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Pseudoconvex Domains over q-Complete Manifolds

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Abstract. Let (D, π) be an unramified Riemann domain over a connected complex manifold X of dimension n. We measure convexity properties of D as a "function" of the convexity properties of X and π. For instance, if (D, π) is pseudoconvex and X is q-complete, then D is q-complete. In particular, for $q = 1$, *i.e.* X is Stein, then D is Stein; thus we recover the theorem of Oka-Docquier-Grauert (with a new proof). More generally, if (D, π) is pseudoconvex of order $n - q$ and X is r-complete with corners, then D is $(q + r - 1)$-complete with corners. As an application of this we get that if X is Stein, $n > q + 1$, and $\pi^{-1}(\Sigma)$ is q-complete with corners for every smooth hypersurface $\Sigma \subset X$, then D is q-complete with corners. For $q = 1$ we obtain the "right" statement of a well-known result due to Lelong in the set-up of Stein manifolds. Finally, if $X = \mathbb{P}^n$ and (D, π) is pseudoconvex of order $n - q$ $(q < n)$, then D is q-complete with corners if D is not biholomorphic to $\mathbb{P}^n$ via π.

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1. – Introduction

It is well-known that geometric properties of complex manifolds imply strong analytic consequences. For instance, Stein manifolds are characterized by the existence of a smooth strongly plurisubharmonic exhaustion function [14].

All the eigenvalues of the *Levi form* of a smooth strongly plurisubharmonic function, or 1*-convex function*, are positive. If we assume only a precise number of these eigenvalues to be positive we get the notion of q*-convex function* and then that of q*-complete manifolds* for which important vanishing theorems hold [2]. The meaning of a q-convex function is that there are directions with respect to which the function becomes strongly plurisubharmonic. If φ_1 and φ_2 are q-convex functions in $\mathbb{C}^n$, then without knowing the eigenvalues, the best one can affirm about their sum $\varphi_1 + \varphi_2$ is its $(2q - 1)$-convexity. However, if at every point there is an $(n - q + 1)$-dimensional vector subspace of $\mathbb{C}^n$ and

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in these directions φ_1 and φ_2 are 1-convex, then so is their sum; thus $\varphi_1 + \varphi_2$ remains q-convex. In this way we arrive at *convexity with respect to a linear set* $\mathcal{M}$ [32] and then at $\mathcal{M}$*-complete manifolds* (Sect. 2.1). If $\operatorname{codim}\mathcal{M} < q$, we recover the usual q-convexity.

On the other hand, there are many examples which appear in complex analysis (for instance taking complements of analytic sets in complex manifolds) where we have to deal with functions that are expressed locally as the maximum of finitely many q-convex functions, the so called functions *q-convex with corners*. It is shown in [7] that such a function q-convex with corners can be always approximated in the C^0-topology by $\widetilde{q}$-convex functions, where $\widetilde{q} = n - [n/q] + 1$, n being the complex dimension of the ambient complex manifold. Therefore we still obtain vanishing or finiteness theorems by [2] if $2q \leq n$ (so that $\widetilde{q} < n$). This q-convexity with corners is more flexible than q-convexity and the normalization is chosen such that manifolds which are 1-complete with corners coincide with Stein manifolds.

The fourth convexity notion to be used in this paper, *pseudoconvexity of general order* (see Sect. 2.4), is very close to convexity with corners and has its roots in Oka's fundamental paper [29]. Roughly speaking, an open set D of $\mathbb{C}^n$ is pseudoconvex of order k if its complement $\mathbb{C}^n \setminus D$ has the same continuity as an analytic set of dimension k, k being an integer, $1 \leq k < n$. For instance, if D is q-complete with corners, then D is pseudoconvex of order $n-q$. By using Hartogs figures of order k one defines *pseudoconvexity of order k* for domains over complex manifolds.

The problem of measuring convexity of manifolds which are domains of local homeomorphisms as a function of the convexity of the image manifolds is very classical and has been central in complex analysis for more than two decades (see [29], [16], [17], [13], [35], [36], [19], [20], [23], [8]). The Levi's problem, or the inverse problem of Hartogs, for domains over a complex manifold, is stated as follows:

"Let (D, π) be a pseudoconvex domain over a complex manifold X. Under what conditions is D a Stein manifold?"

Oka [29] solved affirmatively this problem in the original and fundamental case, i.e., for domains over a Euclidean space $\mathbb{C}^n$.

Since then there has been several extensions of this result for various complex manifolds X which we now resume. First of all, Docquier and Grauert [8] solved the case when X is a Stein manifold, Fujita [13] and Takeuchi [35] considered X the complex projective space $\mathbb{P}^n$, then Hirschowitz ([19], [20]) investigated the case when X is an infinitesimally homogeneous manifold. (See also Ueda [37] which improved upon some result of Hirschowitz for the Grassmannian.) Also Takeuchi's approach [36] revealed geometric aspects for the case when X is a Kähler manifold.

On the other hand, a first attempt to a different set-up was made by Ballico [3] who settled the case of topological coverings of q-complete manifolds, and only recently some progress has been made for pseudoconvexity of general order (see [25], [40]).

It is now a natural question to ask for further convexity properties of pseudoconvex domains over q-complete manifolds, or, more generally, of domains pseudoconvex of order k over manifolds X in a suitable class as mentioned above.

This paper stemmed from answering these questions. Our first main result gives a natural generalization of Oka-Docquier-Grauert theorem which is recovered for $q = 1$ (with a new proof without using Remmert's embedding theorem) and completes Ballico's result quoted above. It is stated as follows:

THEOREM 1. *Let (D, π) be a pseudoconvex domain over a complex manifold X. If X is $\mathcal{M}$-complete with respect to a linear set $\mathcal{M}$ over X, then D is $\pi^*\mathcal{M}$-complete. In particular, if X is q-complete, then D is q-complete, too.*

We remark that Theorem 1 fails if we allow branching. However, if $\pi : D \longrightarrow X$ is a ramified covering and X is q-complete, then D is q-complete. See Section 6.

A characterization of pseudoconvexity of order $n-q$ via q-convexity with corners as well as transitivity properties for pseudoconvexity of general order are shown in Corollaries 3 and 10 respectively. In fact a more general statement holds, namely;

THEOREM 2. *Let (D, π) be a domain pseudoconvex of order $n - q$ over a connected complex manifold X of dimension n. If X is r-complete with corners, then D is $(q + r - 1)$-complete with corners.*

Again this theorem fails if we allow branching, but it holds true if π is a ramified covering. See Section 6 for the correct statement. Note that a weaker assertion is true, namely; if $\pi : Z \longrightarrow X$ is a holomorphic map of complex spaces which is locally q-complete with corners (see Sect. 2.2) and if X is r-complete with corners, then Z is $(q + r)$-complete with corners.

In the same circle of ideas, applying Theorem 2 we obtain:

THEOREM 3. *Let (D, π) be a domain over a Stein manifold X of pure dimension n such that for every smooth hypersurface $\Sigma \subset X$, $\pi^{-1}(\Sigma)$ is q-complete with corners. If $n > q + 1$, then D is q-complete with corners.*

The case $q = 1$ improves on a result from [1] and gives the "right" statement in the set-up of Stein manifolds of the classical work of Lelong [24] in the euclidean space. It is stated as follows:

COROLLARY 1. *If (D, π) is a domain over a connected Stein manifold X of dimension $n > 2$ such that for every smooth hypersurface $\Sigma \subset X$, $\pi^{-1}(\Sigma)$ is Stein, then D is also Stein.*

The next result, *viz.* Theorem 4, may be regarded as a natural generalization of R. Fujita's and Takeuchi's theorem quoted above which is recovered for $q = 1$. For this we use q-plurisubharmonic functions (see Sect. 2.3).

THEOREM 4. *Let (D, π) be a domain pseudoconvex of order $n - q$ over $\mathbb{P}^n$. If D is not biholomorphic to $\mathbb{P}^n$ via π, then D is q-complete with corners.*

(A counterexample when we have branching is shown in [27] for $q = 1$.)

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2. – Preliminaries

Throughout this paper all complex spaces are assumed to be reduced and with countable topology. For practical purposes we abbreviate "usc" and "psh" for "upper semi-continuous" and "plurisubharmonic", a complex manifold of pure dimension n will be referred to as "n-fold", and if U and V are subsets of a topological space (which will be clear from the context) the notation "$U \Subset V$" means that the closure $\overline{U}$ of U is compact and contained in the interior of V.

2.1. – Convexity with respect to linear sets

Let X be a complex space and T_xX denotes the Zariski tangent space of X at x. Set $TX = \cup_{x\in X} T_xX$.

A subset $\mathcal{M} \subset TX$ is said to be a *linear set over* X if for every point $x \in X$, $\mathcal{M}_x := \mathcal{M} \cap T_xX$ is a complex vector subspace. If $\Omega \subset X$ is open, we have an obvious definition for $\mathcal{M}|_\Omega$ as a linear set over Ω. Then we put:

$$\operatorname{codim}_X \mathcal{M} := \sup_{x\in X} \operatorname{codim}_{T_xX} \mathcal{M}_x .$$

For practical purposes, when the ambient space is clear from the context, we write $\operatorname{codim}\mathcal{M}$ instead of $\operatorname{codim}_X\mathcal{M}$.

Let $\pi : Z \longrightarrow X$ be an holomorphic map of complex spaces and $\mathcal{M}$ a linear set over X. For every $z \in Z$ we have an induced $\mathbb{C}$-linear map of complex vector spaces $\pi_{*,z} : T_zZ \longrightarrow T_xX$, where $x = \pi(z)$. We set

$$\pi^*\mathcal{M} := \bigcup_{z\in Z} (\pi_{*,z})^{-1}(\mathcal{M}_x).$$

Clearly, $\pi^*\mathcal{M}$ is a linear set over Z and $\operatorname{codim}_Z \pi^*\mathcal{M} \leq \operatorname{codim}_X \mathcal{M}$.

A (local) chart of X at a point $x \in X$ is a holomorphic embedding $\iota : U \longrightarrow \widehat{U}$, where $U \ni x$ is an open subset of X and $\widehat{U}$ is an open subset of some euclidean space $\mathbb{C}^n$. *Holomorphic embedding* means that $\iota(U)$ is an analytic subset of $\widehat{U}$ and the induced map $\iota : U \longrightarrow \iota(U)$ is biholomorphic.

Suppose $\iota : U \longrightarrow \widehat{U}$ is a local chart at x; then the differential map $\iota_{*,x} : T_xX \longrightarrow \mathbb{C}^n$ of ι at x is an injective homomorphism of complex vector spaces.

Let $D \subset \mathbb{C}^n$ be an open subset. A function $\varphi \in C^\infty(D, \mathbb{R})$ is said to be *q-convex* if its *Levi form*

$$L(\varphi, z)(\xi) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j}(z)\xi_i\overline{\xi}_j,\ \xi \in T_z D = \mathbb{C}^n,$$

has at least $n-q+1$ positive (> 0) eigenvalues for every $z \in D$, or equivalenty, there exists a family $\{M_z\}_{z \in D}$ of $(n-q+1)$-dimensional vector subspaces of $TD = D \times \mathbb{C}^n$ such that $L(\varphi, \cdot)|_{M_z}$ is a positively definite quadratic form for all $z \in D$.

Let X be a complex space. A function $\varphi \in C^\infty(X, \mathbb{R})$ is said to be *q-convex* if every point of X admits a local chart $\iota : U \longrightarrow \widehat{U} \subset \mathbb{C}^n$ such that there is a q-convex function $\widehat{\varphi} \in C^\infty(\widehat{U}, \mathbb{R})$ with $\widehat{\varphi} \circ \iota = \varphi|_U$. (This definition does not depend on the local embeddings.)

We say that X is *q-complete* if there exists a q-convex function $\varphi \in C^{(}X, \mathbb{R})$ which is *exhaustive*, i.e., the sublevel sets $\{x \in X\,;\, \varphi(x) < c\}, c \in \mathbb{R}$, are relatively compact in X. We choose the normalization such that 1-complete spaces correspond to Stein spaces.

The following definition is due to M. Peternell [32].

Definition 1. Let X be a complex space, $\mathcal{M}$ a linear set over X, and $\varphi \in C^\infty(X, \mathbb{R})$.

a) Let $x \in X$. We say that φ is *weakly 1-convex with respect to* $\mathcal{M}_x$ if there are: a local chart $\iota : U \longrightarrow \widehat{U}$ such that $\widehat{\varphi} \circ \iota = \varphi|_U$ and $L(\widehat{\varphi}, \iota(x))(\iota_{*,x}(\xi)) \geq 0$ for every $\xi \in \mathcal{M}_x$. We say that φ is *weakly 1-convex with respect to* $\mathcal{M}$ if φ is 1-convex with respect to $\mathcal{M}_x$ for every $x \in X$.
b) The function φ is said to be *1-convex with respect to* $\mathcal{M}$ if every point of X admits a neighborhood U for which there exists a 1-convex function θ on U such that $\varphi - \theta$ is weakly 1-convex with respect to $\mathcal{M}|_U$.

(It is not difficult to see that the extension $\widehat{\varphi}$ of φ is irrelevant for the above definition!)

Let X be a complex space and $\mathcal{M}$ a linear set over X. A function $\varphi \in C^0(X, \mathbb{R})$ is said to be *$\mathcal{M}$-convex* if every point of X admits a neighborhood D on which there are functions $\varphi_1, \dots, \varphi_k \in C^\infty(D, \mathbb{R})$ which are 1-convex with respect to $\mathcal{M}|_D$ and

$$\varphi|_D = \max(\varphi_1, \dots, \varphi_k).$$

Denote by $C(X, \mathcal{M})$ the set of all $\mathcal{M}$-convex functions on X, where $\mathcal{M}$ is a linear set over X.

We say that X is *$\mathcal{M}$-complete* if there is an exhaustion function $\varphi \in C(X, \mathcal{M})$.

From [39] and [32] we quote respectively:

PROPOSITION 1. *Let $\mathcal{M}$ be a linear set over a complex space X and $\varphi \in C(X, \mathcal{M})$. Then for every $\eta \in C^0(X, \mathbb{R})$, $\eta > 0$, there exists $\widetilde{\varphi} \in C^\infty(X, \mathbb{R})$ which is 1-convex with respect to $\mathcal{M}$ and*

$$\varphi \leq \widetilde{\varphi} < \varphi + \eta.$$

In particular, if codim $\mathcal{M} < q$, then $\widetilde{\varphi}$ is q-convex.

PROPOSITION 2. *Let X be a complex space and φ a q-convex function on X. Then there is a linear set $\mathcal{M}$ over X such that codim $\mathcal{M} < q$ and φ is 1-convex with respect to $\mathcal{M}$.*

As a consequence of these propositions we get:

COROLLARY 2. *Let X be a complex space and φ and ψ two functions on X which are q-convex and r-convex respectively. Then for every $\eta \in C^0(X, \mathbb{R})$, $\eta > 0$, there is a $(q + r - 1)$-convex function θ on X such that*

$$|\theta - \max(\varphi, \psi)| < \eta.$$

2.2. – Convexity with corners

Let X be a complex space. A function $\psi \in C^0(X, \mathbb{R})$ is said to be *q-convex with corners* [30] if every point of X admits a neighborhood $U \subset X$ on which there are finitely many q-convex functions $\psi_1, \dots, \psi_k$ such that

$$\psi|_U = \max(\psi_1, \dots, \psi_k).$$

Denote by $F_q(X)$ the set of all continuous functions on X which are q-convex with corners.

Following [32] denote by $\mathcal{S}_r$ the class of complex spaces X such that $F_r(U) \neq \emptyset$ for every open $U \Subset X$. Also we denote $\mathcal{S}_r^*$ those complex spaces X such that on every relatively compact open subset U of X there are r-convex functions. Clearly, $\mathcal{S}_r^* \subseteq \mathcal{S}_r$ for every r. The equality holds for $r = 1$. As examples we give:

1) Every K-complete space belongs to $\mathcal{S}_1$. (A complex space X is *K-complete* in the sense of Grauert if and only if for every point $x \in X$ the set of all points $y \in X$ for which $f(y) = f(x)$ for all global holomorphic functions f has dimension 0 at x.)
2) If $\pi : X \longrightarrow Y$ is a holomorphic map of complex spaces with fibers in $\mathcal{S}_1$ and $Y \in \mathcal{S}_r^*$ (resp. $Y \in \mathcal{S}_r$), then then $X \in \mathcal{S}_r^*$ (resp. $X \in \mathcal{S}_r$).
3) If A is an analytic subset of a complex space X and $A \in \mathcal{S}_r$ (resp. $A \in \mathcal{S}_r^*$), then there is a neighborhood V of A in X such that $V \in \mathcal{S}_r$ (resp. $V \in \mathcal{S}_r^*$).

We say that X is *q-convex with corners* if there is an exhaustion function φ on X such that $\varphi \in F_q(X \setminus K)$ for some compact subset K of X. If we can choose $K = \emptyset$, then X is sais to be *q-complete with corners.*

Let $\pi : X \longrightarrow Y$ be a holomorphic map of complex spaces. We say that π is *locally q-complete* (resp. *locally q-complete with corners*) if every point of Y

admits a neighborhood V such that $\pi^{-1}(V)$ is q-complete (resp. q-complete with corners). In case $q = 1$ we also call π *locally Stein.*

Following [30], we introduce two subsets $H_q(X)$ and $G_q(X)$ of $C^0(X, \mathbb{R})$ as follows:

$H_q(X) :=$ the set of all $h \in C^0(X, \mathbb{R})$ such that for every $x \in X$ and neighborhood U of x there is a neighborhod V of x with $V \Subset U$ and $f \in F_q(V) \cap C^0(\overline{V}, \mathbb{R})$ such that $f(x) = h(x)$ and $f < h$ on ∂V.

$G_q(X) := C^0(X, \mathbb{R}) \cap \bigcap_{x \in X} G_q(x)$, where for $x \in X$, $G_q(x) :=$ the set of all functions $g : X \longrightarrow \mathbb{R}$ such that there are: a neighborhood U of x and $f \in F_q(U)$ with $f(x) = g(x)$ and $f \leq g$ on U.

Clearly, one has $F_q(X) \subset G_q(X) \subset H_q(X)$. The next lemma completes Lemma 1 in [30].

LEMMA 1. *$F_q(X)$ is dense in $H_q(X)$ with respect to the C^0-topology;* a fortiori, *$F_q(X)$ is dense in $G_q(X)$ with respect to the C^0-topology.*

PROOF. We have to show that for $h \in H_q(X)$ and $\eta \in C^0(X, \mathbb{R})$, $\eta > 0$, there is $\widetilde{h} \in F_q(X)$ with $|\widetilde{h} - h| < \eta$.

STEP 1. We claim that for every compact set $K \subset X$ and open set $\Omega \Subset X$ with $K \subset \Omega$ one has the following property. For every $x \in K$, there are neighborhoods V and U of x, $V \Subset U \Subset \Omega$, and $f \in F_q(U) \cap C^0(\overline{U}, \mathbb{R})$ such that:

$$(\star) \qquad f|_V > h|_V, \; f < h + \eta \text{ on } \overline{U}, \text{ and } f < h \text{ on } \partial U.$$

Put $n = \dim_x X$; then let $\epsilon > 0$ with $2\epsilon < \inf_\Omega \eta$ and W neighborhood of x such that $W \Subset \Omega$, $|h(y) - h(z)| \leq \epsilon$ for $y, z \in W$ and, $\dim_w X \leq n$ for $w \in W$.

By hypothesis there is a neighborhood $U \Subset W$ of x and $g \in F_q(U) \cap C^0(\overline{U}, \mathbb{R})$ such that $g(x) = h(x)$ and $g < h$ on ∂U. If $q \geq n+1$, then $F_{n+1}(W)$ is dense in $C^0(W, \mathbb{R})$ and property $(\star)$ follows easily. If $q \leq n$, then $g < h + \epsilon$ on $\overline{U}$. In order to see this, we let $u \in \partial U$ such that $g \leq g(u)$ on $\overline{U}$ (by applying the maximum principle for functions q-convex with corners); hence for $y \in \overline{U}$, $g(y) - h(y) \leq g(u) - h(u) + h(u) - h(y) < \epsilon$.

Take $f := c + g$ where $c > 0$ is such that $c < \min_{\partial U}(h - g)$ and $c < \epsilon$. Since $f(x) > h(x)$, the existence of V follows by continuity, whence the claim.

STEP 2. To conclude, fix a locally finite open covering $\{\Omega_i\}_{i \in I}$ of X by relatively compact subsets. Then choose compact sets $K_i \subset \Omega_i$ whose union equals X. The above step gives open sets $V_{ij} \Subset U_{ij} \Subset \Omega_i$, j runs over a finite set of indices J_i, and $f_{ij} \in F_q(U_{ij}) \cap C^0(\overline{U_{ij}}, \mathbb{R})$ with property $(\star)$ from the claim in Step 1. Furthermore one may assume that $K_i \subset \cup_j V_{ij}$.

Define $\widetilde{h} : X \longrightarrow \mathbb{R}$ by setting: $\widetilde{h}(x) = \max_{i,j}\{f_{ij}(x)\,;\, x \in U_{ij}\}$. It is easily seen that $\widetilde{h}$ has all the required properties. □

LEMMA 2. *Let U be a complex space, V an r-fold, and $f \in F_{q+r}(U \times V)$, $\sup f < +\infty$. Define $s : U \longrightarrow \mathbb{R}$, $s(x) = \sup\{f(x, y)\,;\, y \in V\}$, $x \in U$, and let $x_0 \in U$ such that there is $y_0 \in V$ with $s(x_0) = f(x_0, y_0)$. Then $s \in G_q(x_0)$.*

PROOF. See [30], Lemma 4, pag. 256. □

LEMMA 3. *Let Y be a complex space and $S \subset Y$ an analytic set. Then there exists $h \in C^\infty(Y, \mathbb{R})$ such that $h \geq 0$, $\{h = 0\} = S$, and $\log h$ is quasi-psh in the sense that for every point $y \in Y$ there exist a neighborhood U and $\theta \in C^\infty(U, \mathbb{R})$ such that $\log h + \theta$ is psh on $U \setminus S$.*

PROOF. See [32]. □

Now we are prepared to prove

PROPOSITION 3. *Let $\pi : E \longrightarrow X$ be a locally trivial holomorphic fibration of complex spaces with fiber Y. Suppose Y is compact and let $r = \dim(Y)$. Then E is $(q+r)$-complete with corners (resp. $(q+r)$-convex with corners) if and only if X is q-complete with corners (resp. q-convex with corners).*

PROOF. Let $\Lambda := \{\chi \in C^\infty(\mathbb{R}, \mathbb{R})\,;\ \chi' > 0, \chi'' \geq 0\}$.

We consider the "if" case. Since Y is r-dimensional, $F_{r+1}(Y)$ is dense in $C^0(Y, \mathbb{R})$; let $\theta \in F_{r+1}(Y)$, $\theta \geq 1$. Then consider open sets $V_i \Subset U_i \Subset W_i \Subset X$ such that $\cup V_i = X$, $\{W_i\}_i$ is locally finite, and $\pi^{-1}(W_i) \simeq W_i \times Y$, where the index i runs through a set I. Let $\rho_i \in C_0^\infty(W_i, \mathbb{R})$ such that $-1 \leq \rho_i \leq 1$, $\rho_i = 1$ on V_i, and $\rho_i = -1$ near ∂U_i. Consider $\psi_i : \pi^{-1}(W_i) \longrightarrow [1, \infty)$ canonically induced by θ via the composition $\pi^{-1}(W_i) \simeq W_i \times Y \longrightarrow Y$. Let $\varphi \in C^0(X, \mathbb{R})$ be exhaustive and $\varphi \in F_q(X \setminus K)$ for some compact set $K \subset X$. Let L be a compact neighborhood of K; if $K = \emptyset$, then we choose $L = \emptyset$.

For $\chi \in \Lambda$ define a function $\Phi_\chi : E \longrightarrow \mathbb{R}$ by setting:

$$\Phi_\chi(\xi) = \sup\{\chi(\varphi(\pi(\xi))) + \psi_i(\xi)\rho_i(\pi(\xi))\,;\ i \text{ such that } U_i \ni \pi(\xi)\}, \xi \in E.$$

Clearly Φ_χ is continuous and exhaustive, and if χ' is sufficiently large, then $\Phi_\chi \in F_{q+r}(E \setminus \pi^{-1}(L))$. (Notice that if Y is smooth, then one can take simply $\Phi = \varphi \circ \pi$.)

Here we consider the "only if" case. Let $S \subset Y$ be an analytic set such that $Y \setminus S$ is a complex manifold of pure dimension r. Let $A \subset E$ be the analytic set canonically induced by S so that $E_x \setminus A_x$ where $E_x = \pi^{-1}(x) \simeq Y$ is the fiber over x and $A_x = A \cap E_x \simeq S$. Let V be a neighborhood of A in E such that $V_x \neq E_x$ for every x. Apply the Lemma 3 and get $h \in C^\infty(E, \mathbb{R})$ such that $h \geq 0$, $\{h = 0\} = A$, and $\log h$ is quasi-psh. Let $\Phi \in C^0(E, \mathbb{R})$ be exhautive such that $\Phi \in F_{q+r}(E \setminus L)$ for some compact $L \subset E$. Let $K \subset X$ be a compact set such that $\pi^{-1}(K)$ is a neighborhod of L (if $L = \emptyset$ we choose $K = \emptyset$). For $\chi \in \Lambda$ with χ' large enough one has $\chi(\Phi) + \log h \in F_{q+r}(E \setminus A)$ and $\chi(\Phi) + \log h$ is exhaustive for $E \setminus V$. Let $\widetilde{\Phi} : E \longrightarrow \mathbb{R}$ given by $\widetilde{\Phi} = h \exp(\chi(\Phi))$ and consider $\varphi : X \longrightarrow \mathbb{R}$ defined by

$$\varphi(x) = \sup\{\widetilde{\Phi}(\xi)\,;\ \xi \in E_x\}, x \in X.$$

Then φ is continuous and exhaustive for X and by Lemma 2 we get $\varphi \in G_q(X \setminus K)$, whence the proof of the proposition applying Lemma 1. □

2.3. – q-plurisubharmonic functions

In order to deal with pseudoconvexity of general order we need q-plurisubharmonic functions which are defined below.

Let X be a complex space, $\varphi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ an upper semicontinuous function (**usc**, for short), and $q \in \mathbb{N}, q > 0$. We say that φ is:

(1) *subpluriharmonic* if for every open set $\Omega \Subset X$ and every pluriharmonic function h on a neighborhood of $\overline{\Omega}$, i.e., h is locally the real part of a holomorphic function, if $\varphi \leq h$ on $\partial\Omega$, then $\varphi \leq h$ on Ω.

(2) *q-plurisubharmonic* (q-**psh**, for short) if for every open set $G \subset \mathbb{C}^q$ and holomorphic map $f : G \longrightarrow X$ the function $\varphi \circ f$ is subpluriharmonic. Denote by $P_q(X)$ the set of all q-psh functions on X.

(3) *strongly q-plurisubharmonic* if for every $\theta \in C_0^\infty(X, \mathbb{R})$ there is $\epsilon > 0$ such that $\varphi + \epsilon\theta \in P_q(X)$. Denote by $SP_q(X)$ the set of all strongly q-psh functions on X.

(In O. Fujita's terminology [11], q-plurisubharmonic functions in $\mathbb{C}^n$ are called "pseudoconvex of order $n-q$" and they coincide with "$(q-1)$-plurisubharmonic functions" in the sense of Hunt and Murray [21]. For the equivalence of these notions see [12].)

As a motivation for our choice we note that a function $\varphi \in C^2(D, \mathbb{R})$, where D is an open subset of $\mathbb{C}^n$, is q-psh iff the Levi form $L(\varphi, z)$ has at least $n-q+1$ non-negative eigenvalues for every $z \in D$. Any function in $F_q(D)$ is strongly q-psh. Also usual plurisubharmonicity equals 1-plurisubharmonicity and an usc function $\varphi : D \longrightarrow \mathbb{R} \cup \{-\infty\}$, $D \subset \mathbb{C}^q$ is open, is q-psh if and only if φ is subpluriharmonic.

Denote by $Gr_q(n)$ the Grassmann variety of q-dimensional planes of $\mathbb{C}^n$. Here we improve on a result from [12] to:

PROPOSITION 4. *Let $D \subset \mathbb{C}^n$ be an open subset and $\varphi : D \longrightarrow \mathbb{R}\cup\{-\infty\}$ an usc function. Assume that for every point $a \in D$ there is a dense subset $\mathcal{H}_a \subset Gr_q(n)$ such that the restriction of φ to $(\{a\}+H)\cap D$ is subpluriharmonic for every $H \in \mathcal{H}_a$. Then φ is q-psh.*

PROOF. For the commodity of the reader we give a proof of this following the arguments in [12] where the case $\mathcal{H}_a = Gr_q(n)$, $a \in D$, is treated.

Consider for $c \in \mathbb{R}$, $\Omega_c := \{(z, w) \in D \times \mathbb{C};\ \varphi(z) + \log|w| < c\}$. By [40], φ is q-psh if and only if $\Omega_0 \subset \mathbb{C}^{n+1}$ is pseudoconvex of order $k := n + 1 - q$.

In order to check this, we let $\Phi : D \times \mathbb{C} \longrightarrow \mathbb{R} \cup \{-\infty\}$ defined by $\Phi(z, w) = \varphi(z) + \log|w|$, $z \in D$, $w \in \mathbb{C}$.

Observe that every q-plane $\Gamma \subset \mathbb{C}^{n+1}$ is: either in the form $\Gamma = \Gamma' \times \mathbb{C}$ for some $(q-1)$-plane $\Gamma' \subset \mathbb{C}^n$, thus the restriction of Φ to $\Gamma \cap (D \times \mathbb{C})$ is subpluriharmonic; or $\Gamma = \{(z, l(z));\ z \in \Gamma''\}$, where $\Gamma'' \subset \mathbb{C}^n$ is a q-plane and $l : \mathbb{C}^n \longrightarrow \mathbb{C}$ is $\mathbb{C}$-linear, hence for arbitrary such l and $\Gamma'' \in \mathcal{H}_0$, the hypothesis implies that the restriction of Φ to $\Gamma \cap (D \times \mathbb{C})$ is subpluriharmonic.

Suppose now, in order to reach a contradiction, that Ω_0 is not pseudoconvex of order k. Hence after scaling and translation, without any loss in generality, we may assume that $\Delta^n(1) \Subset D$, $(0, 0) \notin \Omega_0$, and $\partial\Delta^q(1) \times \{0\} \subset \Omega_0$.

Thus there is $r > 0$ such that $K := \partial\Delta^q(1) \times \overline{\Delta^k(r)} \subset \Omega_0$ and a q-plane $\Gamma \subset \mathbb{C}^{n+1}$ with $\Gamma \cap (\overline{\Delta^q(1)} \times \partial\Delta^k(r)) = \emptyset$ and Φ restricted to $\Gamma \cap (D \times \mathbb{C})$ is subpluriharmonic. Now, as the natural projection from $\Gamma \cap (\Delta^q(1) \times \Delta^k(r))$ into $\Delta^q(1)$ is proper and bijective (standard exercise in linear algebra), the maximum principle for subpluriharmonic functions implies that $\Phi(0,0) \leq \max_K \Phi$. Hence $\Phi(0,0) < 0$. Thus $(0,0) \in \Omega_0$, which is contradictory! □

From [40] we quote the following aproximation result.

Proposition 5. *Let X be a complex manifold. Then the subset $F_q(X)$ of $SP_q(X) \cap C^0(X,\mathbb{R})$ is dense with respect to the C^0-topology.*

2.4. – Domains over complex manifolds

Let X be a n-fold. By a *a domain* over X we mean a pair (D,π) where D is a connected topological space which is Hausdorff and $\pi : D \longrightarrow X$ is a continuous map which is locally homeomorphic. Clearly D inherits a complex structure via π with respect to which π becomes locally biholomorphic.

A *branched Riemann domain over* X is a pair (Y,π), where Y is a connected n-dimensional complex manifold and $\pi : Y \longrightarrow X$ is a holomorphic map with discrete fibers. A branched Riemann domain (Y,π) over X is *locally Stein* if there exists for every $x \in X$ a neighborhood $U(x)$ with $\pi^{-1}(U(x))$ Stein.

For $r > 0$ set $\Delta^k(r) := \{z \in \mathbb{C}^k\,;\ |z_1| < r, \dots, |z_k| < r\}$. A *Hartogs figure of order* q in $\mathbb{C}^n$, or more precisely a $(q, n-q)$ *Hartogs figure* (see [10]), q an integer, $1 \leq q < n$, is defined by:

$$H_q := \left(\Delta^q(1) \times \Delta^{n-q}(r)\right) \bigcup \left((\Delta^q(1) \setminus \overline{\Delta^q(s)}) \times \Delta^{n-q}(1)\right)$$

where $0 < r, s < 1$. Put $\widehat{H}_q := \Delta^n(1)$.

Definition 2. A domain (D,π) over a complex manifold X of pure domension n is *pseudoconvex of order* $n-q$, q an integer, $1 \leq q < n$, if for every injective holomorphic map $f : \widehat{H}_q \longrightarrow X$ such that $f|_{H_q}$ lifts to D, then f lifts to D.

Remark 1. We can also say that a domain (D,π) over X is *locally pseudoconvex of order* $n-q$ if every point of X has a neighborhood U such that the domain $(\pi^{-1}(U), \pi|_{\pi^{-1}(U)})$ is pseudoconvex of order $n-q$ over U. As a matter of fact, by [40], this is equivalent to psedoconvexity of order $n-q$.

Usual pseudoconvexity means pseudoconvexity of order $n-1$. For practical purposes, every domain is pseudoconvex of order 0. Note that a domain (D,π) over X is locally pseudoconvex iff π is locally Stein. A general statement is given below in Corollary 3.

Specialize now to the case (D,π) is a domain over $\mathbb{C}^n$. Denote by S the unit sphere in $\mathbb{C}^n$, i.e., $S = \{w \in \mathbb{C}^n\,;\ \|w\| = 1\}$. For each $w \in S$ define the *Hartogs radius of* (D,π) *in direction* w as a function

$$R_w : D \longrightarrow (0,\infty],$$

where for $\xi \in D$ we set $R_w(\xi) :=$ the supremum of all $r > 0$ such that there is a neighborhood U of ξ in $\tau^{-1}(L_w)$ which is mapped biholomorphically via τ onto the disc in L_w of center $\tau(\xi)$ and radius r, where L_w is the complex line $L_w = \{\tau(\xi) + tw\,;\ t \in \mathbb{C}\}$.

Then R_w is lower semi-continuous and if δ denotes the boundary distance function for the domain (D, π) over $\mathbb{C}^n$, then

$$\delta = \inf_{w \in S} R_w.$$

Let us recall the following more or less known fact (a nice exercise in topology)

Proposition 6. *Let A and B two metric spaces, $A_1 \subset A$ and $B_1 \subset B$ two subsets, and $f : A \longrightarrow B$ a continuous map such that:*

1) *$f|_{A_1}$ is homeomorphic onto B_1;*
2) *For every $x \in A_1$ there exists a neighborhood V_x of x in A such that $f|_{V_x}$ is homeomorphic onto a neighborhood of $f(x)$ in B.*

Then there exists a neighborhood V of A_1 in A such that $f|_V$ is homeomorphic onto a neighborhood of B_1 in B.

Consequently, our Hartogs radii, which are easier to manipulate, equals those from [25]. Now, combining results from [25] and [40] give:

Proposition 7. *Keeping the notation as above, the next statements are equivalent:*

(i) *(D, π) is pseudoconvex of order $n - q$ over $\mathbb{C}^n$.*
(ii) *For every $w \in S$, $-\log R_w$ is q-psh.*
(iii) *$-\log \delta$ is q-psh.*
(iv) *D is q-complete with corners.*

From this and [40] we have:

Corollary 3. *For a domain (D, π) over a n-fold X pseudoconvexity of order $n - q$ is equivalent to π is locally q-complete with corners.*

Corollary 4. *Let (D, π) be a domain pseudoconvex of order $n-q$ over $\mathbb{C}^n$ and δ its boundary distance function. If $\pi(D)$ is bounded, then for every $\rho \in \mathbb{R}$, $\rho > 1$, there is $\widetilde{\delta} \in C^0(D, \mathbb{R})$, $\widetilde{\delta} > 0$, such that $1/\rho < \widetilde{\delta}/\delta < \rho$ and $-\log \widetilde{\delta}$ is q-convex with corners.*

Proof. Let $\theta > 0$ be any 1-convex function on $\mathbb{C}^n$ and positive constants ϵ and c such that $2\epsilon = \log \rho$ and $c\theta < \epsilon$ on $\pi(\Omega)$. Since $-\log \delta$ is continuous and q-psh, by Proposition 5, there is $\widetilde{\psi} \in F_q(\Omega)$ such that $|\widetilde{\psi} + \log \delta - c(\theta \circ \tau)| < \epsilon$. Finally, $\widetilde{\delta} := \exp(-\widetilde{\psi})$ is as desired. □

3. – An abstract patching lemma

LEMMA 4. *Let Z be a topological space and $\mathcal{G}$ a sheaf of germs of continuous functions on Z such that for every open set $U \subset Z$, arbitrary constants $a, b \in \mathbb{R}$, $a > 0$, and arbitray sections $\sigma, \tau \in \mathcal{G}(U)$ one has: $a\sigma, \sigma + \tau, \sigma + b, \max(\sigma, \tau) \in \mathcal{G}(U)$. Let $\Phi \in \mathcal{G}(Z)$ and $\{a_i\}_i$ an increasing sequence tending to ∞. Set $Z_i := \{z \in Z\,;\ \Phi(z) < a_i\}$, $i \in \mathbb{N}$. Assume that*

($\sharp$) *For every $i \in \mathbb{N}$ there exists $\Phi_i \in \mathcal{G}(Z_i)$ such that the subsets of Z_i given by $\{\zeta \in Z_i\,;\ \Phi_i(\zeta) < c\}$, $c \in \mathbb{R}$, are relatively compact in Z.*

Then there is an exhaustion function $\Psi \in \mathcal{G}(Z)$.

REMARK 2. Notice that the above Φ_ν's are not required to be exhaustive for Z_ν. This lemma will be invoked in the following form. Z will be a complex space and for $\mathcal{G}$ we take: (i) the sheaf of germs of functions q-convex with corners; (ii) the sheaf of germs of continuous (strongly) q-psh functions; and (iii) the sheaf of germs of $\mathcal{M}$-convex functions with respect to a linear set $\mathcal{M}$ over Z.

THE PROOF OF THE LEMMA 4. We follow the recipe of Oka [29]. In order to do this, choose an exhaustion $\{\Delta_i\}_i$ of Z by open sets such that $\Delta_i \Subset \Delta_{i+1}$ and $\Delta_i \Subset Z_i$ for $i = 1, 2, \cdots$. Put

$$\Delta_i' := \{\zeta \in Z_i\,;\ \Phi_i(\zeta) < \alpha_i\},\ i = 1, 2, \dots ,$$

and

$$\Delta_i'' := \{\zeta \in Z_i\,;\ \Phi_i(\zeta) < \beta_i\},\ i = 3, 4, \dots ,$$

where the constants α_i, β_j will be determined one after the other in the order

$$\alpha_1, \beta_3, \alpha_2, \beta_4, \dots , \alpha_i, \beta_{i+2}, \alpha_{i+1}, \dots ,$$

so that one has

$$\begin{cases} \Delta_i \subset \Delta_i', \\ \Delta_i' \Subset \Delta_{i+2}'', \\ \Delta_{i+2}'' \cap \{\Phi < (a_i + a_{i+1})/2\} \Subset \Delta_{i+1}'. \end{cases}$$

(Condition ($\sharp$) shows that one can choose the constants α_i, β_j.) Now we construct a sequence of sections $\psi_i \in \mathcal{G}(Z_i)$, $i = 2, 3, \dots$, as follows. Set

$$\psi_2 = \Phi_2.$$

In order to get ψ_3, let $\psi_3' \in \mathcal{G}(Z_3)$ defined by:

$$\psi_3' := c_2 \max\Big(\Phi - (a_1 + a_2)/2, \Phi_3 - \beta_3\Big).$$

If $c_2 > 1$ is large enough, we may arrange things so that

$$\begin{cases} \psi_3' < \psi_2 & \text{on } \Delta_1'; \\ \psi_3' > a_2 & \text{on } Z_3 \setminus \Delta_2'; \\ \psi_3' > \psi_2 & \text{on } \partial\Delta_2'. \end{cases}$$

(This is possible since we have $\Delta_1' \Subset \{\psi_3' < 0\} = \Delta_3'' \cap \{\Phi < (a_1+a_2)/2\} \Subset \Delta_2'$. Hence for some $\epsilon > 0$, $\{\psi' < \epsilon\} \subset \Delta_2'$. Note that here $Z_3 \setminus \Delta_2'$ is not supposed to be relatively compact in Z.) Then we define ψ_3 by setting:

$$\psi_3 := \begin{cases} \max(\psi_2, \psi_3') & \text{on } \Delta_2'; \\ \psi_3' & \text{on } Z_3 \setminus \Delta_2'. \end{cases}$$

We note that $\psi_3 = \psi_2$ in Δ_1' and $\psi_3 > a_2$ in $Z_3 \setminus \Delta_2'$. Now, we iterate this process and obtain the sequence of sections $\psi_2, \psi_3, \ldots, \psi_i, \ldots$, such that:

$$\begin{cases} \psi_i = \psi_{i-1} & \text{on } \Delta_{i-2}'; \\ \psi_i > a_{j-1} & \text{on } Z_i \setminus \Delta_{j-1}', \; i \geq j > 1. \end{cases}$$

Finally, $\Psi := \lim \psi_i$ belongs to $\mathcal{G}(Z)$ and is exhaustive for Z. □

The above lemma gives us the following interesting consequences in the spirit of Docquier and Grauert [8].

COROLLARY 5. *Let Z be a complex space and $\mathcal{M}$ a linear set over Z. Suppose there is a $\mathcal{M}$-convex function Φ on Z such that for every $c \in \mathbb{R}$ the sublevel set $\{\Phi < c\}$ is $\mathcal{M}$-complete. Then Z is $\mathcal{M}$-complete. In particular, if codim $\mathcal{M} < q$, Z is q-complete.*

In order to state another application of the method used in the proof of Lemma 4 we make the following observations.

If Z is a complex manifold and $\Phi \in C^\infty(Z, \mathbb{R})$ is q-convex on $Z \setminus A$, where $A \subset Z$ is a rare closed subset, then for every r-convex function σ defined on a open set $U \subset Z$, the function $\Phi + \sigma$ is $(q + r - 1)$-convex on U. This fails if Z has singularities; for instance, let $Z \subset \mathbb{C}^n (n \geq 2)$ be the complex curve given by $Z = f(\mathbb{C})$, where $f : \mathbb{C} \longrightarrow \mathbb{C}^n, t \mapsto (t^n, \ldots, t^{2n-1})$. Let θ be any 1-convex function on $\mathbb{C}^n$ and set $\Phi := (-\theta)|_Z, \sigma := \theta|_Z$. Clearly Φ is 2-convex on $\mathrm{Reg}(Z) = Z \setminus \{0\}$ and σ is 1-convex on Z. However, $\Phi + \sigma (\equiv 0)$ is not n-convex. In fact, if u is any smooth function on $\mathbb{C}^n$ such that $u|_Z = 0$ then its Levi form $L(u; 0)$ vanishes identically by [38].

The question of studying convexity properties for Z when similar properties for $Z \setminus A$ and A are given (A is, for example an analytic subset of Z) is motivated by an example due to Grauert [15] who, *mutatis mutandis*, produced a complex manifold Z together with a hypersurface $A \subset Z$ such that one has:

(i) There is a smooth proper plurisubharmonic function $\varphi : Z \longrightarrow [0, \infty)$ which is strongly plurisubharmonic on $Z \setminus A$.
(ii) Z is **not** holomorphically convex.

Here we give the following corollary.

COROLLARY 6. *Let Z be a complex manifold and $\Phi \in C^\infty(Z, \mathbb{R})$ exhaustive such that there is a rare closed subset $A \subset Z$ (not necessarily analytic) and Φ is q-convex on $Z \setminus A$. If there is a neighborhood V of A with $V \in \mathcal{S}_r$, then Z is $(q+r-1)$-complete with corners. Moreover, if $r = 1$, then Z is q-complete.*

PROOF. As in Lemma 4, put $Z_i = \{\phi < i\}$ and define $\Phi_i := \epsilon_i \sigma_i + 1/(i - \Phi)$, where $\epsilon_i > 0$ and $\sigma_i \in C^\infty(Z, \mathbb{R})$ is r-convex with corners on a neighborhood V_i of $A \cap Z_{i+1}$. Clearly Φ_i is exhaustive for Z_i and $\Phi_i \in F_{q+r-1}(V_i \cap Z_i)$; if ϵ_i is small enough, then Φ_i is q-convex near $Z_i \setminus V_i \subset Z_i$. Fix such an $\epsilon_i > 0$ and set on Z_3

$$\psi_3' := \max(\sigma_3 + c_2(\Phi - (a_1 + a_2)/2), c_2(\Phi_3 - \beta_3)),$$

for $c_2 \gg 1$, and proceed similarly as in Lemma 4. Then ψ_i are $(q+r-1)$-convex with corners and the first part of the corollary follows.

To show the "moreover" we observe that there is a linear set $\mathcal{M}$ over Z, codim $\mathcal{M} < q$, such that Φ is 1-convex with respect to $\mathcal{M}$ on $Z \setminus A$ and for every open set $U \subset Z$ and any 1-convex function θ on U, $\Phi + \theta$ is 1-convex with respect to $\mathcal{M}$ on U. (As a matter of fact, let $\mathcal{M}^1$ be the linear set over $Z \setminus A$ given by Proposition 2, codim$\mathcal{M}^1 < q$, such that Φ is 1-convex with respect to $\mathcal{M}^1$ on $Z \setminus A$. Since Z is a complex manifold, for every $a \in A$ there is a there is $F_a \subset T_a Z$ a complex vector space of codimension $< q$ such that $L(\Phi, z)$ is semi-positive when restricted to F_a. We define $\mathcal{M}$ over Z by setting $\mathcal{M}_z = \mathcal{M}^1_z$ for $z \in Z \setminus A$ and $\mathcal{M}_z = F_z$ for $z \in A$.) Using this we deduce that ψ_i', $i = 2, 3, \ldots$, are $\mathcal{M}$-convex; hence the proof results as above. □

REMARK 3. The condition on A are fulfilled, for instance, if A is analytic and r-complete (with corners).

On the other hand, Norguet and Siu [28] showed that a complex space X is Stein if it is K-complete and admits a continuous exhaustion function which is psh on $\mathrm{Reg}(X)$.

In the same circle of ideas we give the next result (*viz.*, Corollary 7) which may be regarded as a natural extension of the theorem of Norguet and Siu quoted above which is recovered for $q = 1$ with an elementary proof. As a first step to show this we state the next proposition.

PROPOSITION 8. *Let Z be a complex space, $\Phi \in C^0(Z, \mathbb{R})$ exhaustive, and $A \subset Z$ a rare analytic set such that: either* (a) *Φ is q-psh on $Z \setminus A$ and $Z \in \mathcal{S}_r$; or* (b) *Φ is strongly q-psh on $Z \setminus A$ and $A \in \mathcal{S}_r$. Then Z is $(q + r - 1)$-complete with corners.*

COROLLARY 7. *A K-complete complex space Z admitting a continuous exhaustion function which is q-psh on* $\mathrm{Reg}(Z)$ *is q-complete with corners.*

PROOF OF PROPOSITION 8. The key observation is that for a complex space Y one has $P_q(Y) + P_r(Y) \subseteq P_{q+r-1}(Y)$. This follows, for instance by definitions and Proposition 5.

Now, in case (a), by [42] we get that $\Phi \in P_q(Z)$ and one concludes by Lemma 4. In case (b) we adapt the proof of Corollary 6 in a straightforward manner. We omit the details. □

COROLLARY 8. *Let Z be a complex space and $\Phi \in F_q(Z)$ such that for every $c \in \mathbb{R}$, if we set $Z(c) = \{\Phi < c\}$, then there exists $\varphi_c \in F_q(Z(c))$ with the property that the open subsets of $Z(c)$, $\{\zeta \in Z(c)\,;\ \varphi_c(\zeta) < b\}$, $b \in \mathbb{R}$, are relatively compact in Z. Then Z is q-complete with corners.*

COROLLARY 9. *Let Z be a complex space and $\Phi \in F_q(Z)$ such that for every $c \in \mathbb{R}$ the set $\{\Phi < c\}$ is q-complete with corners. Then Z is q-complete with corners.*

4. – A patching procedure for domains

Consider (D, π) be a domain over a complex manifold X equipped with a complete hermitian metric g. Without any loss in generality, we shall assume that (D, π) is not univalent (otherwise π follows injective and D may be viewed as an open subset of X; in this case Theorems 1 and 2 are easier to prove. See the remark in Sect. 5.)

Let g^* be the hermitian metric canonically induced on D via π. For $x \in X$ and $r > 0$ denote $B_g(x; r) := \{y \in X\,;\ \mathrm{dist}_g(y, x) < r\}$. Define the boundary distance function

$$\delta : D \longrightarrow (0, \infty)$$

as follows. For $\zeta \in D$. Set $x := \pi(\zeta)$ and put $\delta(\zeta) :=$ the supremum of all $r > 0$ for which there is an open set $U \ni \zeta$ in D which is biholomorphic with $B_g(x; r)$ via π. Such an U will also be denoted by $U(\zeta; r)$. Note that for $\zeta \in D$, $U(\zeta; r) \subset \{\xi \in D\,;\ \mathrm{dist}_{g^*}(\xi, \zeta) < r\}$.

For $\epsilon > 0$ set $D_\epsilon := \{\zeta \in D\,;\ \delta(\zeta) > \epsilon\}$.

Also if Z is a differentiable manifold and $\{V_\alpha\}_{\alpha\in\Lambda}$ is a family of open subsets of Z, we say that a family of functions $\{f_\alpha\}_\alpha$, $f_\alpha \in C^\infty(V_\alpha, \mathbb{R})$, has *the real Hessian locally bounded from below* if for every point $a \in Z$ there is a coordinate patch $\Omega \ni a$ and coordiantes $(s_1, \dots, s_m)$ on Ω, $m = \dim(\Omega)$, such that for fome $C \in \mathbb{R}$ one has

$$\frac{\partial^2 f_\alpha}{\partial s_i \partial s_j}(x) \geq C,$$

for all $1 \leq i, j \leq m$, $\alpha \in \Lambda$ and $x \in \Omega \cap V_\alpha$. A standard example is when $\{V_\alpha\}_{\alpha\in\Lambda}$ is locally finite and f_α have compact supports.

If $\mathcal{G}$ is a sheaf over X, then π induces canonically a sheaf $\pi^\star(\mathcal{G})$ over D. Suppose $\mathcal{G}$ is a sheaf of germs of continuous functions on X with the following properties:

(i) For every open set $U \subset X$ and arbitrary sections $\sigma, \tau \in \mathcal{G}(U)$, one has $\sigma + \tau, \max(\sigma, \tau) \in \mathcal{G}(U)$.
(ii) If $\psi \in \mathcal{G}(X)$ is exhaustive, then for every family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, and family of functions $\{f_\alpha\}_\alpha$, $f_\alpha \in C^\infty(V_\alpha, \mathbb{R})$, with the real Hessian

locally bounded from below, there exists $\chi \in C^{\infty}(\mathbb{R}, \mathbb{R})$ strictly increasing and convex such that $\chi(\psi) + f_\alpha \in \mathcal{G}(V_\alpha)$ for all α.

(Typical examples for this situation are: the sheaf of germs of q-convex functions with corners and the sheaf of germs of $\mathcal{M}$-convex functions for a linear set $\mathcal{M}$ over X.) Under the above notation we give:

LEMMA 5. *Let $\psi \in \mathcal{G}(X)$ be exhaustive. Then for every $t > 0$ there exists $\varphi_t \in \pi^{\star}(\mathcal{G})(D_t)$ such that*

$$\{\zeta \in D_t\,;\ \varphi_t(\zeta) < c\} \Subset D,\ \forall c \in \mathbb{R}.$$

PROOF. STEP 1. Fix real numbers ϵ, r and s with $0 < \epsilon \leq r \leq s - \epsilon$. We claim that for every $\zeta \in D_s$ and $x \in X$ with $d(\pi(\zeta), x) < \epsilon$ there exists $\xi \in \pi^{-1}(x)$ such that $U(\xi; r) \subset D_\epsilon$ and $U(\xi; r) \ni \zeta$. Take σ a section of π over $B(\pi(\zeta); s)$ with $\sigma(\pi(\zeta)) = \zeta$. Since $B(x; r) \subset B(\pi(\zeta); s)$, the claim results easily.

In particular, if $\{x_i\}_i$ is a set of points in $\overline{\pi(D_s)}$ such that $\{B(x_i; \epsilon)\}_i$ covers $\overline{\pi(D_s)}$, then $\{U(\zeta_{ij}; r)\}_{ij}$ covers D_s, where $\{\zeta_{ij}\}_{j\in\Lambda_i} = \pi^{-1}(x_i)$.

STEP 2. Put $\epsilon = t/4$. Consider $\{x_i\}_{i\in I}$ be a discrete set (in X) of points in $\overline{\pi(D_t)}$ such that $\{B(x_i; \epsilon)\}_i$ covers $\overline{\pi(D_t)}$. Write $\{\zeta_{ij}\}_{j\in\Lambda_i} = \pi^{-1}(x_i)$, Λ_i being a set of indices.

Therefore the "balls" in D, $\{U(\zeta_{ij}; \epsilon)\}_{ij}$ cover D_t. Moreover $U(\zeta_{ij}; 3\epsilon) \subset D_\epsilon$. Denote $B_i' := B(x_i; \epsilon)$, $B_i := B(x_i; 2\epsilon)$, $B_i'' := B(x_i; 3\epsilon)$. Correspondingly we have U_{ij}', U_{ij}, U_{ij}'', and let $\sigma_{ij} : B_{ij}'' \longrightarrow U_{ij}''$ the sections of π, $\sigma_{ij}(x_i) = \xi_{ij}$.

Let $D_\epsilon^1, D_\epsilon^2, \ldots, D_\epsilon^k, \ldots$ be the connected components of D_ϵ. Fix points $\xi_k \in D_\epsilon^k$ and let $\lambda_k : D_\epsilon^k \longrightarrow \mathbb{R}$ defined by $\lambda_k(\zeta) = \operatorname{dist}(\zeta, \xi_k)$. Then consider $\lambda : D_\epsilon \longrightarrow \mathbb{R}$ defined by $\lambda|_{D_\epsilon^k} = k + \lambda_k$. One checks readily that the subsets $\{\lambda < c\}$, $c \in \mathbb{R}$, of D_ϵ are relatively compact in D.

Consider now the functions $h_{ij} : B_{ij}'' \longrightarrow \mathbb{R}$, $h_{ij} = \lambda \circ \sigma_{ij}$. Clearly, $h_{ij} = k + \lambda_k \circ \sigma_{ij}$ for some k depending on i and j. The key observation is that these are uniformly Lipschitz in the sense that for all indices $i \in I$, $j \in \Lambda_i$, one has:

$$|h_{ij}(x) - h_{ij}(y)| \leq \operatorname{dist}(x, y),\ \forall x, y \in B_i''.$$

STEP 3. Here we recall the standard riemannian convolution smoothing on a riemannian manifold (M, g) of dimension m (see [18]). Let $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth nonnegative function with support in $[-1, 1]$, a positive constant on a neighborhood of 0, and has the property $\int_{v\in\mathbb{R}^m} \mu(\|v\|) = 1$.

If K is a compact subset of M then there is a positive number ϵ_K such that for all $x \in K$ and all $v \in T_xM$ (the tangent space of M at x) with $\|v\| < \epsilon_K$, $\exp_x v$ is defined. Now given a continuous function $\tau : M \longrightarrow \mathbb{R}$ define for each positive ϵ less than $\epsilon_K/3$ the function τ_ϵ by

$$\tau_\epsilon(x) = \frac{1}{\epsilon^m} \int_{v\in T_xM} \mu(\|v\|/\epsilon)\tau(\exp_x v),$$

where the integral is taken relative to the Lebesgue measure on T_xM determined by the riemannian metric g. Then there is a neighborhood U of K on which the functions τ_ϵ are all defined; if U is chosen, as it may always be, to have compact closure in M, then for all sufficiently small positive ϵ, the functions τ_ϵ will be C^∞ on U. Also $\tau_\epsilon \to \tau$ uniformly on U as $\epsilon \to 0$.

Moreover, if τ is Lipschitz on U, i.e., $|\tau(x) - \tau(y)| \leq C\mathrm{dist}(x, y)$ for all $x, y \in U$, then for every coordinate system $(x_1, \cdots, x_n)$ centered at a point of K, the second order derivatives of τ_ϵ are bounded from below by a constant which depends only on ϵ and C.

STEP 4. Consider now smooth functions $\mu_i : X \longrightarrow [-1, 1]$ with compact support contained in B_i'', $\mu_i = 1$ on B_i', and $\mu_i = -1$ on ∂B_i. Note that $\{B_i''\}_i$ is locally finite.

Using Step 3, for every $i \in I$, there are: open neighborhoods $W_i \subset B_i''$ of $\overline{B_i}$, smooth functions $f_{ij} : W_i \longrightarrow \mathbb{R}$, $j \in \Lambda_i$, such that $|f_{ij} - h_{ij}| < 1$ on $\overline{B_i}$ and the family $\{f_{ij}|_{B_{ij}}\}_{ij}$ has the real hessian locally bounded from below. Therefore there exists $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$, $\chi' > 0$, $\chi'' \geq 0$, such that $\chi(\psi) + \mu_i \in \mathcal{G}(X)$ for all i and $\chi(\psi) + f_{ij} \in \mathcal{G}(B_i)$ for all i and $j \in \Lambda_i$. Then define $\theta : D_t \longrightarrow \mathbb{R}$ by setting

$$\theta(\zeta) = \sup\{f_{ij}(\pi(\zeta)) + \mu_i(\pi(\zeta))\,;\ i, j \text{ such that } \pi(\zeta) \in B_i\}.$$

Thus the function $\varphi_t := 2\chi(\psi \circ \pi)|_{D_t} + \theta$ belongs to $\pi^\star(\mathcal{G})(D_t)$ and has the required properties. □

5. – Proofs of Theorems 1 and 2

Here we deduce firstly Theorem 1 using Lemmas 4 and 5 as follows. Fix ψ be a smooth exhaustive function on X which is 1-convex with respect to $\mathcal{M}$; let δ the boundary distance function of (D, π) according to Section 4. In order to apply Lemma 4 we have to define a function Φ on D which is $\pi^\star\mathcal{M}$-convex and such that:

(b) For every $a \in \mathbb{R}$ there exists $\epsilon > 0$ with $D(a) \subset D_\epsilon$, where $D(a) = \{\zeta \in D\,;\ \Phi(\zeta) < a\}$.

Then $\Phi_a := \varphi_\epsilon|_{D(a)}$, where φ_e is given by Lemma 5 for $\mathcal{G}$ equals the sheaf of germs of $\mathcal{M}$-convex functions on X, will conclude the proof.

For the construction of Φ we use a patching technique due of M. Peternell [30]. (See also [25], Lemma 4.) Let $\{(U_j, F_j)\}_{j\in J}$, $U_j \Subset X$, be a locally finite covering of X by coordinate charts with $F_j(U_j)$ a ball in $\mathbb{C}^n$. Set $D_j = \pi^{-1}(U_j)$; hence $(D_j, F_j \circ \pi)$ is a Stein domain over $\mathbb{C}^n$. If δ_j denotes the corresponding boundary distance in the domain $(D_j, F_j \circ \pi)$ over $\mathbb{C}^n$, then $-\log \delta_j$ is a plurisubharmonic continuous function such that for every compact subset $K \subset U_i \cap U_j$, δ_i/δ_j is bounded over K. See Lemma 3 in [25].

Now apply Corollary 4 to get functions $\widetilde{\delta}_j$ on D_j such that $1/2 < \widetilde{\delta}_j/\delta_j < 2$ and $-\log\widetilde{\delta}_j$ is 1-convex.

Choose open sets $W_j \Subset V_j \Subset U_j$ so that $\cup W_j = X$; then select smooth and non-negative functions p_j on X with support contained in V_j, $p_i = 1$ on W_j. There are positive constants $C_j > 0$ such that

$$(\dagger)\qquad C_j(p_j \circ \pi) - \log\widetilde{\delta}_j < C_i(p_i \circ \pi) - \log\widetilde{\delta}_i \text{ on } \pi^{-1}(W_i \cap \partial V_j),$$

for all indices i and j. For $\zeta \in D$ put $I(\zeta) := \{j \in \mathbb{N};\ \zeta \in \pi^{-1}(V_j)\}$. Define a function $h : D \longrightarrow \mathbb{R}$ by setting for $\zeta \in D$,

$$h(\zeta) = \max\{C_i p_i(\pi(\zeta)) - \log\widetilde{\delta}_i(\zeta);\ i \in I(\zeta)\}.$$

The above inequality ($\dagger$) implies easily the continuity of h. On the other hand, if $\chi \geq 0$ is a rapidly increasing convex function on $\mathbb{R}$ so that $\chi(\psi) + C_j p_j$, $j \in \mathbb{N}$, are $\mathcal{M}$-convex, the function $\Phi := \chi(\psi\circ\pi)+h$ results $\pi^*\mathcal{M}$-convex on D and

$$\begin{cases} \Phi \geq -\log\delta_j \text{ on } D \cap \pi^{-1}(V_j); \\ \pi(\{\Phi < a\}) \Subset X, \text{ for every } a \in \mathbb{R}. \end{cases}$$

Moreover, for each index j there are constants $b_j > 0$ and $\epsilon_j > 0$ such that

$$\operatorname{dist}_g(x, y) \geq b_j \|F_j(x) - F_j(y)\|,$$

if $x \in V_j, y \in U_j$ and $\|F_j(x) - F_j(y)\| \leq \epsilon_j$. From this last property, condition (b) can be easily fulfilled.

The proof of Theorem 2 results similarly; we only note that in the above proof we choose $\widetilde{\delta}_j$ with $-\log\widetilde{\delta}_j$ be q-convex with corners; then $\psi \in F_r(X)$ exhaustive. Thus we get $\Phi \in F_{q+r-1}(D)$. We omit the details. Observe that Theorem 2 in case $\pi(D) \Subset X$ holds under the weaker assumption $X \in \mathcal{S}_r$. □

REMARK 4. If (D, π) is univalent, then Φ itself is exhaustive for D.

As a consequence of Theorem 2 one has:

COROLLARY 10. *Let X, Y, and Z be connected complex manifolds of dimension n such that (X, π) is a domain pseudoconvex of order k_1 over Y and (Y, σ) is a domain pseudoconvex of order k_2 over Z. Then $(X, \sigma \circ \pi)$ is pseudoconvex of order k over Z for $k = k_1 + k_2 + 1 - n$.*

Below we discuss the situation when we allow branching. First we state some positive results. In order to do this, we recall from [23] the following.

Let $\pi : Z \longrightarrow X$ be a holomorphic map of complex spaces. We say that π is a *ramified covering* if X and Z have the same dimension and every point $x \in X$ admits an open neighborhood U such that for each connected component W of $\pi^{-1}(U)$ the induced map $\pi|_W : W \longrightarrow U$ is finite (i.e., proper with finite fibers).

More generally, we call π *locally semi-finite* (here we do not assume that X and Z have the same dimension) if every $x \in X$ has a neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of complex spaces $\{W_i\}_{i \in I}$ (I an at most countable set of indices) such that every $\pi|_{W_i} : W_i \longrightarrow U$ is finite.

It is easy to see that if π is locally semi-finite, then π is *locally hyperconvex* in the sense that every point of X has a neighborhood U such that $\pi^{-1}(U)$ is hyperconvex. (A complex space Y is said to be *hyperconvex* [33] if Y is Stein and there is a continuous plurisubharmonic proper function $\psi : Y \longrightarrow [c, 0)$ for some $c < 0$. For instance, if $\sigma : W \longrightarrow Y$ is finite and Y is hyperconvex, then W is hyperconvex. Also a disjoint union of hyperconvex spaces is hyperconvex.)

Now the method of [39] gives readily the following result which generalizes Le Barz's result, with a simpler proof, namely;

PROPOSITION 9. *Let $\pi : Z \longrightarrow X$ be a locally hyperconvex holomorphic map of complex spaces (e.g., π be locally semi-finite). Then one has:*

(a) *Assume $\mathcal{M}$ is a linear set over X such that X is $\mathcal{M}$-complete. Then Z is $\pi^\star\mathcal{M}$-complete. In particular, if X is q-complete, then Z is q-complete.*
(b) *If X is q-complete with corners, then Z is q-complete with corners.*

In the remainder of this section we mention two counter-examples for branched coverings (as alluded to in the introduction).

EXAMPLE 1. For every $q > 1$ there is a branched Riemann domain (D, π) over a q-complete open set $X \subset \mathbb{C}^{q+2}$ such that (D, π) is locally Stein and $H^q(D, \mathcal{O}_D)$ is **not** separated. *A fortiori*, $H^q(D, \mathcal{O}_D)$ does not vanish; hence D is not q-complete by [2].

First we recall that Fornæss [9] produced a branched Riemann domain (Y, τ) over $\mathbb{C}^2$ (τ is $2:1$) which is locally Stein but Y fails to be Stein.

Let $V \subset \mathbb{C}^q$ be an open set such that the cohomology groups $H^j(V, \mathcal{O}_V)$, $j = 1, \ldots, q-1$, are separated and $H^{q-1}(V, \mathcal{O}_V) \neq 0$. For example, $V = \mathbb{C}^q \setminus \{0\}$.

Set $D := Y \times V$, $X := \mathbb{C}^2 \times V$, and $\pi := \tau \times \mathrm{id}$. Clearly (D, π) and X fulfil the required properties but for the non Hausdorff property which we now check using a Künneth formula in the version of Cassa (see [4]).

Let $\mathcal{F}_S$ and $\mathcal{F}_T$ be coherent analytic sheaves on complex manifolds S and T respectively. We denote by $\mathcal{F}_S \star \mathcal{F}_T$ the analytic tensor product $p_S^*\mathcal{F}_S \,\widehat{\otimes}\, p_T^*\mathcal{F}_T$ on $S \times T$, where p_S^* and p_S^* denotes the canonical projections on S and T respectively. (E.g., $\mathcal{O}_S \star \mathcal{O}_T = \mathcal{O}_{S \times T}$.) For a topological vector space E, denote by E_{sep} its canonically associated separated space, i.e., the quotient of E by the closure of $\{0\}$.

If $H^j(T, \mathcal{G}_T)$ is separated $\forall j \in \mathbb{N}$, then $\forall k \in \mathbb{N}$ there exists a *topological* isomorphism

$$H^k(S \times T, \mathcal{F}_S \star \mathcal{F}_T) \cong \bigoplus_{i+j=k} \left((H^i(S, \mathcal{F}_S)_{\mathrm{sep}} \,\widehat{\otimes}\, H^j(T, \mathcal{F}_T)) \oplus R_{ij} \right),$$

where R_{ij} are complex vector spaces of infinite dimension (with the trivial topology) if $H^i(S, \mathcal{F}_S)$ is not-separated and $H^j(T, \mathcal{F}_T)$ does not vanish; otherwise $R_{ij} = \{0\}$.

With these preparations, the example concludes if we notice that $H^1(Y, \mathcal{O}_Y)$ is not separated by a result of Jennane [22]. □

EXAMPLE 2. For every $q > 1$ there is branched Riemann domain (D, π) over a q-complete manifold such that (D, π) is locally Stein and D is q-convex with corners.

We proceed as follows. Let M be a compact connected complex manifold of dimension $q - 1$, for example, $M = \mathbb{P}^{q-1}$. With the notation from Example 1, set $D := Y \times M$, $X := \mathbb{C}^2 \times M$, and $\pi := \tau \times \mathrm{id}$. Since Y is not Stein, Proposition 3 shows that D is not q-convex with corners. The same procedure as in Example 1, gives also in this case that $H^q(D, \mathcal{O}_D) \neq 0$. □

6. – Proof of Theorem 3

This is done in several steps.

STEP 1. As in [8] consider X as a complex submanifold of a Stein open subset $B \subset \mathbb{C}^N$ such that there is a holomorphic retraction $\rho : B \longrightarrow X$. Let $\Omega := D \times_X B$ be the fibre product of (D, π) and (B, ρ) over X and $\tau : \Omega \longrightarrow B$ the canonical projection. Then (Ω, τ) becomes a domain over B, therefore over $\mathbb{C}^N$. Since D is a complex submanifold of Ω, to conclude we establish that Ω is q-complete with corners. This will be accomplished in Step 4 from below.

STEP 2. Let $(z_1, \ldots, z_N)$ be the complex coordinates in $\mathbb{C}^N$ and S the unit sphere in $\mathbb{C}^N$. For $\lambda \in \mathbb{C}^N$, $\lambda \neq 0$, put $H_\lambda = \{z \in \mathbb{C}^N ; z_1\lambda_1 + \cdots + z_N\lambda_N = 0\}$.

We claim that a domain (Ω, τ) over $\mathbb{C}^N$ is q-complete with corners $(q+1 < N)$ if and only if for every $a \in \tau(\Omega)$ and $w \in S$ there is a dense subset $H_w^a \subset H_w$ (depending on a) such that $\tau^{-1}(\{a\} + H_\lambda)$ is q-complete with corners for every $\lambda \in H_w^a$, $\lambda \neq 0$.

In order to verify this, let R_w be the Hartogs radius of (Ω, τ) in direction $w \in S$. By Proposition 6, it is enough to see that $-\log R_w$ is q-psh; thus, by Proposition 4, we have to check that the restriction of $-\log R_w$ to $\tau^{-1}(\{z\} + \Sigma)$ is q-psh for every $z \in \tau(\Omega)$ and every q-plane $\Sigma \subset \mathbb{C}^N$ running through a dense subset of $Gr_q(N)$ (which may depend on z). But this is obvious by hypothesis and Proposition 6 again, since for Γ the plane generated by w and Σ, then $q \leq \dim(\Gamma) \leq q + 1 < N$ and δ_w restricted to $\tau^{-1}(\{z\} + \Gamma)$ equals the Hartogs radius of the domain $(\tau^{-1}(\{z\} + \Sigma), \tau)$ over $\{z\} + \Sigma \cong \Sigma$ in direction w.

STEP 3. Here we give:

LEMMA 6. *Let $B \subset \mathbb{C}^N$ a Stein domain and $X \subset B$ a complex submanifold of pure dimension $n \geq 3$. For $\lambda \in \mathbb{C}^{N-1}$ consider holomorphic functions f_λ on*

X given by $f_\lambda = (\lambda_1 z_1 + \cdots + \lambda_{N-1} z_{N-1})|_X$. Then the set $\{\lambda \in \mathbb{C}^{N-1};\ \exists x \in X$ with $f_\lambda(x) = 0$ and $df_\lambda(x) = 0\}$ has zero Lebesgue measure in $\mathbb{C}^{N-1}$.

Proof. Consider $h : U \longrightarrow \mathbb{C}^N$, $U \subset \mathbb{C}^n$ open, $0 \in U$, $h(0) = x_0$, be a local parametrization of X around a fixed point $x_0 \in X$. Define for $\lambda \in \mathbb{C}^{N-1}$ a holomorphic function $F_\lambda : U \longrightarrow \mathbb{C}$ by $F_\lambda = \lambda_1 h_1 + \cdots + \lambda_{N-1} h_{N-1}$. Then the lemma reduces to the next:

Claim. The set $\Lambda := \{\lambda \in \mathbb{C}^{N-1}; \exists t \in U$ such that $F_\lambda(t) = 0$ and $dF_\lambda(t) = 0\}$ has zero Lebesgue measure in $\mathbb{C}^{N-1}$.

In order to show this let $Y = \{h_1 = \ldots = h_{N-1} = 0\}$. Clearly $Y \subset U$ is analytic of dimension ≤ 1. Let Y' denote the singular set of Y together with the isolated points of Y. Thus $Y_0 := Y \setminus Y'$ is a smooth Riemann surface.

Corresponding to the decomposition of $U = (U \setminus Y) \cup Y_0 \cup Y'$ one has $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. We show that each Λ_i has zero Lebesque measure in $\mathbb{C}^{N-1}$. For Λ_1 and Λ_3 this is easy. (For instance Λ_3 is an at most countable union of proper complex vector subspaces of $\mathbb{C}^{N-1}$ and for Λ_1 we use Bertini-Sard arguments.)

We take now the case of Λ_2. At a point $y \in Y_0$, after some coordinate changes, on a suitable neighborhood W of y in U we may assume Y_0 is given as $\{(0, t_n); t_n \in \Delta(r)\}$ for $r > 0$ and that on W one has: $h_1 = t_1, \ldots, h_{n-1} = t_{n-1}, h_n = t_n$. Denote $t := t_n$. Hence $h_n(0,t) = \cdots = h_{N-1}(0,t) = 0$ for $t \in \Delta(r)$. Thus it suffices to show that the subset T of $\mathbb{C}^{N-1}$ given by

$$T := \left\{\lambda \in \mathbb{C}^{N-1}; \exists t \in \Delta(r), \lambda_1 dt_1 + \cdots + \lambda_{n-1} dt_{n-1} + \sum_{j=n}^{N-1} \lambda_j \sum_{i=1}^{n} \frac{\partial h_j}{\partial t_i}(0,t) dt_i = 0\right\}$$

has zero Lebesgue measure. Reinterpreting the above definition, T equals the set of those $\lambda \in \mathbb{C}^{N-1}$ such that for some $t \in \Delta(r)$ one has:

$$\begin{cases} \lambda_i + \displaystyle\sum_{j=n}^{N-1} \lambda_j \frac{\partial h_j}{\partial t_i}(0,t) = 0, i = 1, \ldots, n-1; \\ \displaystyle\sum_{j=n}^{N-1} \lambda_j \frac{\partial h_j}{\partial t_n}(0,t) = 0. \end{cases}$$

Define holomorphic functions on $a_1, \ldots, a_n$ on $\Delta(r)$ by setting for $i = 1, \ldots, n$:

$$a_i(t) = -\left(\frac{\partial h_n}{\partial t_i}(0,t), \ldots, \frac{\partial h_{N-1}}{\partial t_i}(0,t)\right), t \in \Delta(r).$$

Therefore, it we decompose $\lambda = (\lambda', \lambda'') \in \mathbb{C}^{n-1} \times \mathbb{C}^{N-n}$ one obtains

$$T = \bigcup_{t \in \Delta(r)} \{((\langle\lambda'', a_1(t)\rangle, \ldots, \langle\lambda'', a_{n-1}(t)\rangle, \lambda'');$$
$$\lambda'' \in \mathbb{C}^{N-n}, \langle\lambda'', a_n(t)\rangle = 0\}.$$

Consequently, in view of Fubini's theorem, it suffices to check that $T_{\lambda''} := T \cap (\mathbb{C}^{n-1} \times \{\lambda''\})$, viewed as a subset of $\mathbb{C}^{n-1}$, has zero Lebesque measure for every $\lambda'' \in \mathbb{C}^{N-n}$. But $T_{\lambda''}$ is contained in the complex curve $\{(< \lambda'', a_1(t) >, \dots, < \lambda'', a_{n-1}(t) >) \,;\, t \in \Delta(r)\}$, which has zero Lebesgue measure in $\mathbb{C}^{n-1}$ since $n \geq 3$. □

STEP 4. End of proof of Theorem 3. Here we verify the claim of Step 2. Keeping the notation as in Steps 1 and 2, let $w \in S$ and $a \in B$. For $\lambda \in H_w$, $\lambda \neq 0$, put $f_\lambda = (\lambda_1(z_1 - a_1) + \dots + \lambda_N(z_1 - a_N))|_X$. Then by Lemma 6 the set $H_w' := \{\lambda \in H_w \,;\, \exists x \in X \text{ with } f_\lambda(x) = 0 \text{ and } df_\lambda(x) = 0\}$ has zero Lebesgue measure in H_w. Thus for $\lambda \in H_w \setminus H_w'$, $\{f_\lambda = 0\}$ is a smooth hypersurface of X where f_λ has multiplicity 1. Since $\tau^{-1}(\{a\} + H_\lambda)$ is a complex submanifold of $\pi^{-1}(\{f_\lambda = 0\}) \times B$ which is q-complete with corners by hypothesis, then $\tau^{-1}(\{a\} + H_\lambda)$ is q-complete with corners, whence the claim. This concludes the proof of Theorem 3. □

7. – Domains over P^n

In this section, in order to prove Theorem 4, we need criteria for q-plurisubharmonicity, so we recall the following definitions.

Let X be a complex space, $\varphi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ usc, and $x_0 \in X$. We say [11] that φ is:

a) *subpluriharmonic at* x_0 if there is a Stein neighborhood U of x_0 (sufficiently small) such that:
   (i) $\varphi(x_0) \leq \sup_{\partial V} \varphi$ for every neighborhood V of x_0, $V \Subset U$.
   (ii) The above inequality persists if we replace φ by $\varphi + \mathrm{Re} f$, $\forall f \in \mathcal{O}(U)$.

b) *q-plurisubharmonic at* x_0 if for every open set $G \subset \mathbf{C}^q$, $G \ni 0$, and holomorphic map $f : G \longrightarrow X$ with $f(0) = x_0$, the function $\varphi \circ f$ is subpluriharmonic at 0.

(If $X \in \mathcal{S}_1$, then φ if subpluriharmonic if and only if φ is subpluriharmonic at every point of X. See [42].) Note that if $D \subset \mathbb{C}^n$ is an open set, $\varphi : D \longrightarrow \mathbb{R} \cup \{-\infty\}$ is usc, and $\psi \in C^0(D, \mathbb{R})$ psh (in the usual sense), if φ is q-psh at a point $z_0 \in D$, then $\varphi + \psi$ is also q-psh at z_0. (For smooth ψ, write on an open ball B around z_0, $\psi = \psi_0 + \mathrm{Re} h$ with h holomorphic on B, ψ_0 be 1-convex on B and $\psi_0 \geq \psi_0(z_0) = 0$. Otherwise, we choose a sequence $\psi^{(k)}$ of smooth psh functions which converges uniformly to ψ and proceed by standard arguments.)

Suppose now that X is a Kähler manifold with Kähler form ω and $\varphi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ usc. For every $x \in X$ and open ball B around x (in some local coordinates) we may write $\omega = i\partial\bar{\partial}\Psi$ with Ψ be 1-convex on B. Set

$$\Lambda_x := \{\lambda \in \mathbb{R} \,;\, \varphi - \lambda\Psi \text{ is } q\text{-plurisubharmonic at } x\}.$$

It is easy to see that Λ_x does not depend on Ψ as above and if $\lambda \in \Lambda_x$, then $(-\infty, \lambda] \subset \Lambda_x$. The *modulus of q-plurisubharmonicity of φ at x*, $W_q[\varphi](x)$, is defined by setting:

$$W_q[\varphi](x) := \sup \Lambda_x,$$

where by convention $\sup \emptyset = -\infty$. ($W_q[\varphi](x)$ might be $+\infty$.) $W_q[\varphi](x)$ is intrinsically defined. $W_q[\varphi]$ is therefore an extended real-valued function on X. We observe that if φ is of class C^2 near x and $\lambda_1 \leq \cdots \leq \lambda_q \leq \cdots \leq \lambda_n$ are the eigenvalues of $L(\varphi; x)$ with respect to some normal coordinates $(z_1, \ldots, z_n)$ centered at x, then $W_q[\varphi](x) = \lambda_q$. (Another modulus was introduced in [26]. If we denote this by $M_q[\varphi](x)$, then $M_q[\varphi](x) \geq W_q[\varphi](x)$. We do not know if they are equal.)

The following are straightforward consequences of the definitions and summarize the elementary properties of $W_q[\cdot]$. Let X be a Kähler manifold, $x_0 \in X$, and $\varphi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ usc. Then one has:

1) $\varphi \in P_q(X)$ iff $W_q[\varphi] \geq 0$. Therefore $\varphi \in SP_q(X)$ iff every $x \in X$ admits an open neighborhood U for which there is $\lambda > 0$ with $W_q[\varphi] \geq \lambda$ on U.
2) Let u be psh near x_0. Then $W_q[\varphi + u](x_0) \geq W_q[\varphi](x_0)$ with equality if u is pluriharmonic near x_0.
3) If ψ is usc on X and supports φ at x_0, i.e., $\psi \leq \varphi$ near x_0 and $\psi(x_0) = \varphi(x_0)$, then $W_q[\varphi](x_0) \geq W_q[\psi](x_0)$. In particular, if $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ is increasing and convex, then $W_q[\chi(\varphi)](x_0) \geq \chi'(\varphi(x_0)) W_q[\varphi](x_0)$. (We set $\chi'(-\infty) = \lim \chi'(t)$ as t tends to $-\infty$. By convention $0 \cdot (-\infty) = -\infty$.)
4) Let $\varphi_k, k \in \mathbb{N}$, be usc functions such that for some $\lambda \in \mathbb{R}$, $W_q[\varphi_k](x_0) \geq \lambda$ for all $k \in \mathbb{N}$. If the sequence $\{\varphi_k\}_k$ either decreases pointwise to φ or else converges uniformly on compact sets to φ, then $W_q[\varphi](x_0) \geq \lambda$.
5) Let ψ be usc on X. Then $W_q[\max(\varphi, \psi)] \geq \max(W_q[\varphi], W_q[\psi])$.

Property 3) will be used in a crucial way and we would like to describe it explicitely. Suppose we wish to show φ is q-psh. By 1) this is equivalent to showing $W_q[\varphi](x) \geq 0$ for each x. However, it is sometimes possible to find a function ψ suporting φ at x for which we compute easily $W_q[\psi](x)$ (for instance if ψ is of class C^2 near x).

From Proposition 4 we get:

COROLLARY 11. *Let (D, π) be a domain over $\mathbb{P}^n$, $\varphi : D \longrightarrow \mathbb{R} \cup \{-\infty\}$ usc, and $\lambda \in \mathbb{R}$. If $W_q[\varphi|_{\pi^{-1}(\Sigma)}] \geq \lambda$ on $\pi^{-1}(\Sigma)$ for every q-dimensional linear subspace Σ of $\mathbb{P}^n$, then $W_q[\varphi] \geq \lambda$ on D.*

Also one has the next

LEMMA 7. *Let X be an n-dimensional Kähler manifold, $x_0 \in X$, $\varphi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ usc, and $\lambda \in \mathbb{R}$ such that there exists a $(n - q + 1)$-dimensional submanifold Y near x_0 with $Y \ni x_0$ and $W_1[\varphi|_Y](x_0) \geq \lambda$. Then $W_q[\varphi](x_0) \geq \lambda$.*

PROOF. This is obvious since for $f : G \longrightarrow X$ holomorphic, $G \subset \mathbb{C}^n$ open with $G \ni 0$ and $f(0) = x_0$, there is a holomorphic map $h : \Delta \longrightarrow G$ (Δ the unit disc in $\mathbb{C}$) such that $h(0) = 0$ and $f(h(\Delta)) \subset Y$. Then we use the definitions. □

EXAMPLE 3. For $a \in \mathbb{P}^q$ set $\Omega := \mathbb{P}^q \setminus \{a\}$ and let δ be the boundary distance of Ω, $\delta(z) = \operatorname{dist}(z, a)$ for $z \in \Omega$. Then $\delta : \Omega \longrightarrow (0, \pi/2]$ is continuous and there is $c > 0$ (independent of a) such that on Ω one has

$$W_q[-\log \delta] \geq c.$$

Note that if $\tau : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$ is the natural projection and $\langle u, v\rangle = \sum_j u_j \overline{v_j}$ is the scalar product in $\mathbb{C}^{n+1}$, then for two points $u^\star = \tau(u)$ and $v^\star = \tau(v)$ of $\mathbb{P}^n$ one has $\operatorname{dist}(u^\star, v^\star) \leq \pi/2$ and

$$\sin^2 \operatorname{dist}(u^\star, v^\star) = 1 - \frac{|\langle u, v\rangle|^2}{\|u\|^2 \cdot \|v\|^2}.$$

In order to verify the example, in view of the above lemma, it suffices to settle the case $q = 1$ which is an easy exercise.

The key fact in proving Theorem 4 is the next result.

THEOREM 5. *Let (D, σ) be a domain pseudoconvex of order $n - q$ over $\mathbb{P}^n$ and δ its boundary distance function. If D is not biholomorphic to $\mathbb{P}^n$ via σ, then there exists $c > 0$ (which does not depend on the given domain) such that:*

$$W_q[-\log \delta] \geq c.$$

PROOF. Let $\zeta_0 \in D$; put $x_0 := \sigma(\zeta_0) \in \mathbb{P}^n$. There exists $y_0 \in \mathbb{P}^n$ with $y_0 \notin \sigma(D)$ and $\delta(\zeta_0) = \operatorname{dist}(x_0, y_0)$. Let L be a q-dimensional linear vector subspace of $\mathbb{P}^n$, $L \ni x_0$. We shall find $c > 0$ (independent of L and (D, σ)) such that $W_q[(-\log \delta)|_{\sigma^{-1}(L)}](\zeta_0) \geq c$, which, in view of Corollary 11, concludes the proof of the theorem.

We distinguish two cases. If L passes through y_0; then let δ_L be the boundary distance of the domain $(\sigma^{-1}(L), \sigma)$ over L. Therefore $\delta|_L \geq \delta_L$ with equality at ζ_0; hence it suffices to settle the case $q = n$ and the assertion follows using Example 3 and Property 3 of $W_q[\cdot](\cdot)$. (If $d : \mathbb{P}^q \setminus \{y_0\} \longrightarrow (0, \pi/2]$, $d(x) = \operatorname{dist}(x, y_0)$, then $\delta(\zeta) \leq d(\sigma(\zeta))$ near ζ_0 with equality for $\zeta = \zeta_0$.)

Consider now the case $L \not\ni y_0$. We reduce ourselves readily to $n = q + 1$. Assume this and take inhomogeneous coordinates $(z_1, \dots, z_{q+1})$ on $\mathbb{P}^{q+1}$ such that its domain $U_0 (\simeq \mathbb{C}^{q+1})$ contains x_0 and y_0, the line passing through x_0 and y_0 is given by $z_1 = \dots = z_q = 0$, and the z_{q+1} coordinates of x_0 and y_0 are a and $-a$ respectively with $|a| \leq 1$.

The domain $(\sigma^{-1}(U_0), \sigma)$ over U_0 being pseudoconvex of order $n - q$, if $R(\zeta)$ is the Hartogs radius with respect to z_{q+1}, by Proposition 6, $-\log R$ is q-psh on $\sigma^{-1}(U_0)$. Since $L_0 := L \cap U_0$ is given by $z_{q+1} = a + t_1 z_1 + \dots + t_q z_q$ for some constants $t_j \in \mathbb{C}$, consider for $w \in L_0$ the line $\Gamma(w)$ defined by $z_1 = w_1, \dots, z_q = w_q$. Take $\zeta \in D$ such that $z = \sigma(\zeta) \in L_0$ and $R(\zeta) < +\infty$. Thus there is a point $w \in \Gamma(z)$ with $R(\zeta) = |z_{q+1} - w_{q+1}|$. Observe that $R(\zeta_0) = 2|a|$. Using the inequality

$$\arcsin \frac{|s - t|}{\sqrt{(1 + |s|^2)(1 + |t|^2)}} \leq 2 \arctan \frac{|s - t|}{2}$$

for $s \neq t$ in $\mathbb{C}$ (equality holds for $s = -t$ and $|s| \leq 1$), and the distance formula in $\mathbb{P}^n$, we get

$$\delta(\zeta) \leq 2 \arctan \frac{R(\zeta)}{2\sqrt{1 + |\sigma_1(\zeta)|^2 + \cdots + |\sigma_q(\zeta)|^2}}$$

for $\zeta \in D$ with $\sigma(\zeta) \in L_0$ and $R(\zeta) < +\infty$, with equality if $\zeta = \zeta_0$. Here $\sigma = (\sigma_1, \ldots, \sigma_q, \sigma_{q+1})$ in $\sigma^{-1}(U_0)$ over $U_0 \cong \mathbb{C}^{q+1}$. Note that $\sigma_1(\zeta_0) = \cdots = \sigma_q(\zeta_0) = 0$. Obviously the above inequality holds true if $R(\zeta) = \infty$. Consider $\psi : \sigma^{-1}(U_0) \longrightarrow \mathbb{R} \cup \{-\infty\}$, defined by

$$\psi(\zeta) = -\log R(\zeta) + \log 2 + \frac{1}{2} \log(1 + |\sigma_1(\zeta)|^2 + \cdots + |\sigma_q(\zeta)|^2), \zeta \in \sigma^{-1}(U_0)$$

and $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ defined by $\chi(t) = -\log \arctan(\exp(-t)), t \in \mathbb{R}$. Since χ is strictly increasing and convex, $-\log \delta \geq -\log 2 + \chi(\psi)$ on $\sigma^{-1}(L_0)$ with equality at ζ_0, and $W_q[\psi](\zeta_0) \geq 1/2$, we obtain applying property 3) from above

$$W_q[-\log \delta](\zeta_0) \geq \frac{|a|}{(1 + |a|^2) \arctan |a|},$$

whence the desired estimation (note that $|a| \leq 1$). □

Finally, the proof of Theorem 4 concludes similarly to that of Theorem 1 using Lemma 4, Proposition 5, and Lemma 5. Note that in Lemma 5, instead of $\psi \circ \pi$ we use $-\log \delta$. The key estimation is given by Theorem 5 and we omit details. □

For the sake of completeness, we add a complement to the above set-up. So for the rest of the paper we suppose X is a connected Kähler n-fold, (D, π) a domain over X, and δ its boundary distance function (see Section 4).

LEMMA 8. *Let $\zeta_0 \in D$ be such that there exists $y_0 \in X$ with the following properties:*

(i) *$\delta(\zeta_0) = \operatorname{dist}(x_0, y_0)$, where $x_0 = \pi(\zeta_0)$.*
(ii) *There exists a geodesic Γ joining x_0 and y_0.*
(iii) *There exists an $(n - q)$-dimensional complex submanifold Y near y_0 such that $Y \ni y_0$ and $Y \subset X \setminus \pi(D)$.*

Then we have the inequality

$$W_q[-\log \delta](\zeta_0) \geq \min(K/12, K/4),$$

where K is the minimum of the holomorphic bisectional curvature of X along the geodesics Γ as in (ii).

PROOF. This is a small variation of [26], Lemma 4.1, p. 96, and we omit it. □

LEMMA 9. *Assume that X is not compact and let (D, π) be pseudoconvex of order $n-q$ over X. If $\pi(D) \Subset X$, then there are constants ϵ_0 and c, $\epsilon_0 > 0$, such that*

$$W_q[-\log \delta](\zeta) \geq c,$$

for $\zeta \in D$, $\delta(\zeta) < \epsilon_0$. The constants ϵ_0 and c depend only on the compact set $\overline{\pi(D)}$. Moreover, if the holomorphic bisectional curvature of X is positive, then one can choose $c > 0$.

PROOF. One deduces this by similar arguments as those in [36] and [26]. □

Assume from now on that X has positive holomorphic bisectional curvature. If X is compact, then by a theorem due to Siu and Yau [34], X is biholomorphic to $\mathbb{P}^n$, thus there is no loss in generality if we take X noncompact. We show:

THEOREM 6. *Let (D, π) be a pseudoconvex domain of order $n-q$ over X as above. If $\pi(D) \Subset X$, then there exists a constant $c > 0$ such that :*

$$W_q[-\log \delta] \geq c.$$

PROOF. In order to apply Lemma 9, we recall from ([30], p. 257) the following

LEMMA 10. *Let Z be a complex manifold and $\varphi \in F_q(Z)$. Then for λ running through a dense subset of $\mathbb{R}$, the open subset $Z(\lambda) := \{\varphi < \lambda\}$ of Z has the following property: For every $z_0 \in \partial Z(\lambda)$ there is a $(n-q)$-dimensional complex submanifold Y near z_0 such that $Y \ni z_0$ and $Y \subset Z \setminus Z(\lambda)$.*

Now we let ϵ_0 and $c > 0$ be as in Lemma 9. Set $\Omega := \{\zeta \in D\,;\ \delta(\zeta) < \epsilon_0\}$. Since $-\log \delta \in C^0(\Omega, \mathbb{R}) \cap SP_q(\Omega)$, by Proposition 6, for every $\eta > 0$ there is $\varphi \in F_q(\Omega)$ such that $|\log \delta + \varphi| < \eta$. Consider $\epsilon > 0$ such that $2\epsilon < \epsilon_0$. For $\eta > 0$ and $\lambda \in \mathbb{R}$ define

$$D'_\epsilon := D_{2\epsilon} \cup \{\zeta \in \Omega\,;\ \varphi(\zeta) < \lambda\},$$

where η and λ are such that $\eta - \log \epsilon > \lambda > \eta - \log 2\epsilon$, in order that the boundary of $D_{2\epsilon}$ relative to D is contained in $\{\zeta \in \Omega\,;\ \varphi(\zeta) < \lambda\}$, $D'_\epsilon \subset D_\epsilon$, and λ chosen according to Lemma 10; hence D'_ϵ fulfils (i), (ii), and (iii) of Lemma 8.

Therefore, if δ'_ϵ denotes the boundary distance of the domain (D'_ϵ, π) over X, we have $W_q[-\log \delta'_\epsilon] \geq c$. As ϵ tends to 0, δ'_ϵ tends to δ. The assertion follows immediately by using the properties of $W_q[\cdot](\cdot)$. □

We, finally, obtain:

THEOREM 7. *Let X be a connected Kähler manifold of dimension n which is non-compact and has positive holomorphic bisectional curvature. Let (D, π) be a pseudoconvex domain of order $n-q$ over X such that $\pi(D) \Subset X$. Then D is q-complete with corners.*

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