Then there exists a nowhere dense subset A of H and an $n \in \mathbb{N}$ such that $A^n = H$. Since G is pseudobounded, it follows that there is an $m \in \mathbb{N}$ such that $H^m = G$. Hence $(A^n)^m = H^m = G = A^{nm}$. However, the set A is a nowhere dense subset of G, which is a contradiction. □

Next, we mainly discuss when a paratopological group is a topological group.

Lemma 1. [1] *Assume that G is a paratopological group and not a topological group. Then there is an open neighborhood U of the neutral element e of G such that $U \cap U^{-1}$ is nowhere dense in G.*

Theorem 4. *Suppose that G is a pseudobounded topological group, H is a premeager paratopological group, and that $f : G \to H$ is a continuous onto homomorphism. Then H is a topological group.*

Proof. Assume that H is not a topological group. By Lemma 1, there is an open neighborhood U of the neutral element e of H such that $U \cap U^{-1}$ is nowhere dense. Let W be a symmetric neighborhood of the identity in G such that $f(W) \subset U$. Since $f(W) = f(W^{-1}) = f^{-1}(W) \subset U^{-1}$. Hence we have $f(W) \subset U \cap U^{-1}$. Since G is pseudobounded, there exists an $n \in \mathbb{N}$ such that $W^n = G$, and it follows that $f(W^n) = f(G) = [f(W)]^n = H \subset (U \cap U^{-1})^n$. Thus $H = (U \cap U^{-1})^n$, and H does not have the premeager property, which is a contradiction. Therefore, H is a topological group. □

For a paratopological group G with a topology τ, one defines the *conjugate topology* τ^{-1} on G by $\tau^{-1} = \{U^{-1} : U \in \tau\}$. The upper bound $\tau^* = \tau \vee \tau^{-1}$ is a group topology on G. Then we call $G^* = (G, \tau^*)$ the group *associated to* G.

Definition 4. [11] Let $\mathcal{P}$ be a topological property. A paratopological group G is called *totally* $\mathcal{P}$ if the associated topological group G^* has property $\mathcal{P}$.

Obvious, if a paratopological group is totally pseudobounded then G is pseudobounded.

Theorem 5. *Let G be a totally pseudobounded paratopological group. If G has the premeager property, then G is a topological group.*

Proof. Suppose that G is not a topological group. It follows from Lemma 1 that there is an open neighborhood U of the neutral element e of G such that $U \cap U^{-1}$ is nowhere dense in G. Since G is totally pseudobounded and $U \cap U^{-1}$ in open in the associated topological group G^*, there exists an $n \in \mathbb{N}$ such that $(U \cap U^{-1})^n = G$. Therefore, G does not have the premeager property, which is a contradiction. □

Corollary 2. *If G is a totally pseudobounded paratopological Lusin group, then G is a topological group.*

Proof. It is easy to see from Proposition 2 and Theorem 5. □

4 ω-pseudobounded topological groups and paratopological groups

Let G be a locally compact topological group. Then G is connected if and only if, for every neighborhood V of the identity element e of G, we have $G = \bigcup_{n \in \mathbb{N}} V^n$, see [8, Corollary 7.9]. Therefore, we have the following definition.

Definition 5. Let G be a paratopological group and $A \subset G$. We say that A is an *ω-pseudobounded* subset of G, if for every neighborhood V of the identity element e of G, we have $A \subset \bigcup_{n \in \mathbb{N}} V^n$. If G is an ω-pseudobounded subset of G then we say that G is *ω-pseudobounded*.

Obviously, a pseudobounded paratopological group is ω-pseudobounded. However, there exists an ω-pseudobounded topological group which is not pseudobounded, see Example 3.

The proof of the next proposition is an easy exercise.

Proposition 6. *Let G be a paratopological group. Then we have the following four assertions.*

1. *If $A \subset G$ is pseudobounded (resp. ω-pseudobounded), then $\overline{A}$ is a pseudobounded (resp. ω-pseudobounded) subset of G;*
2. *If G is ω-pseudobounded, then G has no proper open subgroups;*
3. *If G is locally compact and pseudobounded (resp. ω-pseudobounded), then G is compact (resp. σ-compact);*
4. *If A, B are pseudobounded (resp. ω-pseudobounded) subsets in G, then AB and A^{-1} are also pseudobounded (resp. ω-pseudobounded) subsets in G.*

Example 3. There exists an ω-pseudobounded and non-pseudobounded topological group.

Proof. Let $(\mathbb{R}, +)$ be the real line endowed with the Euclidean topology, where '+' is the additive operation. Obviously, the $\mathbb{R}$ with the additive operation is ω-pseudobounded. However, the $\mathbb{R}$ with the additive operation is not pseudobounded. □

In Example 3, the Euclidean topology $\mathbb{R}$ is connected. So a question arises as follow: Is every connected topological group ω-pseudobounded? The answer is affirmative. Indeed, we have the following proposition.

Proposition 7. *If G is a connected topological group, then G is ω-pseudobounded.*

Proof. For any open neighborhood U of the neutral element e of G, there exists an open symmetric neighborhood V of e in G such that $V \subset U$. Clearly, the set $W = \bigcup_{n=1}^{\infty} V^n$ is an open subgroup of G. Since every open subgroup of a topological group is closed, the set W is closed in G. It follows from the connectedness of G that $G = W$. Since $V \subset U$, we have $G = \bigcup_{n=1}^{\infty} U^n$, that is, the space G is ω-pseudobounded. □

Example 4. There exists a T_1, connected and non-ω-pseudobounded paratopological group.

Proof. Let $G = (\mathbb{R}, +)$ be the group of real numbers with the usual addition, and let $\tau = \{\{x\} \cup [y, +\infty) : x, y \in \mathbb{R}\} \cup \{\emptyset, X\}$ be the topology of G. Then the operation '+' is jointly continuous, hence $(G, +)$ is a T_1 paratopolgical group. Moreover, it is easy to see that G is connected and non-ω-pseudobounded. □

Remark 1. It follows from Example 4 that one cannot generalize Proposition 7 to T_1 paratopological groups. However, we don't know if there exists a Hausdorff connected and non-ω-pseudobounded paratopological group, see Question 10.

Proposition 8. [8, Corollary 7.9] *Let G be a locally compact topological group. Then the following conditions are equivalent:*

1. *G is connected;*
2. *G has no proper open subgroups;*
3. *G is ω-pseudobounded.*

Since an 0-dimensional space is non-connected, we have the following corollary by to Proposition 8.

Corollary 3. *If G is a locally compact O-dimensional topological group, then G is non-ω-pseudobounded.*

We cannot omit the condition "locally compact" in Proposition 8, see Example 5.

Example 5. There exists an ω-pseudobounded, nowhere locally compact and non-connected topological group.

Proof. Let Q be the set of rational numbers with the topology inherited from $\mathbb{R}$. Then $(Q, +)$ is a topological group with the additive operation. It is easy to see that Q is ω-pseudobounded and non-connected. □

Remark 2. In [4], K.H. Azar proved that every pseudobounded topological group is connected, see [4, Theorem 2.6]. But, the proof has a gap. It is still an open problem if every pseudobounded topological group is connected, see Question 9.

The following proposition generalizes a result in [2], see Corollary 4.

Proposition 9. *Suppose that H is a discrete invariant subgroup of an ω-pseudobounded topological group G. Then each element of H commutes with each element of G, that is, H is contained in the center of the group G.*

Proof. If $H = \{e\}$, there is nothing to prove. Assume that H is a non-trivial subgroup of G. Choose an arbitrary point $x \in H \setminus \{e\}$. Since H is discrete, there exists an open neighborhood U of x in G such that $U \cap H = \{x\}$. It follows from the continuity of the multiplication in G that there is an open symmetric neighborhood V of e in G such that $VxV \subset U$.

Claim: For each $y \in V$, we have $xy = yx$.

Indeed, for each $y \in V$, since H is an invariant subgroup, we have $yxy^{-1} \in H$. Moreover, we have $yxy^{-1} \in VxV^{-1} = VxV \subset U$. Thus $yxy^{-1} \in H \cap U = \{x\}$, that is, $yxy^{-1} = x$.

Since G is ω-pseudobounded, we have $G = \bigcup_{n=1}^{n=\infty} V^n$. For each $g \in G$, there exists an $n \in \mathbb{N}$ such that $g \in V^n$, that is, the element g can be written in the form $g = y_1 \cdots y_n$, where $y_1, \cdots, y_n \in V$. Since x commutes with each element of V by Claim, we have

$$gx = y_1 \cdots y_n x = y_1 \cdots x y_n = \cdots = y_1 x \cdots y_n = x y_1 \cdots y_n = xg.$$

Therefore, the element $x \in H$ is in the center of the group G. Because x is an arbitrary element in H, we conclude that the center of G contains H. $\square$

It follows from Propositions 7 and 9 that we have following corollary.

Corollary 4. *Suppose that H is a discrete invariant subgroup of a connected topological group G. Then each element of H commutes with each element of G, that is, H is contained in the center of the group G.*

Theorem 6. *Let G be a paratopological group and let H be a normal subgroup of G. If H and G/H are ω-pseudobounded, then G is ω-pseudobounded.*

Proof. Let U be a neighborhood of e in G. Obviously, the set $V = U \cap H$ is an open neighborhood of e in H. Since G/H and H are ω-pseudobounded, we have $\bigcup_{n=1}^{\infty} V^n = H$ and $\bigcup_{n=1}^{\infty} (U/H)^n = G/H$. We claim that $\bigcup_{n=1}^{\infty} U^n = G$. In fact, let $x \in G$.

Case 1: $x \in H$.

Since $x \in H$, there exists an $n \in \mathbb{N}$ such that $x \in V^n$. Therefore, we have $x \in H \cap V^n \subset U^n \subset \bigcup_{n=1}^{\infty} U^n$.

Case 2: $x \notin H$.

Then we have $xH \in G/H$, and hence there exists an $m \in \mathbb{N}$ such that $xH \in (U/H)^m$. Therefore, there exist points $x_1, \cdots, x_m \in U$ such that $xH = x_1 \cdots x_m H$.

Hence, there exist an $h \in H$ and a $l \in \mathbb{N}$ such that $xh \in U^m$ and $h \in V^l$. It follows that $x \in U^m H = U^m V^l \subset U^m U^l = U^{m+l} \subset \bigcup_{n=1}^{\infty} U^n$.

Therefore, we have $\bigcup_{n=1}^{\infty} U^n = G$, that is, G is ω-pseudobounded. □

5 Open problems

The Sorgenfrey line is a paratopological group which is first-countable, non-pseudobounded and does not have the premeager property. However, the Sorgenfrey line is not a topological group. Therefore, we have the following three questions.

Question 1. *If G is a pseudobounded and premeager paratopological group, is G a topological group?*

Question 2. *Is every first-countable and pseudobounded paratopological group a topological group?*

Question 3. *Is every first-countable paratopological group with the premeager property a topological group?*

It follows from Corollary 2 that it is natural to pose the following question.

Question 4. *If G is a pseudobounded paratopological Lusin group, is G a topological group?*

Since a first-countable topological group is metrizable, we have the following question.

Question 5. *Is every first-countable and pseudobounded paratopological group metrizable?*

Note 2. If the answer to Question 2 is positive, then the answer to Question 5 is also positive.

In [10], O.V. Ravsky proved that every paratopological group has a left invariant quasi-metric if and only if it is first-countable. So we have the following question:

Question 6. *Is every first-countable and pseudobounded paratopological group bounded with respect to a left invariant quasi-metric?*

In [10], O.V. Ravsky posed the following question:

Question 7. *Does every first-countable paratopological group have a continuous and left invariant quasi-metric?*

Note 3. It follows from Theorem 2 that if the answer to Question 7 is positive then the answer to Question 6 is also positive.

It follows from Theorem 5 that we have the following question.

Question 8. *If G is a totally ω-pseudobounded paratopological group with the premeager property, is G a topological group?*

Question 9. *Is every pseudobounded topological group connected?*

Question 10. *Is every Hausdorff (regular) connected paratopological group ω-pseudobounded?*

Acknowledgements. We wish to thank the reviewers for the detailed list of corrections, suggestions to the paper, and all her/his efforts in order to improve the paper.

References

[1] A.V. Arhangel'skii, E.A. Reznichenko, *Paratopological and semitopological groups versus topological groups*, Topology Appl. 151 (2005), 107–119.

[2] A.V. Arhangel'shiĭ, M. Tkachenko, *Topological Groups and Related Structures*, Atlantis Press and World Sci., 2008.

[3] C. Alegre, S. Romaguera, *On paratopological vector spaces*, Acta Math. Hungarica 101 (2003), 237–261.

[4] K.H. Azar, *Bounded topological groups*, arXiv: 1003.2876.

[5] R. Engelking, *General Topology* (revised and completed edition), Heldermann Verlag, Berlin, 1989.

[6] G. Gruenhage, *Generalized metric spaces*, In: K. Kunen, J.E. Vaughan, eds., Handbook of Set-Theoretic Topology, North-Holland, 1984, pp. 423–501.

[7] O. Gutik, D. Pagon, D. Repovš, *The continuity of the inversion and the structure of maximal subgroups in countably compact topological semigroups*, Acta Math. Hungarica 124 (2009), 201–214.

[8] E. Hewitt, K.A. Ross, *Abstract Harmonic Analysis*, Springer, Berlin, 1963.

[9] A.B. Kharazishvili, *Nonmeasurable sets and functions*, Mathematical Studies 195, North-Holland, 2004.

[10] O.V. Ravsky, *Paratopological groups I*, Matem. Studii 16 (2001), 37–48.

[11] M. Sanchis, M. Tkachenko, *Totally Lindelöf and totally ω-narrow paratopological groups*, Topology Appl. 155 (2008), 322–334.

Fucai Lin (corresponding author): Department of Mathematics and Information Science, Zhangzhou Normal University, Zhangzhou 363000, P. R. China
E-mail: `linfucai2008@yahoo.com.cn`

Shou Lin: Institute of Mathematics, Ningde Teachers' College, Fujian 352100, P. R. China
E-mail: `linshou@public.ndptt.fj.cn`