J. Math. Soc. Japan
Vol. 62, No. 1 (2010) pp. 269–303
doi: 10.2969/jmsj/06210269

Pseudoharmonic maps and vector fields on *CR* manifolds

By Sorin Dragomir and Yoshinobu Kamishima

(Received Jan. 7, 2009)

Abstract. Building on the work by J. Jost and C.-J. Xu [**32**], and E. Barletta et al. [**3**], we study smooth *pseudoharmonic maps* from a compact strictly pseudoconvex *CR* manifold and their generalizations e.g. pseudoharmonic unit tangent vector fields.

Introduction.

The purpose of this paper is to study several analogs to differential geometric objects appearing in Riemannian geometry and admitting a treatment based on elliptic theory e.g. the Laplace-Beltrami operator (cf. [**40**]), harmonic maps among Riemannian manifolds (cf. [**49**]), and harmonic vector fields (regarded as smooth maps of a Riemannian manifold into the total space of the tangent bundle endowed with the Sasaki metric, cf. [**51**] and [**52**]). We obtain the following results. Boundary values of Bergman-harmonic maps $\phi : \Omega \to S$ from a smoothly bounded strictly pseudoconvex domain $\Omega \subset \boldsymbol{C}^n$ into a Riemannian manifold S are shown to be pseudoharmonic maps, provided their normal derivatives vanish. We prove that $\overline{\partial}_b$-pluriharmonic maps are pseudoharmonic maps. A pseudoharmonic map $\phi : M \to S^\nu$ from a compact strictly pseudoconvex *CR* manifold into a sphere is shown either to link or to meet any codimension 2 totally geodesic sphere in S^ν. Also we prove that a smooth vector field $X : M \to T(M)$ from a strictly pseudoconvex *CR* manifold M is a pseudoharmonic map if and only if X is parallel (with respect to the Tanaka-Webster connection) along the maximally complex, or Levi, distribution. We start a theory of *pseudoharmonic* vector fields i.e. unit vector fields $X \in \mathscr{U}(M,\theta)$ which are critical points of the energy functional $E(X) = \frac{1}{2}\int_M \mathrm{trace}_{G_\theta}(\pi_H X^* S_\theta)\, \theta \wedge (d\theta)^n$ relative to variations through unit vector fields. Any such critical point X is shown to satisfy the nonlinear subelliptic system $\Delta_b X + \|\nabla^H X\|^2 X = 0$. Also $\inf_{X \in \mathscr{U}(M,\theta)} E(X) = n\,\mathrm{Vol}(M,\theta)$ yet E is unbounded from above. We establish first and second variation formulae for $E : \mathscr{U}(M,\theta) \to [0,+\infty)$ and give applications.

2000 *Mathematics Subject Classification.* Primary 32V15; Secondary 35H20, 53C12, 53C43.
Key Words and Phrases. Graham-Lee connection, Bergman harmonic map, pseudoharmonic map, total bending, pseudoharmonic vector field.

1. Boundary values of Bergman-harmonic maps.

In their seminal 1998 paper J. Jost and C.-J. Xu studied (cf. [**32**]) the existence and regularity of weak solutions $\phi : \overline{\Omega} \to (S, h)$ to the nonlinear subelliptic system

$$H\phi^i + \sum_{a=1}^{m} \left(\Gamma^i_{jk} \circ \phi\right) X_a(\phi^j) X_a(\phi^k) = 0, \quad 1 \le i \le \nu, \tag{1}$$

where $H = -\sum_{a=1}^{m} X_a^* X_a$ is the Hörmander operator associated to a system $X = \{X_1, \dots, X_m\}$ of smooth vector fields on a open set $\Omega \subseteq \boldsymbol{R}^n$, verifying the Hörmander condition on Ω, (S, h) is a Riemannian manifold and Γ^i_{jk} are the Christoffel symbols associated to the Riemannian metric h. Their study is part of a larger program aiming to the study of hypoelliptic nonlinear systems of variational origin similar to the harmonic maps system, although but degenerate elliptic. Indeed, if $X = b^A_a(x)\,\partial/\partial x^A$ then $X_a^* f = -\partial(b^A_a(x) f)/\partial x^A$ for any $f \in C^1_0(\Omega)$ hence

$$Hu = \sum_{A,B} \frac{\partial}{\partial x^A} \left(a^{AB}(x) \frac{\partial u}{\partial x^B} \right)$$

where $a^{AB}(x) = \sum_{a=1}^{m} b^A_a(x) b^B_a(x)$ so that in general $[a^{AB}]$ is only semi positive definite. Hence H is degenerate elliptic (in the sense of J. M. Bony [**7**]). As successively observed (cf. E. Barletta et al. [**3**]) solutions of systems of the form (1) may be built within *CR* geometry as S^1-invariant harmonic maps $\Phi : C(M) \to S$ where $S^1 \to C(M) \xrightarrow{\pi} M$ is the canonical circle bundle over a strictly pseudoconvex *CR* manifold M and harmonicity is meant with respect to the Fefferman metric F_θ (associated to a choice of contact form θ on M, cf. J. M. Lee [**36**]). Base maps $\phi : M \to S$ corresponding (i.e. $\Phi = \phi \circ \pi$) to such Φ were termed *pseudoharmonic maps* and shown to satisfy

$$\Delta_b \phi^i + \sum_{a=1}^{2n} \left(\Gamma^i_{jk} \circ \phi\right) X_a(\phi^j) X_a(\phi^k) = 0, \quad 1 \le i \le \nu, \tag{2}$$

where Δ_b is the sublaplacian associated to (M, θ) and $\{X_a : 1 \le a \le 2n\}$ is a local orthonormal frame of the maximal complex, or Levi, distribution $H(M)$ of M. The sublaplacian may be locally written $\Delta_b = -\sum_{a=1}^{2n} X_a^* X_a$ hence the similarity among the systems (1) and (2). The formal adjoint X_a^* of X_a is however meant with respect to the L^2 inner product $(u, v) = \int u\overline{v}\, \theta \wedge (d\theta)^n$, while in [**32**] one integrates with respect to the Lebesgue measure on Ω (the precise quantitative relationship among the two notions is explained in the next section).

The derivation of (1) by analogy to the harmonic map system (replacing the Laplace-Beltrami operator with the Hörmander operator) is nevertheless rather formal. Indeed *CR* manifolds appear mainly as boundaries of smooth domains Ω in $\boldsymbol{C}^n$ and it is not known so far whether boundary values of harmonic maps from Ω extending smoothly up to $\partial\Omega$ are pseudoharmonic. One of the results in this paper is the following

THEOREM 1. *Let $\Omega \subset \boldsymbol{C}^n$ $(n \geq 2)$ be a smoothly bounded strictly pseudoconvex domain and g the Bergman metric on Ω. Let S be a complete ν-dimensional $(\nu \geq 2)$ Riemannian manifold of sectional curvature* $\mathrm{Sect}(S) \leq \kappa^2$ *for some $\kappa > 0$. Assume that S may be covered by one coordinate chart $\chi = (y^1, \dots, y^\nu) : S \to \boldsymbol{R}^\nu$. Let $f \in W^{1,2}(\Omega, S) \cap C^0(\overline{\Omega}, S)$ be a map such that $f(\overline{\Omega}) \subset B(p, \mu)$ for some $p \in S$ and some $0 < \mu < \min\{\pi/(2\kappa),\ i(p)\}$ where $i(p)$ is the injectivity radius of p. Let $\phi = \phi_f : \overline{\Omega} \to S$ be the solution to the Dirichlet problem*

$$\tau_g(\phi) = 0 \quad \text{in } \Omega, \quad \phi = f \quad \text{on } \partial\Omega. \tag{3}$$

If $f \in C^\infty(\partial\Omega, S)$ then

$$N(f^i) = -\frac{1}{2(n-1)} (H_b f)^i, \quad 1 \leq i \leq \nu, \tag{4}$$

for any local coordinate system (ω, y^i) on S such that $\phi(\overline{\Omega}) \cap \omega \neq \emptyset$ $(f^i = y^i \circ f)$. Also $N = -JT$ and T is the characteristic direction of $\partial\Omega$. In particular if $N(f^i) = 0$ then $f : \partial\Omega \to S$ is a pseudoharmonic map.

Here $\tau_g(\phi) \in \Gamma^\infty(\phi^{-1}TS)$ is the tension field of ϕ as a map among the Riemannian manifolds (Ω, g) and S (cf. Section 3 for definitions). The key idea in the proof of Theorem 1 is (as first observed by A. Korányi and H. M. Reimann [**34**]) that the Kählerian geometry of the interior of Ω and the contact geometry of the boundary $\partial\Omega$ may be effectively related through the use of the Bergman kernel $K(z, \zeta)$ of Ω. The main technical ingredient in the proof is an ambient linear connection ∇ (the *Graham-Lee connection*, cf. R. Graham et al. [**22**], or Appendix A in [**4**]) defined on a neighborhood of $\partial\Omega$ in Ω and inducing the Tanaka-Webster connection (cf. [**47**], [**50**]) on each level set of $\varphi(z) = -K(z,z)^{-1/(n+1)}$ $(z \in \Omega)$. See also Section 5.3 in [**5**, pp. 87–95]. The proof of Theorem 1 is relegated to Section 3 of this paper.

2. Pseudoharmonic maps.

Let $(M, T_{1,0}(M))$ be a $(2n+1)$-dimensional orientable *CR* manifold, of *CR* dimension n. The *maximally complex distribution* is $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$. It carries the complex structure $J : H(M) \to H(M)$ given by $J(Z + \overline{Z}) = i(Z - \overline{Z})$ for any $Z \in T_{1,0}(M)$. A *pseudohermitian structure* is a globally defined nowhere zero C^∞ section θ in the conormal bundle $H(M)^\perp \subset T^*(M)$. The *Levi form* is $G_\theta(X,Y) = (d\theta)(X, JY)$ for any $X, Y \in H(M)$. Throughout $(M, T_{1,0}(M))$ is assumed to be *strictly pseudoconvex* i.e. G_θ is positive definite for some pseudohermitian structure θ. Then θ is a *contact form*, that is to say $\Psi = \theta \wedge (d\theta)^n$ is a volume form on M. Let T be the *characteristic direction* of $d\theta$ i.e. the globally defined nowhere zero tangent vector field on M, everywhere transverse to $H(M)$, determined by $\theta(T) = 1$ and $T \rfloor d\theta = 0$. Strictly pseudoconvex *CR* manifolds are equipped with a natural second order differential operator (similar to the Laplace-Beltrami operator on a Riemannian manifold)

$$\Delta_b u = \mathrm{div}\big(\nabla^H u\big), \quad u \in C^2(M), \tag{5}$$

the *sublaplacian* of (M, θ). Here div is the divergence with respect to Ψ i.e. $\mathscr{L}_X \Psi = \mathrm{div}(X)\Psi$ where $\mathscr{L}_X$ is the Lie derivative, and $\nabla^H u = \pi_H \nabla u$ (the *horizontal gradient* of u). Also ∇u is the gradient of u with respect to the Riemannian metric g_θ given by

$$g_\theta(X,Y) = G_\theta(X,Y), \quad g_\theta(X,T) = 0, \quad g_\theta(T,T) = 1, \tag{6}$$

for any $X, Y \in H(M)$ (the *Webster metric* of (M,θ)) and $\pi_H : T(M) \to H(M)$ is the projection relative to the decomposition $T(M) = H(M) \oplus \boldsymbol{R}T$. The sublaplacian is degenerate elliptic (in the sense of J. M. Bony [**7**]) and subelliptic of order 1/2 (cf. G. B. Folland [**16**]) hence hypoelliptic (cf. L. Hörmander [**29**]). Let us assume that M is compact and consider the energy functional

$$E(\phi) = \frac{1}{2} \int_M \mathrm{trace}_{G_\theta}(\pi_H \phi^* h)\ \Psi \tag{7}$$

where $\pi_H B$ denotes the restriction to $H(M)$ of the bilinear form B. Here E is defined on the set of all C^∞ maps $\phi : M \to S$ from M into a ν-dimensional Riemannian manifold (S, h). A *pseudoharmonic* map is a C^∞ map $\phi : M \to S$ such that $\{dE(\phi_t)/dt\}_{t=0} = 0$ for any smooth 1-parameter variation $\phi_t : M \to S$ of ϕ i.e. $\phi_0 = \phi$. Let us set

$$(H_b\phi)^i \equiv \Delta_b \phi^i + \sum_{a=1}^{2n} \left(\Gamma^i_{jk} \circ \phi\right) X_a(\phi^j) X_a(\phi^k), \quad 1 \leq i \leq \nu.$$

The Euler-Lagrange equations of the variational principle $\delta E(\phi) = 0$ are $H_b(\phi) = 0$ (cf. [**3**]). Let $\phi : M \to S$ be a pseudoharmonic map. Let (U, x^A) and (V, y^i) be a local coordinate systems on M and S such that $\phi(U) \subseteq V$. Let $\{X_a : 1 \leq a \leq 2n\}$ be a local G_θ-orthonormal frame in $H(M)$ defined on the open set U. As a consequence of the nondegeneracy of M the vector fields $\{(d\varphi)X_a : 1 \leq a \leq 2n\}$ form a Hörmander system on $\Omega = \varphi(U) \subseteq \boldsymbol{R}^{2n+1}$ where $\varphi = (x^1, \dots, x^{2n+1})$. As the formal adjoint of $X_a = b^A_a \partial/\partial x^A$ with respect to Ψ is given by $X^*_a u = -\partial(b^A_a u)/\partial x^A - b^B_a \Gamma^A_{AB} u$ one may conclude that $f \equiv \phi \circ \varphi^{-1} : \Omega \to S$ is a subelliptic harmonic map if and only if $L f^i = 0$ in Ω where L is the (purely local) first order differential operator $Lu = \sum_{a=1}^{2n} b^B_a \Gamma^A_{AB} X_a u$ and Γ^A_{BC} are the local coefficients of the Tanaka-Webster connection of (M, θ) with respect to (U, x^A). If for instance $M = \boldsymbol{H}_n$ (the *Heisenberg group*, cf. e.g. [**14**, pp. 11–12]) then $L \equiv 0$ and the two notions coincide.

To demonstrate a class of pseudoharmonic maps we look at $\overline{\partial}_b$-pluriharmonic maps of a nondegenerate *CR* manifold into a Riemannian manifold. We need a few additional notions of pseudohermitian geometry (cf. e.g. [**14**, Chapter 1]). The *tangential Cauchy-Riemann* operator is the first order differential operator

$$\overline{\partial}_b : C^\infty(M) \to \Gamma^\infty(T_{0,1}(M)^*)$$

defined by $(\overline{\partial}_b f)\overline{Z} = \overline{Z}(f)$ for any C^∞ function $f : M \to \boldsymbol{C}$ and any $Z \in T_{1,0}(M)$. A $(0,1)$-*form* is a $\boldsymbol{C}$-valued differential 1-form η on M such that $T_{1,0}(M) \rfloor \eta = 0$ and $T \rfloor \eta = 0$. Also a $(1,1)$-*form* is a $\boldsymbol{C}$-valued differential 2-form ω on M such that

$$\omega(Z, W) = \omega(\overline{Z}, \overline{W}) = 0, \quad T \rfloor \omega = 0,$$

for any $Z, W \in T_{1,0}(M)$. Let $\Lambda^{0,1}(M) \to M$ and $\Lambda^{1,1}(M) \to M$ be the corresponding vector bundles. Besides from $\overline{\partial}_b$ we need the differential operator

$$\partial_b : \Gamma^\infty(\Lambda^{0,1}(M)) \to \Gamma^\infty(\Lambda^{1,1}(M))$$

defined as follows. Let η be a $(0,1)$-form. Then $\partial_b \eta$ is the unique $(1,1)$-form on M coinciding with $d\eta$ on $T_{1,0}(M) \otimes T_{0,1}(M)$.

A C^2 function $u : M \to \boldsymbol{R}$ is said to be $\overline{\partial}_b$-*pluriharmonic* if $\partial_b \overline{\partial}_b u = 0$ (cf. [**11**] or Section 5.6 in [**5**, p. 112]).

The notion of a $\overline{\partial}_b$-pluriharmonic function admits a natural generalization to smooth maps $\phi : M \to S$ with values in a Riemannian manifold. The *second fundamental form* of ϕ is given by (cf. R. Petit [**45**])

$$\beta_\phi(X,Y) = (\phi^{-1}\nabla^h)_X \phi_* Y - \phi_* \nabla_X Y, \quad X, Y \in \mathscr{X}(M). \tag{8}$$

As to the notation adopted in (8), ∇^h is the Levi-Civita connection of (S,h), ∇ is the Tanaka-Webster connection of (M,θ), and $\phi_* X$ is the cross-section in the pullback bundle $\phi^{-1}TS \to M$ given by $(\phi_* X)_x = (d_x\phi)X_x$ for any $x \in M$. Also $\phi^{-1}\nabla^h$ is the connection in $\phi^{-1}TS \to M$ induced by ∇^h i.e. locally

$$(\phi^{-1}\nabla^h)_{\partial_A} X_k = \frac{\partial \phi^j}{\partial x^A}\Big(\Gamma^i_{jk} \circ \phi\Big) X_i.$$

Here (U, x^A) and (ω, y^i) are local coordinate systems on M and S respectively (with $\phi(U) \subseteq \omega$), ∂_A is short for $\partial/\partial x^A$, $\phi^i = y^i \circ \phi$, and X_i is the natural lift of $\partial/\partial y^i$ i.e. $X_i(x) = (\partial/\partial y^i)_{\phi(x)}$ (so that $\{X_i : 1 \le i \le \nu\}$ is a local frame in $\phi^{-1}TS \to M$ defined on the open set $\phi^{-1}(V)$). We say $\phi : M \to S$ is $\overline{\partial}_b$*-pluriharmonic* if

$$\beta_\phi(X,Y) + \beta_\phi(JX,JY) = 0, \quad X, Y \in H(M). \tag{9}$$

Equivalently $\beta_\phi(Z,\overline{W}) = 0$ for any $Z, W \in T_{1,0}(M)$. This may be locally written

$$(\partial_b\overline{\partial}_b\phi^i)(Z,\overline{W}) + Z(\phi^j)\overline{W}(\phi^k)\ \Gamma^i_{jk} \circ \phi = 0$$

hence if $S = \boldsymbol{R}^\nu$ then $\partial_b\overline{\partial}_b\phi^i = 0$ i.e. each ϕ^i is a $\overline{\partial}_b$-pluriharmonic function.

PROPOSITION 1. *Let M be a strictly pseudoconvex CR manifold and S a Riemannian manifold. Every $\overline{\partial}_b$-pluriharmonic map $\phi : M \to S$ is a pseudoharmonic map.*

PROOF. Let $\{W_\alpha : 1 \le \alpha \le n\}$ be a local orthonormal (that is $G_\theta(W_\alpha, W_{\overline{\alpha}}) = \delta_{\alpha\beta}$) frame of $T_{1,0}(M)$ so that locally

$$\Delta_b u = \sum_{\alpha=1}^n \{W_\alpha W_{\overline{\alpha}} u + W_{\overline{\alpha}} W_\alpha u - (\nabla_{W_\alpha} W_{\overline{\alpha}})u - (\nabla_{W_{\overline{\alpha}}} W_\alpha)u\}$$

for any $u \in C^2(M)$. As ϕ is $\overline{\partial}_b$-pluriharmonic

$$W_\alpha W_{\overline{\beta}}\phi^i - (\nabla_{W_\alpha} W_{\overline{\beta}})\phi^i + W_\alpha(\phi^j) W_{\overline{\beta}}(\phi^k)\,\Gamma^i_{jk}\circ\phi = 0$$

hence

$$\Delta_b\phi^i + 2\sum_{\alpha=1}^{n} W_\alpha(\phi^j) W_{\overline{\alpha}}(\phi^k)\,\Gamma^i_{jk}\circ\phi = 0$$

which is easily seen to be equivalent to (2). □

A theory of harmonic vector fields on a Riemannian manifold M was started by G. Wiegmink [**51**], and C. M. Wood [**52**], starting from the observation that the total space $T(M)$ of the tangent bundle over a Riemannian manifold (M, g) carries a Riemannian metric g_S naturally associated to g (the *Sasaki metric*, cf. e.g. D. E. Blair [**6**]). Then one may consider the ordinary Dirichlet energy functional $E(X) = \frac{1}{2}\int_M \operatorname{trace}_g(X^* g_S)\, d\operatorname{vol}(g)$ defined on $C^\infty(M, T(M))$. As it turns out a vector field $X : M \to T(M)$ is a harmonic map, i.e. a critical point of E for *arbitrary* smooth 1-parameter variations of X if and only if X is absolutely parallel. Hence the space $C^\infty(M, T(M))$ is intuitively too "large" for ones purposes. The same result is got however when looking for critical points of E restricted to the space of all smooth vector fields $\mathscr{X}(M)$.

A new and wider notion of harmonicity is however obtained by looking at *unit* vector fields X and restricting oneself to variations of X through unit vector fields. Precisely, let $U(M, g)_x = \{v \in T_x(M) : g_x(v, v) = 1\}$ $(x \in M)$. A unit vector field $X : M \to U(M, g)$ is *harmonic* if $\{dE(X_t)/dt\}_{t=0} = 0$ for any smooth 1-parameter family of smooth unit vector fields $X_t : M \to U(M, g)$ such that $X_0 = X$. The corresponding Euler-Lagrange equations are

$$\Delta X + \|\nabla^g X\|^2 X = 0, \tag{10}$$

where $\Delta = -(\nabla^g)^* \nabla^g$ is the *rough Laplacian* and ∇^g is the Levi-Civita connection of (M, g). A rather different theory (of *harmonic vector fields*) arises, aspects of which (e.g. stability of Hopf vector fields on spheres, the interplay with contact geometry) were subsequently investigated by many authors (cf. F. C. Brito [**8**], D.-S. Han et al. [**23**], A. Higuchi et al. [**26**], C. Oniciuc [**39**], D. Perrone [**42**]–[**44**], A. Yampolsky [**54**]). A similar approach also led to the more general theory of harmonic sections in vector bundles (cf. K. Hasegawa [**24**], J. J. Konderak [**33**], O. Gil-Medrano [**18**]).

Inspired by the geometric interpretation of subelliptic harmonic maps (in terms of the Fefferman metric, cf. [**3**]) together with the extension of the harmonic vector field theory to semi-Riemannian geometry (cf. O. Gil-Medrano et al. [**19**]) D. Perrone et al. studied (cf. [**13**]) a subelliptic analog to harmonic vector fields. There one considers vector fields $X \in H(M)$ on a strictly pseudoconvex CR manifold M endowed with a contact form θ (with G_θ positive definite) such that $G_\theta(X, X) = 1$ and the horizontal lift $X^\uparrow : C(M) \to T(C(M))$ (with respect to the connection 1-form $\sigma \in \Gamma^\infty(T^*(C(M)) \otimes L(S^1))$ in C. R. Graham [**21**]) is harmonic with respect to the Fefferman metric F_θ (which is a Lorentzian metric on $C(M)$, cf. [**36**]). By a result in [**13**] any such X satisfies

$$\Delta_b X + 4\nabla_T JX + 2\tau JX + 6\phi JX = \lambda(X)\ X \tag{11}$$

where

$$\begin{aligned}\lambda(X) = &-\|\pi_H \nabla X\|^2 + 4g_\theta(\nabla_T JX, X)\\ &+ 2g_\theta(\tau JX, X) + 6g_\theta(\phi JX, X).\end{aligned}$$

This is a nonlinear subelliptic system of variational origin (actually (11) are the Euler-Lagrange equations associated to the functional $\mathscr{B}(X) = -\int_M \lambda(X)\, \theta \wedge (d\theta)^n/2)$ yet formally rather dissimilar from the harmonic vector fields system (10) in Riemannian geometry. In the present paper we build (cf. Section 6) another subelliptic analog to the theory of harmonic vector fields, starting from the functional (7) restricted to the space of all unit vector fields (with respect to the Webster metric g_θ) and allowing only for variations through unit vector fields.

3. The Graham-Lee connection and C^∞ regularity up to the boundary of Bergman-harmonic maps.

Let $\Omega \subset \boldsymbol{C}^n$ ($n \geq 2$) be a bounded domain and g its Bergman metric (cf. e.g. S. Helgason [**25**, p. 369]). A smooth map $\phi : \Omega \to S$ into a Riemannian manifold S is *Bergman-harmonic* if it is a critical point of the energy functional

$$E(\phi) = \frac{1}{2}\int_\Omega \|d\phi\|^2 \ d\,\mathrm{vol}(g)$$

where $\|d\phi\|$ is the Hilbert-Schmidt norm of $d\phi$. The Euler-Lagrange system of the variational principle $\delta E(\phi) = 0$ is $\tau_g(\phi) = 0$ where $\tau_g(\phi) \in \Gamma^\infty(\phi^{-1}TS)$ is locally given by

$$\tau_g(\phi)^i = \Delta_g \phi^i + \left(\Gamma^i_{jk} \circ \phi\right) \frac{\partial \phi^j}{\partial x^A} \frac{\partial \phi^k}{\partial x^B} G^{AB}, \quad 1 \leq i \leq \nu,$$

where $(x^1, \dots, x^{2n})$ are the Cartesian coordinates in $\boldsymbol{R}^{2n}$ and $[G^{AB}] = [G_{AB}]^{-1}$, $G_{AB} = g(\partial_A, \partial_B)$, $\partial_A \equiv \partial/\partial x^A$. Also Δ_g is the Laplace-Beltrami operator of (Ω, g). For instance if Ω is the unit ball $\boldsymbol{B}^n = \{z \in \boldsymbol{C}^n : |z| < 1\}$ then

$$\Delta_g u = 4(1 - |z|^2) \sum_{i,j=1}^{n} (\delta_{ij} - z_i \overline{z}_j) \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}$$

hence Δ_g is elliptic in $\boldsymbol{B}^n$ yet its coefficients vanish at $\partial \boldsymbol{B}^n$ (also the second order differential operator $\sum_{i,j=1}^{n} (\delta_{ij} - z_i \overline{z}_j) \, \partial^2 / \partial z_i \partial \overline{z}_j$ is not elliptic at $\partial \boldsymbol{B}^n$). The degeneracy of the ellipticity of Δ_g at $\partial \boldsymbol{B}^n$ is responsible for pathologies such as the failure of C^∞ regularity up to the boundary of the solution to the Dirichlet problem for Bergman-harmonic functions.

From now on we assume that the Riemannian manifold S satisfies the requirements adopted in Theorem 1. As S is covered by one coordinate chart $\chi = (y^1, \dots, y^\nu) : S \to \boldsymbol{R}^\nu$ the Sobolev space $W^{1,2}(\Omega, S)$ is unambiguously defined as

$$W^{1,2}(\Omega, S) = \{\phi : \Omega \to S : \phi^i \equiv y^i \circ \phi \in W^{1,2}(\Omega), \quad 1 \le i \le \nu\}.$$

Let $p \in S$ and $\mu > 0$ be chosen as in Theorem 1. Let d be the distance function on S associated to the given Riemannian metric. The metric ball $B(p, \mu) = \{q \in S : d(p, q) < \mu\}$ is usually referred to as a *regular ball* and maps $f : \overline{\Omega} \to S$ satisfying a convexity condition $f(\overline{\Omega}) \subset B(p, \mu)$ behave very much like maps with values in $\boldsymbol{R}^\nu$. Indeed, by a classical result of S. Hildebrand, H. Kaul and K. Widman [**27**], for any $f \in W^{1,2}(\Omega, S) \cap C^0(\overline{\Omega}, S)$ with values in a regular ball $B(p, \mu)$, by exploiting the variational origin of $\tau_g(\phi) = 0$ the Dirichlet problem (3) may be solved i.e. there is a unique $\phi_f \in W^{1,2}(\Omega, S) \cap L^\infty(\Omega, S)$ such that $\phi_f|_{\partial\Omega} = f$ (that is $\phi_f^i - f^i \in W_0^{1,2}(\Omega)$, $1 \le i \le \nu$), $\phi_f(\overline{\Omega}) \subset B(p, \mu)$, ϕ_f minimizes E among all such maps, and ϕ_f is a weak solution to $\tau_g(\phi) = 0$ that is

$$\int_\Omega \left\{ g^*(d\phi_f^i, d\varphi) - \left(\Gamma^i_{jk} \circ \phi_f \right) \frac{\partial \phi_f^j}{\partial x^A} \frac{\partial \phi_f^k}{\partial x^B} G^{AB} \varphi \right\} d \,\mathrm{vol}(g) = 0$$

for any $\varphi \in C_0^\infty(\Omega)$. It is also well known (cf. S. Hildebrand and K. Widman [**28**]) that $\phi_f \in C^0(\Omega, S)$. Moreover continuous solutions to a class of quasilinear elliptic systems including $\tau_g(\phi) = 0$ are known (cf. e.g. S. Campanato [**9**], M. Giaquinta [**17**]) to be smooth hence $\phi_f \in C^\infty(\Omega, S)$.

Assuming additionally that $f \in C^\infty(\Omega, S)$ it is a natural question whether $\phi_f \in C^\infty(\overline{\Omega}, S)$. As we shall briefly recall, due to the fact that the ellipticity of the system $\tau_g(\phi) = 0$ degenerates at $\partial\Omega$, the C^∞ regularity of ϕ_f up to the boundary fails in general. For instance if $\Omega = \boldsymbol{B}^n$ and $S = \boldsymbol{R}$ then (by a result of C. R. Graham [**20**]) $\phi_f \in C^\infty(\overline{\Omega}, \boldsymbol{R})$ implies that f must be the boundary values of a pluriharmonic function. Therefore, in general one expects that whenever $\phi_f \in C^\infty(\overline{\Omega}, S)$ the datum f must satisfy additional compatibility relations which one may indicate formally as $\mathscr{C}(f) = 0$ on $\partial\Omega$. A differential geometric approach to deriving $\mathscr{C}(f) = 0$ in the scalar case (i.e. $S = \boldsymbol{R}$) was proposed by C. R. Graham and J. M. Lee [**22**], and relies on the construction of a canonical ambient connection (the *Graham-Lee connection*, cf. Theorem 2 below). It is then natural to ask 1) what is the geometric nature of the compatibility conditions $\mathscr{C}(f) = 0$ and of course 2) are $\mathscr{C}(f) = 0$ sufficient (to conclude that ϕ_f is C^∞ up to $\partial\Omega$)? Theorem 1 gives a partial answer to the first question while the second is an open problem (except for the scalar case, cf. [**20**], [**22**]).

The remainder of this section is devoted to the proof of Theorem 1. Let Ω be a smoothly bounded strictly pseudoconvex domain in $\boldsymbol{C}^n$ and $K(z,\zeta)$ its Bergman kernel (cf. e.g. [**25**, pp. 364–371]). By a result of C. Fefferman [**15**], $\varphi(z) = -K(z,z)^{-1/(n+1)}$ is a defining function for Ω (and $\Omega = \{\varphi < 0\}$). Let us set $\theta = \frac{i}{2}(\overline{\partial} - \partial)\varphi$. Then

Lemma 1 (E. Barletta [**1**]). *For any smoothly bounded strictly pseudoconvex domain $\Omega \subset \boldsymbol{C}^n$ the Bergman metric g is given by*

$$g(X,Y) = \frac{n+1}{\varphi}\left\{\frac{i}{\varphi}\,(\partial\varphi \wedge \overline{\partial}\varphi)(X, JY) - d\theta(X, JY)\right\}, \tag{12}$$

for any $X, Y \in \mathscr{X}(\Omega)$, where J is the complex structure on $\boldsymbol{C}^n$.

For each $\epsilon > 0$ we set $M_\epsilon = \{z \in \Omega : \varphi(z) = -\epsilon\}$. There is $\epsilon_0 > 0$ such that M_ϵ is a strictly pseudoconvex CR manifold of CR dimension $n-1$ for any $0 \le \epsilon \le \epsilon_0$. Hence there is a one-sided neighborhood V of $\partial\Omega$ carrying a tangentially CR foliation $\mathscr{F}$ (cf. also S. Nishikawa et al. [**12**]) by level sets of φ such that

$$V/\mathscr{F} = \{M_\epsilon : 0 < \epsilon < \epsilon_0\}.$$

Let $H(\mathscr{F}) \to V$ (respectively $T_{1,0}(\mathscr{F}) \to V$) be the bundle whose portion over M_ϵ is the Levi distribution $H(M_\epsilon)$ (respectively the CR structure $T_{1,0}(M_\epsilon)$) of M_ϵ. Note that

$$T_{1,0}(\mathscr{F}) \cap T_{0,1}(\mathscr{F}) = (0),$$
$$[\Gamma^\infty(T_{1,0}(\mathscr{F})), \Gamma^\infty(T_{1,0}(\mathscr{F}))] \subseteq \Gamma^\infty(T_{1,0}(\mathscr{F})).$$

Here $T_{0,1}(\mathscr{F}) = \overline{T_{1,0}(\mathscr{F})}$. We need the following

LEMMA 2 (J. M. Lee and R. Melrose [**35**]). *There is a unique complex vector field ξ on V, of type $(1,0)$, such that $\partial\varphi(\xi) = 1$ and ξ is orthogonal to $T_{1,0}(\mathscr{F})$ with respect to $\partial\overline{\partial}\varphi$ i.e. $\partial\overline{\partial}\varphi(\xi, \overline{Z}) = 0$ for any $Z \in T_{1,0}(\mathscr{F})$. Let $r = 2\,\partial\overline{\partial}\varphi(\xi, \overline{\xi})$ be the transverse curvature of φ. Then $r \in C^\infty(\overline{\Omega})$ i.e. r is smooth up to the boundary.*

Let $\xi = (N - iT)/2$ be the real and imaginary parts of ξ. Then

$$(d\varphi)(N) = 2, \quad \theta(N) = 0, \quad \partial\varphi(N) = 1,$$
$$(d\varphi)(T) = 0, \quad \theta(T) = 1, \quad \partial\varphi(T) = i.$$

In particular T is tangent to $\mathscr{F}$. Let g_θ be given by

$$g_\theta(X, Y) = (d\theta)(X, JY), \; g_\theta(X, T) = 0, \; g_\theta(T, T) = 1, \tag{13}$$

for any $X, Y \in H(\mathscr{F})$. Then g_θ is a tangential Riemannian metric for the foliation $\mathscr{F}$ i.e. a Riemannian metric in $T(\mathscr{F}) \to V$. As a consequence of (12) we may state

LEMMA 3. *The Bergman metric g of $\Omega \subset \boldsymbol{C}^n$ is given by*

$$g(X, Y) = -\frac{n+1}{\varphi}\, g_\theta(X, Y), \quad X, Y \in H(\mathscr{F}). \tag{14}$$

$$g(X, T) = 0, \;\; g(X, N) = 0, \quad X \in H(\mathscr{F}), \tag{15}$$

$$g(T, N) = 0, \;\; g(T, T) = g(N, N) = \frac{n+1}{\varphi}\left(\frac{1}{\varphi} - r\right). \tag{16}$$

In particular $1 - r\varphi > 0$ everywhere in Ω.

Using (14)–(16) one may relate the Levi-Civita connection ∇^g of (V, g) to the Graham-Lee connection of (V, φ). The latter has the advantage of staying finite at the boundary (in the limit it gives the Tanaka-Webster connection of $\partial\Omega$).

Let us recall the Graham-Lee connection. Let ∇ be a linear connection on V. Let us consider the $T(V)$-valued 1-form τ on V defined by

$$\tau(X) = T_\nabla(T, X), \quad X \in T(V),$$

where T_∇ is the torsion tensor field of ∇. We say T_∇ is *pure* if

$$T_\nabla(Z, W) = 0, \; T_\nabla(Z, \overline{W}) = 2ig_\theta(Z, \overline{W})T, \tag{17}$$

$$T_\nabla(N, W) = r\,W + i\,\tau(W), \tag{18}$$

for any $Z, W \in T_{1,0}(\mathscr{F})$, and

$$\tau(T_{1,0}(\mathscr{F})) \subseteq T_{0,1}(\mathscr{F}), \tag{19}$$

$$\tau(N) = -J\,\nabla^H r - 2r\,T. \tag{20}$$

Here $\nabla^H r$ is defined by $\nabla^H r = \pi_H \nabla r$ and $g_\theta(\nabla r, X) = X(r)$, $X \in T(\mathscr{F})$. Also $\pi_H : T(\mathscr{F}) \to H(\mathscr{F})$ is the projection associated to the direct sum decomposition $T(\mathscr{F}) = H(\mathscr{F}) \oplus \boldsymbol{R}T$. We recall

Theorem 2 (C. R. Graham and J. M. Lee [**22**]). *There is a unique linear connection ∇ on V such that* i) *$T_{1,0}(\mathscr{F})$ is parallel with respect to ∇,* ii) *$\nabla g_\theta = 0$, $\nabla T = 0$, $\nabla N = 0$, and* iii) *T_∇ is pure.*

∇ given by Theorem 2 is the *Graham-Lee connection* of (V, φ). The proof of Theorem 2 follows from (cf. also [**4**])

Lemma 4. *Let $\phi : T(\mathscr{F}) \to T(\mathscr{F})$ be the bundle morphism given by $\phi(X) = JX$, for any $X \in H(\mathscr{F})$, and $\phi(T) = 0$. Then*

$$\phi^2 = -I + \theta \otimes T,$$
$$g_\theta(X, T) = \theta(X),$$
$$g_\theta(\phi X, \phi Y) = g_\theta(X, Y) - \theta(X)\theta(Y),$$

for any $X, Y \in T(\mathscr{F})$. Moreover, if ∇ is a linear connection on V satisfying the axioms (i)–(iii) *in Theorem 2 then*

$$\phi \circ \tau + \tau \circ \phi = 0 \tag{21}$$

along $T(\mathscr{F})$. Consequently τ may be computed as

$$\tau(X) = -\frac{1}{2}\phi(\mathscr{L}_T \phi)X, \tag{22}$$

for any $X \in H(\mathscr{F})$.

A calculation (relying on Lemma 3) leads to

Theorem 3. *Let $\Omega \subset \boldsymbol{C}^n$ be a smoothly bounded strictly pseudoconvex domain, $K(z, \zeta)$ its Bergman kernel, and $\varphi(z) = -K(z,z)^{-1/(n+1)}$. Then the Levi-Civita connection ∇^g of the Bergman metric g and the Graham-Lee connection of (V, φ) are related by*

$$\nabla^g_X Y = \nabla_X Y + \left\{ \frac{\varphi}{1-\varphi r}\, g_\theta(\tau X, Y) + g_\theta(X, \phi Y) \right\} T$$
$$- \left\{ g_\theta(X,Y) + \frac{\varphi}{1-\varphi r}\, g_\theta(X, \phi\,\tau\, Y) \right\} N, \tag{23}$$

$$\nabla^g_X T = \tau X - \left(\frac{1}{\varphi} - r\right)\phi X - \frac{\varphi}{2(1-r\varphi)}\,\{X(r)T + (\phi X)(r)N\}, \tag{24}$$

$$\nabla^g_X N = -\left(\frac{1}{\varphi} - r\right) X + \tau\,\phi\, X + \frac{\varphi}{2(1-r\varphi)}\,\{(\phi X)(r)\, T - X(r)\, N\}, \tag{25}$$

$$\nabla^g_T X = \nabla_T X - \left(\frac{1}{\varphi} - r\right)\phi X - \frac{\varphi}{2(1-r\varphi)}\,\{X(r)T + (\phi X)(r)N\}, \tag{26}$$

$$\nabla^g_N X = \nabla_N X - \frac{1}{\varphi}\, X + \frac{\varphi}{2(1-r\varphi)}\,\{(\phi X)(r)T - X(r)N\}, \tag{27}$$

$$\nabla^g_N T = -\frac{1}{2}\,\phi\,\nabla^H r - \frac{\varphi}{2(1-r\varphi)}\left\{ \left(N(r) + \frac{4}{\varphi^2} - \frac{2r}{\varphi}\right) T + T(r) N \right\}. \tag{28}$$

$$\nabla^g_T N = \frac{1}{2}\,\phi \nabla^H r - \frac{\varphi}{2(1-r\varphi)}\left\{ \left(N(r) + \frac{4}{\varphi^2} - \frac{6r}{\varphi} + 4r^2\right) T + T(r) N \right\}, \tag{29}$$

$$\nabla^g_T T = -\frac{1}{2}\;\nabla^H r - \frac{\varphi}{2(1-r\varphi)}\left\{ T(r)T - \left(N(r) + \frac{4}{\varphi^2} - \frac{6r}{\varphi} + 4r^2\right) N \right\}, \tag{30}$$

$$\nabla^g_N N = -\frac{1}{2}\;\nabla^H r + \frac{\varphi}{2(1-r\varphi)}\left\{ T(r)T - \left(N(r) + \frac{4}{\varphi^2} - \frac{2r}{\varphi}\right) N \right\}, \tag{31}$$

for any $X, Y \in H(\mathscr{F})$.

Using Theorem 3 we may attack the proof of Theorem 1 (a similar technique was used in [**2**]). To this end let $\{W_\alpha : 1 \le \alpha \le n-1\}$ be a local orthonormal frame of $T_{1,0}(\mathscr{F})$ i.e. $g_\theta(W_\alpha, W_{\overline{\beta}}) = \delta_{\alpha\beta}$. Here $W_{\overline{\alpha}} = \overline{W}_\alpha$. Let us set $\psi = \varphi/(1-r\varphi)$ for simplicity. Then

$$E_\alpha = \sqrt{-\frac{\varphi}{n+1}}\, W_\alpha, \quad E_n = \sqrt{\frac{2\psi\varphi}{n+1}}\, \xi,$$

is a local orthonormal frame of $T^{1,0}(V)$ i.e. $g(E_j, E_k) = \delta_{jk}$. For any $u \in C^2(\Omega)$

$$\begin{aligned}\Delta_g u &= \sum_{j=1}^{n}\Big\{E_jE_{\bar j}u + E_{\bar j}E_ju - \Big(\nabla^g_{E_j}E_{\bar j}\Big)u - \Big(\nabla^g_{E_{\bar j}}E_j\Big)u\Big\}\\ &= -\frac{\varphi}{n+1}\sum_{\alpha=1}^{n-1}\Big\{W_\alpha W_{\overline{\alpha}}u + W_{\overline{\alpha}}W_\alpha u - \Big(\nabla^g_{W_\alpha}W_{\overline{\alpha}}\Big)u - \Big(\nabla^g_{W_{\overline{\alpha}}}W_\alpha\Big)u\Big\}\\ &\quad + \frac{2\psi\varphi}{n+1}\Big\{\xi\overline{\xi}u + \overline{\xi}\xi u - \Big(\nabla^g_{\xi}\overline{\xi}\Big)u - \Big(\nabla^g_{\overline{\xi}}\xi\Big)u\Big\}.\end{aligned}$$

On the other hand (by (28)–(31) in Theorem 3)

$$\nabla^g_{\xi}\overline{\xi} = -\frac{1}{4}\nabla^H r - \frac{r}{2}(N+iT)$$

hence

$$\xi\overline{\xi} + \overline{\xi}\xi - \nabla^g_{\xi}\overline{\xi} - \nabla^g_{\overline{\xi}}\xi = \frac{1}{2}\left(N^2+T^2\right) + \frac{1}{2}\nabla^H r + rN. \tag{32}$$

Moreover (by (23) in Theorem 3)

$$\nabla^g_{W_\alpha}W_{\overline{\alpha}} = \nabla_{W_\alpha}W_{\overline{\alpha}} - (N+iT) \tag{33}$$

for any $1\le\alpha\le n-1$. Let us substitute from (32)–(33) so that to obtain

$$\Delta_g = -\frac{\varphi}{n+1}\Delta_b - \frac{2\varphi(n-1)}{n+1}N + \frac{\psi\varphi}{n+1}\left\{N^2+T^2+\nabla^H r + 2rN\right\} \tag{34}$$

where Δ_b is given by

$$\Delta_b u = \sum_{\alpha=1}^{n-1}\left(W_\alpha W_{\overline{\alpha}} + W_{\overline{\alpha}}W_\alpha - \nabla_{W_\alpha}W_{\overline{\alpha}} - \nabla_{W_{\overline{\alpha}}}W_\alpha\right)u.$$

For each $z\in V$ the definition of $(\Delta_b u)(z)$ doesn't depend upon the choice of local orthonormal frame $\{W_\alpha : 1\le\alpha\le n-1\}$ of $T_{1,0}(\mathscr{F})$ at z. Also Δ_b restricts to each leaf of $\mathscr{F}$ as the sublaplacian of the leaf.

Let $g_{j\bar k} = g(\partial_j, \partial_{\bar k})$ where $\partial_j = \partial/\partial z^j$ and $\partial_{\bar k} = \partial/\partial \bar z^k$. If $[g^{j\bar k}] = [g_{j\bar k}]^{-1}$ then

$$[G^{AB}] = \begin{pmatrix} \frac{1}{4}\left(g^{j\bar k} + g^{\bar j k}\right) & -\frac{1}{4i}\left(g^{j\bar k} - g^{\bar j k}\right)\\ \frac{1}{4i}\left(g^{j\bar k} - g^{\bar j k}\right) & \frac{1}{4}\left(g^{j\bar k} + g^{\bar j k}\right)\end{pmatrix}.$$

Consequently

$$\big(\Gamma^a_{bc}\circ\phi\big)\,\frac{\partial\phi^b}{\partial x^A}\,\frac{\partial\phi^c}{\partial x^B}\,G^{AB} = 2\big(\Gamma^a_{bc}\circ\phi\big)\,\frac{\partial\phi^b}{\partial z^j}\,\frac{\partial\phi^c}{\partial\overline{z}^k}\,g^{j\overline{k}} = 2\sum_{j=1}^{n}\big(\Gamma^a_{bc}\circ\phi\big)E_j(\phi^b)E_{\overline{j}}(\phi^c).$$

The last equality follows from $\partial/\partial z^j = \lambda^k_j E_k$ and $g^{j\overline{k}} = \sum_{\ell=1}^n \mu^j_\ell \mu^{\overline{k}}_{\overline{\ell}}$ where $\mu = \lambda^{-1}$. We may conclude that

$$\Big(\Gamma^i_{jk}\circ\phi\Big)\,\frac{\partial\phi^j}{\partial x^A}\,\frac{\partial\phi^k}{\partial x^B}\,G^{AB} = -\,\frac{2\varphi}{n+1}\,\Big(\Gamma^i_{jk}\circ\phi\Big)\left\{\sum_{\alpha=1}^{n-1}W_\alpha(\phi^j)W_{\overline{\alpha}}(\phi^k) - 2\psi\xi(\phi^j)\overline{\xi}(\phi^k)\right\}. \tag{35}$$

Taking into account (34)–(35) the system $\tau_g(\phi)^i = 0$ may be written

$$\Delta_b\phi^i + 2(n-1)N\phi^i - \psi\big(N^2 + T^2 + \nabla^H r + 2rN\big)\phi^i + 2\Big(\Gamma^i_{jk}\circ\phi\Big)\left\{\sum_{\alpha=1}^{n-1}W_\alpha(\phi^j)W_{\overline{\alpha}}(\phi^k) - 2\psi\xi(\phi^j)\overline{\xi}(\phi^k)\right\} = 0. \tag{36}$$

Let $\phi = \phi_f$ be the solution to the Dirichlet problem (3) with $f \in C^\infty(\partial\Omega, S)$. Let us assume that $\phi \in C^\infty(\overline{\Omega}, S)$. Note that $\psi = O(\varphi)$. Therefore as $\varphi \to 0$ the equation (36) leads to

$$(H_b f)^i + 2(n-1)Nf^i = 0, \quad 1 \le i \le \nu,$$

which is (4) in Theorem 1. Note that, in opposition to the elliptic case, the normal derivatives of the map $f : \partial\Omega \to S$ may be determined in terms of purely tangential quantities.

4. An alternative expression for the first variation.

Let $\phi : M \to S$ be a smooth map of a compact strictly pseudoconvex *CR* manifold M of *CR* dimension n endowed with a contact form θ (such that the Levi form G_θ is positive definite) into a ν-dimensional Riemannian manifold (S, h). Let $\phi_t : M \to S$, $|t| < \epsilon$, be a smooth 1-parameter variation of ϕ (with $\phi_0 = \phi$). By a result in [**3**] the first variation formula is

$$\frac{d}{dt}\{E(\phi_t)\}_{t=0} = -\int_M \hat{h}(V, H_b(\phi))\, \theta \wedge (d\theta)^n$$

where $V \in \Gamma^\infty(\phi^{-1}TS)$ is the infinitesimal variation associated to $\{\phi_t\}_{|t|<\epsilon}$ and $\hat{h}$ is the Riemannian bundle metric in $\phi^{-1}TS \to M$ induced by h and $H_b(\phi) = \mathrm{trace}_{G_\theta}(\pi_H \beta_\phi)$. In this section we derive an alternative expression for the first variation formula which is imitative of the approach in [**49**, pp. 132–139]. By a classical result of J. C. Nash [**37**], there is an isometric immersion $\iota : S \to \boldsymbol{R}^K$ for some $K > \nu$, hence $h = \iota^* g_0$ where g_0 is the Euclidean metric on $\boldsymbol{R}^K$. Let $\Phi = \iota \circ \phi$. If $\{\phi_t\}_{|t|<\epsilon}$ is a 1-parameter variation as above we set $\Phi_t = \iota \circ \phi_t$. Also let us consider

$$\psi : M \times (-\epsilon, \epsilon) \to S, \quad \psi(x,t) = \phi_t(x), \quad x \in M, \ |t| < \epsilon,$$

and $\Psi = \iota \circ \psi$. We wish to compute $e(\phi) = 1/2\, \mathrm{trace}_{G_\theta}(\pi_H \phi^* h)$. To this end let $\{E_a : 1 \leq a \leq 2n\}$ be a local G_θ-orthonormal frame of $H(M)$, defined on an open set $U \subseteq M$. Then

$$e(\phi)_x = \frac{1}{2}\sum_{a=1}^{2n} g_{0,\Phi(x)}\big((d_x\Phi)E_{a,x},\, (d_x\Phi)E_{a,x}\big), \quad x \in U.$$

If we adopt the customary identifications $F_v : T_v(\boldsymbol{R}^K) \to \boldsymbol{R}^K$ given by $F_v(\partial/\partial u^A)_v = e_A$ for $1 \leq A \leq K$ (so that $g_{0,v}(F_v^{-1}(\xi), F_v^{-1}(\eta)) = \langle \xi, \eta \rangle$ for any $\xi, \eta \in \boldsymbol{R}^K$) then

$$e(\phi)_x = \frac{1}{2}\sum_{a,A}\big[u^A\big(F_{\Phi(x)}(d_x\Phi)E_{a,x}\big)\big]^2.$$

Here $(u^1, \ldots, u^K)$ and $\{e_A : 1 \leq A \leq K\}$ are the Cartesian coordinates and canonical basis of $\boldsymbol{R}^K$. Let $\iota^A = u^A \circ \iota$ and let $B^A_j = \partial \iota^A / \partial y^j$ be the Jacobian of the given immersion, where (V, y^i) are local coordinates on S such that $\phi^{-1}(V) \subseteq U$. We may assume without loss of generality that U is the domain of a local coordinate chart $(x^1, \ldots, x^{2n+1}) : U \to \boldsymbol{R}^{2n+1}$ on M. If $E_a = \sum_{p=1}^{2n+1} U^p_a\, \partial/\partial x^p$ for some $U^p_a \in C^\infty(U)$ then

$$e(\phi) = \frac{1}{2}\sum_{a,A}\left[U^p_a (B^A_i \circ \phi)\, \frac{\partial \phi^i}{\partial x^p}\right]^2 \tag{37}$$

on U. Then (by (37))

$$e(\phi_t)_x = \frac{1}{2}\sum_{a,A} f_a^A(x,t)^2$$

where $f_a^A(x,t) = U_a^p(x)B_i^A(\psi(x,t))(\partial\psi^i/\partial x^p)(x,t)$. Let $\mathscr{E}(t) = E(\phi_t)$ hence

$$\mathscr{E}'(0) = \sum_{a,A}\int f_a^A(x,0)\,\frac{\partial f_a^A}{\partial t}\,(x,0)\,dx$$

where $dx = (\theta\wedge(d\theta)^n)(x)$. Also $f_a^A(x,0) = B_i^A(\phi(x))(E_a\phi^i)_x$ and

$$\frac{\partial f_a^A}{\partial t} = \left(U_a^p\circ\pi\right)\left[\left(B_{ij}^A\circ\psi\right)\frac{\partial\psi^j}{\partial t}\,\frac{\partial\psi^i}{\partial x^p} + \left(B_i^A\circ\psi\right)\frac{\partial^2\psi^i}{\partial x^p\partial t}\right]. \tag{38}$$

Here $\pi : M\times(-\epsilon,\epsilon)\to M$ is the natural projection and $B_{ij}^A = \partial B_i^A/\partial y^j$. Let $W\in\Gamma^\infty(\phi^{-1}TS)$ be given by

$$W_x = (d_{(x,0)}\psi)\frac{\partial}{\partial t}\bigg|_{(x,0)} = \frac{\partial\psi^i}{\partial t}\,(x,0)\frac{\partial}{\partial y^i}\bigg|_{\phi(x)}.$$

Let us consider the functions $W^A : U\to \boldsymbol{R}$, $1\le A\le K$, given by

$$W^A(x) = u^A(F_{\Phi(x)}(d_{\phi(x)}\iota)W_x),\quad x\in U.$$

We shall need the following

Lemma 5. *Let $\Phi^A = \iota^A\circ\phi$. Then*

$$\mathscr{E}'(0) = -\int\sum_A W^A\,(\Delta_b\Phi^A)\,dx. \tag{39}$$

Proof. Note that $(\partial\psi^i/\partial x^p)(x,0) = (\partial\phi^i/\partial x^p)(x)$. Similarly if we set $g^i(x) = (\partial\psi^i/\partial t)(x,0)$ then $(\partial^2\psi^i/\partial x^p\partial t)(x,0) = (\partial g^i/\partial x^p)(x)$. Next (by (38))

$$\frac{\partial f_a^A}{\partial t}\bigg|_{t=0} = U_a^p\,\frac{\partial}{\partial x^p}\left((B_i^A\circ\phi)\,g^i\right)$$

so that

$$\mathscr{E}'(0) = \sum_{a,A} \int (B_i^A \circ \phi) E_a(\phi^i) E_a(W^A) dx \tag{40}$$

due to $W^A = g^i(B_i^A \circ \phi)$. Let us consider the tangent vector field $X \in \mathscr{X}(U)$ given by

$$X = \sum_A W^A (B_i^A \circ \phi)\, \nabla^H \phi^i.$$

By taking into account the identity

$$\sum_a E_a(G_\theta(X, E_a)) = \operatorname{div}(X) + \sum_a G_\theta(X, \nabla_{E_a} E_a)$$

we may integrate by parts in (40)

$$\begin{aligned}
\mathscr{E}'(0) = \sum_{a,A} \int \{ & E_a(W^A (B_i^A \circ \phi)\, E_a \phi^i) \\
& - W^A\, E_a((B_i^A \circ \phi)\, E_a \phi^i)\}\, dx = \int \Big\{ \sum_a E_a(G_\theta(X, E_a)) \\
& - \sum_{a,A} W^A \Big[(B_i^A \circ \phi)\, E_a^2 \phi^i + (B_{ij}^A \circ \phi)\, (E_a \phi^i)(E_a \phi^j) \Big] \Big\}\, dx
\end{aligned}$$

so that to get

$$\mathscr{E}'(0) = - \int \sum_A W^A \Big\{ (B_i^A \circ \phi)\, \Delta_b \phi^i + \sum_a (B_{ij}^A \circ \phi)\, (E_a \phi^i)(E_a \phi^j) \Big\}\, dx. \tag{41}$$

Since $E_a(\Phi^A) = (B_i^A \circ \phi)\, E_a \phi^i$ the identity (5) yields

$$\Delta_b \Phi^A = (B_i^A \circ \phi)\, \Delta_b \phi^i + \sum_a (B_{ij}^A \circ \phi)\, (E_a \phi^i)(E_a \phi^j) \tag{42}$$

and (39) follows from (41). Lemma 5 is proved. $\square$

At this point we need to recall the Gauss formula for the immersion $\iota : S \to \boldsymbol{R}^K$ (cf. e.g. [**10**])

$$B^A_{ij} = B^A_k \, \Gamma^k_{ij} + A^A_{ij} \tag{43}$$

where Γ^i_{jk} are the coefficients of the induced connection, $A(\partial_i,\ \partial_j) = A^A_{ij}\,\partial/\partial u^A$ is the second fundamental form of ι, and ∂_i is short for $\partial/\partial y^i$. By (43) one may write (42) as

$$\begin{aligned} &\big(\Delta_b \Phi^A\big)(x)\,\frac{\partial}{\partial u^A}\bigg|_{\Phi(x)} \\ &\quad = (d_{\phi(x)}\iota)\left[\big(\Delta_b \phi^i\big)(x)\,\frac{\partial}{\partial y^i}\bigg|_{\phi(x)} + \Big((\phi^{-1}\nabla^N)_{E_a}\phi_* E_a\Big)_x\right] \\ &\qquad + \sum_a A_{\phi(x)}\big((\phi_* E_a)_x,\, (\phi_* E_a)_x\big) \end{aligned} \tag{44}$$

for any $x \in U$. We may state

THEOREM 4. *Let $\phi : M \to S$ be a smooth map of a strictly pseudoconvex CR manifold M into a Riemannian manifold S. Let $\iota : S \to \boldsymbol{R}^K$ be an isometric immersion of S in some Euclidean space $\boldsymbol{R}^K$. Then ϕ is a pseudoharmonic map if and only if*

$$(\Delta_b \Phi)(x) = F_{\Phi(x)}(\mathrm{trace}_{G_\theta}\pi_H(\phi^* A))_x \tag{45}$$

for any $x \in M$, where $\Delta_b \Phi = (\Delta_b \Phi^1, \ldots, \Delta_b \Phi^K)$ and A is the second fundamental form of ι.

PROOF. We may choose a variation $\{\phi_t\}_{|t|<\epsilon}$ such that

$$W_x = \tan_{\phi(x)}\big(\Delta_b \Phi^A\big)(x)\,\frac{\partial}{\partial u^A}\bigg|_{\Phi(x)}, \quad x \in M,$$

where $\tan_y : T_{\iota(y)}(\boldsymbol{R}^K) \to T_y(S)$, is the projection associated with the direct sum decomposition

$$T_{\iota(y)}(\boldsymbol{R}^K) = [(d_y \iota) T_y(S)] \oplus E(\iota)_y, \quad y \in S,$$

and $E(\iota) \to S$ is the normal bundle of ι. Then (by (39) in Lemma 5) $W = 0$ so that (44) yields (45). □

COROLLARY 1. *Let M be a compact connected strictly pseudoconvex CR manifold. Then any pseudoharmonic map $\phi : M \to \boldsymbol{R}^L$ is constant.*

Proof. We may embed $\boldsymbol{R}^L$ as a totally geodesic submanifold into some Euclidean space $\boldsymbol{R}^K$ hence (by (45)) $\Delta_b \Phi^A = 0$. Then $\Delta_b(\Phi^A)^2 = 2\|\nabla^H \Phi^A\|^2$ so that (by integrating over M and applying Green's lemma) $\nabla^H \Phi^A = 0$. In particular $\overline{\partial}_b \Phi^A = 0$ i.e. Φ^A is a $\boldsymbol{R}$-valued CR function. $\square$

On the other hand

Lemma 6. *On a nondegenerate connected CR manifold any real valued CR function is constant.*

Proof. Let $\{T_\alpha : 1 \leq \alpha \leq n\}$ be a local frame of $T_{1,0}(M)$ defined on the open set $U \subseteq M$. Let $u : M \to \boldsymbol{R}$ be a CR function i.e. a C^1 solution to $T_{\overline{\alpha}}(u) = 0$ where $T_{\overline{\alpha}} = \overline{T}_\alpha$. By complex conjugation $T_\alpha(u) = 0$ as well. Finally by the purity axiom satisfied by the torsion of the Tanaka-Webster connection of (M, θ) (cf. (1.37) in [**14**, p. 25])

$$2 g_{\alpha\overline{\beta}} T(u) = \Gamma^{\overline{\gamma}}_{\alpha\overline{\beta}} T_{\overline{\gamma}}(u) - \Gamma^{\gamma}_{\overline{\beta}\alpha} T_\gamma(u) - [T_\alpha, T_{\overline{\beta}}](u) = 0$$

for any local frame $\{T_\alpha : 1 \leq \alpha \leq n\}$ of $T_{1,0}(M)$. Here $g_{\alpha\overline{\beta}} = G_\theta(T_\alpha, T_{\overline{\beta}})$ and Γ^a_{bc} are the local coefficients of the Tanaka-Webster connection $(a, b, c \in \{1, \dots, n, \overline{1}, \dots, \overline{n}\})$. Hence $T(u) = 0$ so that u is locally constant. $\square$

5. Pseudoharmonic maps into spheres.

Let $S^\nu = \{x = (x_1, \dots, x_{\nu+1}) \in \boldsymbol{R}^{\nu+1} : x_1^2 + \cdots + x_{\nu+1}^2 = 1\}$ and let $\Sigma = \{x \in S^\nu : x_1 = x_2 = 0\}$. Let M be a topological space. A continuous map $\phi : M \to S^\nu$ *meets* Σ if $\phi(M) \cap \Sigma \neq \emptyset$. Let $\phi : M \to S^\nu$ be a map that doesn't meet Σ. We say ϕ *links* Σ if the map $\phi : M \to S^\nu \setminus \Sigma$ is not null-homotopic. Our purpose in this section is to establish the following

Theorem 5. *A nonconstant pseudoharmonic map $\phi : M \to S^\nu$ of a compact strictly pseudoconvex CR manifold M into a sphere S^ν either links or meets Σ.*

Theorem 5 is the subelliptic counterpart of a result by B. Solomon [**46**]. As in [**46**] the proof relies on the observation that $S^\nu \setminus \Sigma$ is isometric to the warped product $S^{\nu-1}_+ \times_v S^1$ where $v : S^{\nu-1}_+ \times S^1 \to (0, +\infty)$ is given by $v(y, z) = y_\nu$ for any $y \in S^{\nu-1}_+$ and any $z \in S^1 \subset \boldsymbol{C}$ where $S^{\nu-1}_+ = \{y = (y', y_\nu) \in \boldsymbol{R}^\nu : y \in S^\nu,\ \ y_\nu > 0\}$. Indeed $I(y, u + iv) = (y_\nu u, y_\nu v, y')$ is an isometry of $S^{\nu-1}_+ \times S^1$ endowed with the warped product metric $\pi_1^* g_{\nu-1} + v^2\, \pi_2^* g_1$ into $S^\nu \setminus \Sigma$. Here $\pi_1 : S^{\nu-1}_+ \times S^1 \to S^{\nu-1}_+$ and $\pi_2 : S^{\nu-1}_+ \times S^1 \to S^1$ are the natural projections. Also we denote by g_k the standard metric on $S^k \subset \boldsymbol{R}^{k+1}$. The first step is to establish

LEMMA 7. *Let $\phi : M \to S$ be a pseudoharmonic map of a compact strictly pseudoconvex CR manifold into a Riemannian warped product $S = L \times_w \boldsymbol{R}$ with $w \in C^\infty(L)$. Then $\phi(M) \subset L \times \{t_\phi\}$ for some $t_\phi \in \boldsymbol{R}$.*

PROOF. We set $F = p_1 \circ \phi$ and $u = p_2 \circ \phi$ where $p_1 : S \to L$ and $p_2 : S \to \boldsymbol{R}$ are the natural projections. The target manifold carries the Riemannian metric $h = p_1^* g_L + (w \circ p_1)^2 \, dt \otimes dt$ where g_L is a Riemannian metric on L and $t = p_2$. Let $\varphi \in C^\infty(M)$ and let $\phi_t : M \to S$ be given by $\phi_t(x) = (F(x), u(x) + t\varphi(x))$ for any $x \in M$ and any $|t| < \epsilon$. Then $\{\phi_t\}_{|t|<\epsilon}$ is a smooth 1-parameter variation of ϕ and

$$\mathrm{trace}_{G_\theta}\big(\pi_H \phi_t^* h\big) = \mathrm{trace}_{G_\theta}(\pi_H F^* g_L) + (w \circ F)^2 \, \|\nabla^H u_t\|^2,$$

where $u_t = p_2 \circ \phi_t$, so that

$$E(\phi_t) = E(F) + \frac{1}{2} \int_M (w \circ F)^2 \|\nabla^H u_t\|^2 \, \theta \wedge (d\theta)^n.$$

As ϕ is pseudoharmonic (integrating by parts)

$$\begin{aligned} 0 = \frac{d}{dt} \{E(\phi_t)\}_{t=0} &= \int_M (w \circ F)^2 G_\theta(\nabla^H u, \nabla^H \varphi) \, \theta \wedge (d\theta)^n \\ &= - \int_M \varphi \, \mathrm{div}\Big((w \circ F)^2 \, \nabla^H u\Big) \, \theta \wedge (d\theta)^n \end{aligned}$$

hence u satisfies the subelliptic equation

$$\mathrm{div}\Big((w \circ F)^2 \, \nabla^H u\Big) = 0. \tag{46}$$

Finally (by (46)) $\mathrm{div}((w \circ F)^2 \, \nabla^H u^2) = 2(w \circ F)^2 \, \|\nabla^H u\|^2$ and then (by Green's lemma) $\int_M (w^2 \circ F) \, \|\nabla^H u\|^2 \, \theta \wedge (d\theta)^n = 0$ so that u is a $\boldsymbol{R}$-valued CR function and hence $u =$ constant. Lemma 7 is proved. $\square$

We shall apply Lemma 7 for $L = S_+^{\nu-1}$ and $w \in C^\infty(S_+^{\nu-1})$ given by $w(y) = y_\nu$. The proof of Theorem 5 is by contradiction. Precisely we assume that $\phi : M \to S^\nu \setminus \Sigma$ is a null-homotopic pseudoharmonic map and show that ϕ must be constant.

Let $p : \boldsymbol{R} \to S^1$ be the exponential map $p(t) = e^{it}$. Also let us consider the warped product metric $p_1^* g_{\nu-1} + (w \circ p_1)^2 \, dt \otimes dt$ on $S_+^{\nu-1} \times \boldsymbol{R}$. Then $\pi = (\mathrm{id}, p)$ is a local isometry of $S_+^{\nu-1} \times_w \boldsymbol{R}$ onto $S_+^{\nu-1} \times_v S^1$. Let $F = \pi_1 \circ \psi$ and $u = \pi_2 \circ \psi$ where

$\psi = I^{-1} \circ \phi$. Let $x_0 \in M$ and $z_0 = u(x_0) \in S^1$. Let $t_0 \in \boldsymbol{R}$ such that $p(t_0) = z_0$. As $\psi : M \to S_+^{\nu-1} \times S^1$ is null-homotopic it follows that $u_* \pi_1(M, x_0) = 0$ hence by standard homotopy theory (cf. e.g. Proposition 5.3 in [**30**, p. 43]) there is a unique function $\tilde{u} : M \to \boldsymbol{R}$ such that $\tilde{u}(x_0) = t_0$ and $p \circ \tilde{u} = u$. As u is smooth it follows that $\tilde{u} \in C^\infty(M)$ as well. The following result is immediate

Lemma 8. *Let $\phi : M \to S$ be a pseudoharmonic map from a compact strictly pseudoconvex CR manifold into a Riemannian manifold S. If $\pi : \tilde{S} \to S$ is a local isometry then any smooth map $\tilde{\phi} : M \to \tilde{S}$ such that $\pi \circ \tilde{\phi} = \phi$ is pseudoharmonic.*

By Lemma 8 it follows that $\tilde{\psi} = (F, \tilde{u}) : M \to S_+^{\nu-1} \times_w \boldsymbol{R}$ is a pseudoharmonic map. Then (by Lemma 7) $\tilde{\psi}(M) \subset S_+^{\nu-1} \times \{t\}$ for some $t \in \boldsymbol{R}$. We may conclude the proof of Theorem 5 by using

Lemma 9. *Any pseudoharmonic map $\phi : M \to S_+^{\nu-1}$ of a compact strictly pseudoconvex CR manifold M into an open upper hemisphere $S_+^{\nu-1}$ is constant.*

Proof. Let $\iota : S^{\nu-1} \to \boldsymbol{R}^\nu$ be the inclusion and $\Phi = \iota \circ \phi$. It is a standard matter to compute the second fundamental form of ι. As a corollary of (45) in Theorem 4

$$\Delta_b \Phi^A = |E\Phi|^2 \, \Phi^A, \quad 1 \le A \le \nu, \tag{47}$$

where $|E\Phi|^2 = \sum_{a,A} (E_a \Phi^A)^2$ with respect to a local G_θ-orthonormal frame $\{E_a : 1 \le a \le 2n\}$ of $H(M)$. Cf. also (4.52) in [**14**, p. 257]. As $\phi(M) \subset S_+^{\nu-1}$ one has $\Phi^\nu > 0$. Let us set $A = \nu$ in (47), integrate over M and use Green's lemma. It follows that $\int_M \Phi^\nu \, |E\Phi|^2 \, \theta \wedge (d\theta)^n = 0$ hence Φ is a $\boldsymbol{R}^\nu$-valued CR function on M. As M is nondegenerate Φ must be constant. $\square$

6. Pseudoharmonic vector fields.

6.1. Total bending.

Related to (5) we consider the sublaplacian on vector fields. Let M be a strictly pseudoconvex CR manifold and θ a contact form on M such that G_θ is positive definite. If X is a C^2 vector field on M then $\Delta_b X$ is the vector field locally given by

$$(\Delta_b X)^i = \Delta_b X^i + 2a^{jk} \Gamma^i_{j\ell} \frac{\partial X^\ell}{\partial x^k} + a^{jk} \left(\frac{\partial \Gamma^i_{j\ell}}{\partial x^k} + \Gamma^s_{j\ell} \Gamma^i_{ks} - \Gamma^s_{jk} \Gamma^i_{s\ell} \right) X^\ell \tag{48}$$

where $X = X^i \, \partial/\partial x^i$ and Γ^i_{jk} are the coefficients of the Tanaka-Webster connection ∇ of (M, θ) with respect to the local coordinate system (U, x^i) on M. Also $a^{ij} = g^{ij} - T^i T^j$ and $[g^{ij}] = [g_{ij}]^{-1}$ where $g_{ij} = g_\theta(\partial_i, \partial_j)$, $\partial_i = \partial/\partial x^i$. Let

$$\mathscr{U}(M, \theta) = \{X \in \mathscr{X}^\infty(M) : g_\theta(X, X) = 1\}$$

be the set of all C^∞ unit vector fields on (M, g_θ). The pseudohermitian *biegung*, or *total bending*, is the functional $\mathscr{B} : \mathscr{U}(M, \theta) \to [0, +\infty)$ given by

$$\mathscr{B}(X) = \frac{1}{2} \int_M \|\nabla^H X\|^2 \, \Psi, \quad X \in \mathscr{U}(M, \theta). \tag{49}$$

Here $\nabla^H X \in \Gamma^\infty(H(M)^* \otimes T(M))$ is the restriction of ∇X to $H(M)$. The biegung (49) is a pseudohermitian analog to R. Wiegmink's total bending (cf. [**51**]) of a vector field on a Riemannian manifold (and $\mathscr{B}(X)$ measures the failure of X to satisfy $\nabla_Y X = 0$ for any $Y \in H(M)$). We adopt the following definition. A *pseudoharmonic vector field* is a C^∞ unit vector field $X \in \mathscr{U}(M, \theta)$ which is a critical point of $\mathscr{B}$ with respect to 1-parameter variations of X through unit vector fields. For simplicity we assume that M is compact (otherwise we may modify the definition (49) by integrating over a relatively compact domain $\Omega \subset M$ and consider only variations supported in Ω). Pseudoharmonic vector fields will be shown to satisfy the nonlinear subelliptic system

$$\Delta_b X + \|\nabla^H X\|^2 X = 0, \tag{50}$$

(the Euler-Lagrange equations of the variational principle associated to (49)). The pseudohermitian biegung (49) is related to the functional (7). To see this we need the *CR* analog to the Sasaki metric (on the total space of the tangent bundle of a Riemannian manifold).

6.2. Geometry of the tangent bundle over a CR manifold.

Let $\pi^{-1}TM \to T(M)$ be the pullback of the tangent bundle, where $\pi : T(M) \to M$ is the projection. If X is a vector field on M then $\hat{X} = X \circ \pi$ is its *natural lift* (a cross section in $\pi^{-1}TM \to T(M)$). Let θ be a contact form on M with G_θ is positive definite. The Tanaka-Webster connection ∇ of (M, θ) induces a connection $\hat{\nabla}$ in $\pi^{-1}TM \to T(M)$ which is easiest to describe in local coordinates. Let $(U, \tilde{x}^i)$ be a local coordinate system on M and $(\pi^{-1}(U), x^i, y^i)$ the naturally induced local coordinates on $T(M)$. Let X_i be the natural lifts of $\partial/\partial \tilde{x}^i$ (a local frame in $\pi^{-1}TM \to T(M)$ defined on the open set $\pi^{-1}(U)$). Let Γ^i_{jk} be the local coefficients of ∇ with respect to $(U, \tilde{x}^i)$. Then $\hat{\nabla}$ is locally given by

$$\hat{\nabla}_{\partial_j} X_k = (\Gamma^i_{jk} \circ \pi) X_i, \quad \hat{\nabla}_{\dot{\partial}_j} X_k = 0, \tag{51}$$

where $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial y^i$ for simplicity. Let $\mathscr{L}$ be the *Liouville vector* i.e. locally $\mathscr{L} = y^i X_i$. A tangent vector field $\mathscr{X}$ on $T(M)$ is *horizontal* if $\hat{\nabla}_{\mathscr{X}} \mathscr{L} = 0$. A calculation based on (51) shows that $\mathscr{X} = \mathscr{X}^i \partial_i + \mathscr{X}^{i+m} \dot{\partial}_i$ is horizontal if and only if $\mathscr{X}^{i+m} = -N^i_j \mathscr{X}^j$. Here $N^i_j = \Gamma^i_{jk} y^k$ and $m = 2n+1$. Let

$$\mathscr{H}_u = \{\mathscr{X}_u : \mathscr{X} \text{ horizontal}\}, \quad u \in T(M).$$

Then

$$\delta_i = \partial_i - N^i_j \dot{\partial}_j, \quad 1 \le i \le m, \tag{52}$$

is a local frame of $\mathscr{H} \to T(M)$ on $\pi^{-1}(U)$ hence $\mathscr{H}$ is a C^∞ distribution of rank m on $T(M)$ and

$$T(T(M)) = \mathscr{H} \oplus \operatorname{Ker}(d\pi). \tag{53}$$

Thus the restriction to $\mathscr{H}$ of

$$L : T(T(M)) \to \pi^{-1} TM, \quad L_u \mathscr{X} = (d_u \pi) \mathscr{X},$$
$$\mathscr{X} \in T_u(T(M)), \quad u \in T(M),$$

is a bundle isomorphism whose inverse is denoted by $\beta : \pi^{-1} TM \to \mathscr{H}$ (the *horizontal lift* associated to ∇). Let $\gamma : \pi^{-1} TM \to \operatorname{Ker}(d\pi)$ be the *vertical lift* i.e. locally $\gamma X_i = \dot{\partial}_i$. The *Dombrowski map* is the bundle morphism

$$K : T(T(M)) \to \pi^{-1} TM, \quad K = \gamma^{-1} \circ Q,$$

where $Q : T(T(M)) \to \operatorname{Ker}(d\pi)$ is the projection associated to the decomposition (53). The given data induces a Riemannian metric S_θ on $T(M)$ given by

$$S_\theta(\mathscr{X}, \mathscr{Y}) = g_\theta(L\mathscr{X}, L\mathscr{Y}) + g_\theta(K\mathscr{X}, K\mathscr{Y}), \ \mathscr{X}, \mathscr{Y} \in T(T(M)).$$

As well as in Riemannian geometry (cf. D. E. Blair [**6**]) S_θ is referred to as the *Sasaki metric* of (M, θ). The total space of the tangent bundle of a strictly pseudoconvex *CR* manifold possesses a rich geometric structure whose investigation is (as opposed to the Riemannian case, cf. [**6**] and references therein) far from being complete. For instance, note that the Riemannian manifold $(T(M), S_\theta)$ carries the compatible almost complex structure

$$J(\beta X) = \gamma X, \quad J(\gamma X) = -\beta X, \quad X \in \pi^{-1}TM.$$

A simple calculation shows that the Nijenhuis tensor field of J is given by

$$N_J(\beta X, \beta Y) = \gamma R(X,Y)\mathscr{L} + \beta T(X,Y),$$
$$N_J(\gamma X, \beta Y) = \beta R(X,Y)\mathscr{L} - \gamma T(X,Y),$$
$$N_J(\gamma X, \gamma Y) = -\gamma R(X,Y)\mathscr{L} - \beta T(X,Y),$$

for any $X, Y \in \pi^{-1}TM$. Here

$$R(X,Y)Z = R^{\hat\nabla}(\beta X, \beta Y)Z, \quad T(X,Y) = T^{\hat\nabla}(\beta X, \beta Y).$$

Also $R^{\hat\nabla}$ is the curvature tensor field of $\hat\nabla$ and $T^{\hat\nabla}$ is defined by

$$T^{\hat\nabla}(\mathscr{X}, \mathscr{Y}) = \hat\nabla_{\mathscr{X}} L\mathscr{Y} - \hat\nabla_{\mathscr{Y}} L\mathscr{X} - L[\mathscr{X}, \mathscr{Y}]$$

for any tangent vector fields $\mathscr{X}, \mathscr{Y}$ on $T(M)$. As the Tanaka-Webster connection has torsion J is never integrable.

6.3. The first variation formula.

Let us consider the functional $E : C^\infty(M, T(M)) \to \boldsymbol{R}$ given by

$$E(\phi) = \frac{1}{2}\int_M \operatorname{trace}_{G_\theta}(\pi_H \phi^* S_\theta)\ \Psi$$

where $\pi_H \phi^* S_\theta$ denotes the restriction of the bilinear form $\phi^* S_\theta$ to $H(M) \otimes H(M)$. We shall show that

THEOREM 6. *Let M be a compact strictly pseudoconvex CR manifold and θ a contact form with G_θ positive definite. Let X be a smooth vector field on M. Then*

$$E(X) = n \operatorname{Vol}(M, \theta) + \mathscr{B}(X), \tag{54}$$

where $\operatorname{Vol}(M,\theta) = \int_M \Psi$. *Consequently* i) $E(X) \geq n \operatorname{Vol}(M,\theta)$ *with equality if and only if* $\nabla^H X = 0$. *Also* ii) $X : (M, \theta) \to (T(M), S_\theta)$ *is a pseudoharmonic map if and only if* $\nabla^H X = 0$. *Let us assume additionally that* $X \in \mathscr{U}(M,\theta)$ *and let*

$\mathscr{X} : M \times (-\delta, \delta) \to T(M)$ be a smooth 1-parameter variation of X through unit vector fields ($\mathscr{X}(x, 0) = X(x)$, $x \in M$) and let us set $V = (\partial X_t/\partial t)_{t=0}$ where $X_t(x) = \mathscr{X}(x, t)$, $x \in M$, $|t| < \delta$. Then iii) *$g_\theta(V, X) = 0$ and*

$$\frac{d}{dt}\{E(X_t)\}_{t=0} = -\int_M g_\theta(V, \Delta_b X)\,\Psi. \tag{55}$$

Consequently iv) *a C^∞ unit vector field X on M is a pseudoharmonic vector field if and only if X is a C^∞ solution to* (50).

Statement (ii) extends a result by T. Ishihara [**31**], and O. Nouhaud [**38**], to the subelliptic case.

PROOF OF THEOREM 6. Let $x \in M$ and $\{E_a : 1 \le a \le 2n\}$ be a local orthonormal (with respect to G_θ) frame of $H(M)$, defined on an open neighborhood of x. If $E_a = \lambda_a^j\,\partial/\partial\tilde{x}^j$ then

$$(d_x X)E_{a,x} = \lambda_a^j(x)\{\delta_j + [(\nabla_j X^i)\circ\pi]\dot{\partial}_i\}_{X(x)}$$

where δ_j are given by (52) and $\nabla_j X^i = \partial X^i/\partial\tilde{x}^j + \Gamma^i_{jk}X^k$. Then

$$\begin{aligned}
(\mathrm{trace}_{G_\theta}\pi_H X^* S_\theta)_x &= \sum_{a=1}^{2n}(X^*S_\theta)(E_a, E_a)_x \\
&= \sum_a S_{\theta, X(x)}((d_x X)E_{a,x},\ (d_x X)E_{a,x}) \\
&= \sum_a \lambda_a^j(x)\lambda_a^k(x)\{S_\theta(\delta_j,\ \delta_k) + (\nabla_j X^r)(\nabla_k X^s)S_\theta(\dot{\partial}_r,\ \dot{\partial}_s)\}_{X(x)} \\
&= \sum_a \{g_\theta(E_a, E_a) + \lambda_a^j\lambda_a^k(\nabla_j X^r)(\nabla_k X^s)g_{rs}\}_x.
\end{aligned}$$

Let $T = T^i\,\partial/\partial\tilde{x}^i$. As $\sum_{a=1}^{2n}\lambda_a^i\lambda_a^j = g^{ij} - T^iT^j$ it follows that

$$\mathrm{trace}_{G_\theta}(\pi_H X^* S_\theta) = 2n + \|\nabla X\|^2 - \|\nabla_T X\|^2$$

where $\|\nabla X\|^2 = g^{ij}(\nabla_i X^k)(\nabla_j X^\ell)g_{k\ell}$. As $\{E_j : 0 \le j \le 2n\}$ (with $E_0 = T$) is a local g_θ-orthonormal frame of $T(M)$ one also has $\|\nabla X\|^2 = \sum_{j=0}^{2n} g_\theta(\nabla_{E_j}X,\ \nabla_{E_j}X)$ hence (54). Clearly (54) yields statement (i) in Theorem 6.

Let us prove (ii). If $\nabla^H X = 0$ then X is a pseudoharmonic map and an absolute minimum for E in $\Gamma^\infty(M, T(M))$. Viceversa let us assume X is a pseudoharmonic map of M into the Riemannian manifold $(T(M), S_\theta)$. Thus $\{dE(X_t)/dt\}_{t=0} = 0$ for any smooth 1-parameter variation $X_t : M \to T(M)$ of X $(X_0 = X)$. In particular for the variation $X_t(x) = (1-t)X_x$, $x \in M$, $|t| < \epsilon$ (by (54))

$$0 = \frac{dE(X_t)}{dt}\bigg|_{t=0} = \frac{d}{dt}\{n\,\mathrm{Vol}(M,\theta) + \mathscr{B}(X_t)\}_{t=0}$$
$$= \frac{d}{dt}\left\{\frac{(1-t)^2}{2}\int_M \|\nabla^H X\|^2 \Psi\right\}_{t=0} = -\int_M \|\nabla^H X\|^2 \Psi.$$

Let $X \in \mathscr{U}(M,\theta)$. To prove the first variation formula (55) we need some preparation. Let $N = M \times (-\delta, \delta)$ and let $p : N \to M$ be the projection. Let $p^{-1}TM \to N$ be the pullback of the tangent bundle $T(M) \to M$ by p. Then $\mathscr{X}$ may be thought of as a C^∞ section in $p^{-1}TM \to N$. If Y is a tangent vector field on M we set $\hat{Y} = Y \circ p$. The Webster metric g_θ induces a bundle metric $\hat{g}_\theta$ in $p^{-1}TM \to N$ uniquely determined by $\hat{g}_\theta(\hat{Y}, \hat{Z}) = g_\theta(Y,Z) \circ p$. Also let D be the connection in $p^{-1}TM \to N$ induced by the Tanaka-Webster connection ∇. Precisely let $\tilde{Y}$ be the tangent vector field on $T(M)$ given by

$$\tilde{Y}_{(x,t)} = (d_x i_t) Y_x, \quad x \in M, \quad |t| < \delta,$$

where $i_t : M \to N$, $i_t(x) = (x,t)$. Then D is determined by

$$D_{\tilde{Y}} \hat{Z} = \widehat{\nabla_Y Z}\,, \quad D_{\partial/\partial t} \hat{Z} = 0, \quad Y, Z \in T(M).$$

Moreover a simple calculation shows that $D\hat{g}_\theta = 0$ and

$$(D_{\tilde{Y}} \mathscr{X})_{(x,t)} = (\nabla_Y X_t)_x, \quad (x,t) \in N.$$

Then

$$\mathscr{B}(X_t) = \frac{1}{2}\int_M \sum_{a=1}^{2n} \hat{g}_\theta(\nabla_{E_a} X_t,\, \nabla_{E_a} X_t)_x\ \Psi(x)$$
$$= \frac{1}{2}\int_M \sum_a \hat{g}_\theta(D_{\tilde{E}_a} \mathscr{X},\, D_{\tilde{E}_a} \mathscr{X})_{(x,t)}\ \Psi(x)$$

hence

$$\begin{aligned}\frac{d}{dt}\mathscr{B}(X_t) &= \int_M \sum_a \hat{g}_\theta(D_{\partial/\partial t} D_{\tilde{E}_a}\mathscr{X},\, D_{\tilde{E}_a}\mathscr{X})_{(x,t)}\ \Psi(x)\\ &= \int_M \sum_a \hat{g}_\theta(D_{\tilde{E}_a} D_{\partial/\partial t}\mathscr{X},\, D_{\tilde{E}_a}\mathscr{X})_{(x,t)}\ \Psi(x)\end{aligned}$$

as $R^D(\partial/\partial t, \tilde{E}_a)\mathscr{X} = 0$ and $[\partial/\partial t, \tilde{E}_a] = 0$. Moreover (by $D\hat{g}_\theta = 0$)

$$\begin{aligned}\frac{d}{dt}\mathscr{B}(X_t) = \int_M \sum_a \{&\tilde{E}_a(\hat{g}_\theta(D_{\partial/\partial t}\mathscr{X},\, D_{\tilde{E}_a}\mathscr{X}))\\ &- \hat{g}_\theta(D_{\partial/\partial t}\mathscr{X},\, D_{\tilde{E}_a} D_{\tilde{E}_a}\mathscr{X})\}_{(x,t)}\ \Psi(x).\end{aligned}$$

For each fixed $|t| < \delta$ we define $Y_t \in H(M)$ by setting

$$G_\theta(Y_t, Y)_x = \hat{g}_\theta(D_{\partial/\partial t}\mathscr{X},\, D_{\tilde{Y}}\mathscr{X})_{(x,t)}$$

for any $Y \in H(M)$ and any $x \in M$. Then (by $\nabla g_\theta = 0$)

$$\begin{aligned}\tilde{E}_a(\hat{g}_\theta(D_{\partial/\partial t}\mathscr{X},\, D_{\tilde{E}_a}\mathscr{X})) &= E_a(g_\theta(Y_t, E_a))\\ &= g_\theta(\nabla_{E_a} Y_t,\ E_a) + g_\theta(Y_t, \nabla_{E_a} E_a).\end{aligned}$$

As $\nabla\Psi = 0$ the divergence operator (see Section 2) is also given by

$$\operatorname{div}(Y) = \operatorname{trace}\{Z \mapsto \nabla_Z Y\} = \sum_{j=0}^{2n} g_\theta(\nabla_{E_j} Y,\ E_j).$$

Finally (by Green's lemma)

$$\begin{aligned}&\frac{d}{dt}\{E(X_t)\}_{t=0}\\ &\quad = -\int_M \hat{g}_\theta\left(D_{\partial/\partial t}\mathscr{X},\, \sum_a \{D_{\tilde{E}_a} D_{\tilde{E}_a}\mathscr{X} - D_{\widetilde{\nabla_{E_a} E_a}}\mathscr{X}\}\right)_{(x,0)} \Psi(x)\\ &\quad = -\int_M g_\theta(V, \Delta_b X)_x \Psi(x)\end{aligned}$$

and (55) is proved. If X is a critical point i.e. $\{dE(X_t)/dt\}_{t=0} = 0$ then (55) together with the constraint $g_\theta(V, X) = 0$ (obtained by differentiating $g_\theta(X_t, X_t) = 1$ at $t = 0$) imply that $\Delta_b X = \lambda X$ for some $\lambda \in C^\infty(M)$ and taking the inner product with X shows that $\lambda = g_\theta(\Delta_b X, X) = -\|\nabla^H X\|^2$. Theorem 6 is proved. $\square$

6.4. Unboundedness of the energy functional.

Under the assumptions of Theorem 6 we may prove the following

COROLLARY 2. *The characteristic direction T of $d\theta$ is a pseudoharmonic vector field and an absolute minimum of the energy functional $E : \mathscr{U}(M,\theta) \to [0,+\infty)$. Moreover, for any nonempty open subset $\Omega \subseteq M$ and any unit vector field X on M such that $X \in H(M)$ there is a sequence $\{Y_\nu\}_{\nu\geq 1}$ of unit vector fields such that each Y_ν coincides with X outside Ω and $E(Y_\nu) \to \infty$ for $\nu \to \infty$. In particular the energy functional E is unbounded from above.*

PROOF. The first statement in Corollary 2 follows from $\nabla T = 0$ (and then $E(T) = \inf_{X\in\mathscr{U}(M,\theta)} E(X) = n\mathrm{Vol}(M,\theta)$). To prove the second statement let $h = (x^1,\ldots,x^m) : U \to \boldsymbol{R}^m$ be a local coordinate system on M such that $U \subseteq \Omega$, $h(U) \supset [-2\pi, 2\pi]^m$ and $X = \partial/\partial x^1$ on U (cf. the proof of the classical Frobenius theorem, e.g. [**41**, pp. 91–92]). Moreover let $\varphi \in C^\infty_0(M)$ be a test function such that i) $0 \leq \varphi(x) \leq 1$ for any $x \in M$, ii) $\varphi = 1$ in a neighborhood V of the compact set $K = h^{-1}([-\pi,\pi]^m)$ such that $\overline{V} \subset U$, and iii) $\varphi = 0$ outside $h^{-1}([-2\pi, 2\pi]^m)$. For each $\nu \in \boldsymbol{Z}$, $\nu \geq 1$, let f_ν be the C^∞ extension to M of the function $\sin(\nu x^1)$ (thought of as defined on the closed set $\overline{V}$) and let us set $\alpha_\nu = \varphi f_\nu$. Next let us consider the C^∞ vector field

$$Y_\nu = (\cos\alpha_\nu)X + (\sin\alpha_\nu)T, \quad \nu \geq 1.$$

Then Y_ν is a unit vector field coinciding with X outside Ω. As we may complete X to a local frame of $H(M)$ (and $\nabla T = 0$, $\theta(\nabla_X X) = 0$)

$$\|\nabla^H Y_\nu\|^2 \geq g_\theta(\nabla_X Y_\nu, \nabla_X Y_\nu) = X(\alpha_\nu)^2 + (\cos^2\alpha_\nu)\|\nabla_X X\|^2 \geq X(\alpha_\nu)^2.$$

On the other hand $X(\alpha_\nu) = X(\varphi)f_\nu + \varphi\nu(\cos\nu x^1)$ on U so that $X(\alpha_\nu) = \nu\cos\nu x^1$ on $V \supset K$. Hence

$$2E(Y_\nu) \geq \int_K \|\nabla^H Y_\nu\|^2\,\Psi \geq \int_K X(\alpha_\nu)^2\,\Psi = \nu^2\int_K \cos^2(\nu x^1)\,\Psi.$$

If $d\,\mathrm{vol}(g_\theta) = \sqrt{G(x)}\,dx^1 \wedge \cdots \wedge dx^m$ is the Riemannian volume form of (M, g_θ) (with $G(x) = \det[g_{ij}(x)]$) there is a constant $c_n > 0$ such that $\Psi = c_n\,d\,\mathrm{vol}(g_\theta)$ (and $c_n = 2^n n!$, cf. [**48**]). Let us set $a = \inf_{x\in K}\sqrt{G(x)}$. Then $a > 0$ and

$$\int_K \cos^2(\nu x^1)\,\Psi \geq ac_n \int_{[-\pi,\pi]^m} \cos^2(\nu t^1)\,dt^1\cdots dt^m = a2^{n-1}n!(2\pi)^m$$

Hence $E(Y_\nu) \geq a2^{n-2}n!(2\pi)^m\nu \to \infty$ for $\nu \to \infty$. □

6.5. The second variation formula.

Let $X \in \mathscr{U}(M,\theta)$ and let us consider a smooth 2-parameter variation of X

$$\mathscr{Y} : M \times I_\delta^2 \to T(M), \quad I_\delta = (-\delta,\delta), \quad \delta > 0,$$
$$X_{t,s} = \mathscr{Y} \circ i_{t,s}, \quad t,s \in I_\delta, \quad X_{0,0} = X.$$

Here we set $N = M \times I_\delta^2$ and $i_{t,s} : M \to N$, $i_{t,s}(x) = (x,t,s)$ for any $x \in M$. We shall prove the following

Theorem 7. *Let* $V = (\partial X_{t,s}/\partial t)_{t=s=0}$ *and* $W = (\partial X_{t,s}/\partial s)_{t=s=0}$. *Let us assume that* $X_{t,s} \in \mathscr{U}(M,\theta)$ *for any* $t,s \in I_\delta$. *If* $X \in \mathscr{U}(M,\theta)$ *is a smooth pseudoharmonic vector field then*

$$\frac{\partial^2}{\partial t\partial s}\left\{\mathscr{B}(X_{t,s})\right\}_{t=s=0} = -\int_M g_\theta(V,\, \Delta_b W + \|\nabla^H X\|^2 W)\Psi. \tag{56}$$

In particular for any smooth 1*-parameter variation of* X

$$\frac{d^2}{dt^2}\left\{\mathscr{B}(X_t)\right\}_{t=0} = \int_M \{\|\nabla^H V\|^2 - \|\nabla^H X\|^2\|V\|^2\}\,\Psi. \tag{57}$$

The identity (56) is the *second variation formula* (of the pseudohermitian biegung). To prove Theorem 7 let $p : N \to M$ be the projection and $p^{-1}TM \to N$ the pullback of $T(M)$ by p. Then $\mathscr{Y}$ is a C^∞ section in $\pi^{-1}TM \to N$. Let $\hat{g}_\theta$ and D be respectively the Riemannian bundle metric induced by g_θ and the connection induced by the Tanaka-Webster connection ∇ in $\pi^{-1}TM \to N$. Similar to the conventions adopted in the proof of Theorem 6 we set

$$\tilde{Y}_{(x,t,s)} = (d_x i_{t,s})Y_x, \quad x \in M, \quad t,s \in I_\delta.$$

For simplicity we set $\boldsymbol{T} = \partial/\partial t$ and $\boldsymbol{S} = \partial/\partial s$ $(\boldsymbol{T},\boldsymbol{S} \in \mathscr{X}^\infty(N))$. Then (by $D\hat{g}_\theta = 0$)

$$\begin{aligned}\frac{\partial}{\partial t}\mathscr{B}(X_{t,s}) &= \int_M \sum_{a=1}^{2n} \hat{g}_\theta(D_{\boldsymbol{T}} D_{\tilde{E}_a}\mathscr{Y},\, D_{\tilde{E}_a}\mathscr{Y})\Psi \\ &= \int_M \sum_a \hat{g}_\theta(D_{\tilde{E}_a} D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a}\mathscr{Y})\end{aligned}$$

due to

$$[\boldsymbol{T},\ \tilde{E}_a] = 0, \quad R^D(\boldsymbol{T},\ \tilde{E}_a)\mathscr{Y} = 0.$$

Then

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}\mathscr{B}(X_{t,s}) &= \int_M \sum_a \frac{\partial}{\partial s}\hat{g}_\theta(D_{\tilde{E}_a} D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a}\mathscr{Y})\Psi \\ &= \int_M \sum_a \{\hat{g}_\theta(D_{\boldsymbol{S}} D_{\tilde{E}_a} D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a}\mathscr{Y}) + \hat{g}_\theta(D_{\tilde{E}_a} D_{\boldsymbol{T}}\mathscr{Y},\, D_{\boldsymbol{S}} D_{\tilde{E}_a}\mathscr{Y})\}\,\Psi \\ &\quad (\text{as } [\boldsymbol{S}, \tilde{E}_a] = 0 \text{ and } R^D(\boldsymbol{S}, \tilde{E}_a)\mathscr{Y} = 0) \\ &= \int_M \sum_a \{\hat{g}_\theta(D_{\tilde{E}_a} D_{\boldsymbol{S}} D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a}\mathscr{Y}) + \hat{g}_\theta(D_{\tilde{E}_a} D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a} D_{\boldsymbol{S}}\mathscr{Y})\}\,\Psi \\ &= \int_M \sum_a \{\tilde{E}_a(\hat{g}_\theta(D_{\boldsymbol{S}} D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a}\mathscr{Y})) - \hat{g}_\theta(D_{\boldsymbol{S}} D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a} D_{\tilde{E}_a}\mathscr{Y}) \\ &\quad + \tilde{E}_a(\hat{g}_\theta(D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a} D_{\boldsymbol{S}}\mathscr{Y})) - \hat{g}_\theta(D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a} D_{\tilde{E}_a} D_{\boldsymbol{S}}\mathscr{Y})\}\Psi. \qquad (58)\end{aligned}$$

For each fixed $(t, s) \in I_\delta^2$ we define $Y_{t,s} \in H(M)$ by

$$G_\theta(Y_{t,s},\, Z)_x = \hat{g}_\theta(D_{\boldsymbol{S}} D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{Z}}\mathscr{Y})_{(x,t,s)}, \quad Z \in H(M).$$

Then

$$\begin{aligned}\sum_a \tilde{E}_a(\hat{g}_\theta(D_{\boldsymbol{S}} D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a}\mathscr{Y})) &= \sum_a E_a(g_\theta(Y_{t,s},\, E_a)) \circ p \\ &= \sum_a \{g_\theta(\nabla_{E_a} Y_{t,s},\ E_a) + g_\theta(Y_{t,s},\, \nabla_{E_a} E_a)\} \circ p \\ &= \operatorname{div}(Y_{t,s}) \circ p + \hat{g}_\theta(D_{\boldsymbol{S}} D_{\boldsymbol{T}}\mathscr{Y},\, \sum_a D_{\widetilde{\nabla_{E_a} E_a}}\mathscr{Y}).\end{aligned}$$

Similarly, given $Z_{t,s} \in H(M)$ determined by

$$G_\theta(Z_{t,s},\, Z)_x = \hat{g}_\theta(D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{Z}} D_{\boldsymbol{S}}\mathscr{Y})_{(x,t,s)}$$

one has

$$\begin{aligned}
&\sum_a \tilde{E}_a(\hat{g}_\theta(D_{\boldsymbol{T}}\mathscr{Y},\, D_{\tilde{E}_a} D_{\mathscr{S}}\mathscr{Y})) \\
&\quad = \operatorname{div}(Z_{t,s}) \circ p + \hat{g}_\theta\Big(D_{\boldsymbol{T}}\mathscr{Y},\, \sum_a D_{\widetilde{\nabla_{E_a} E_a}} D_{\boldsymbol{S}}\mathscr{Y}\Big).
\end{aligned}$$

Going back to (58) one has (by Green's lemma)

$$\begin{aligned}
&\frac{\partial^2}{\partial s \partial t}\{\mathscr{B}(X_{t,s})\}_{t=s=0} \\
&\quad = \int_M \Big\{\hat{g}_\theta\Big(D_{\boldsymbol{S}} D_{\boldsymbol{T}}\mathscr{Y},\, \sum_a \{D_{\widetilde{\nabla_{E_a} E_a}}\mathscr{Y} - D_{\tilde{E}_a} D_{\tilde{E}_a}\mathscr{Y}\}\Big) \\
&\qquad + \hat{g}_\theta\Big(D_{\boldsymbol{T}}\mathscr{Y},\, \sum_a \{D_{\widetilde{\nabla_{E_a} E_a}} D_{\boldsymbol{S}}\mathscr{Y} - D_{\tilde{E}_a} D_{\tilde{E}_a} D_{\boldsymbol{S}}\mathscr{Y}\}\Big)\Big\}_{t=s=0} \Psi \\
&\quad = -\int_M \{g_\theta(U,\, \Delta_b X) + g_\theta(V,\, \Delta_b W)\}\Psi
\end{aligned}$$

where we have set $U = (\partial^2 X_{t,s}/\partial t \partial s)_{t=s=0}$. Moreover (by differentiating $\hat{g}_\theta(\mathscr{Y}, \mathscr{Y}) = 1$)

$$\begin{aligned}
g_\theta(U, X)_x &= \hat{g}_\theta(D_{\boldsymbol{S}} D_{\boldsymbol{T}}\mathscr{Y},\, \mathscr{Y})_{(x,0,0)} \\
&= \{\boldsymbol{S}(\hat{g}_\theta(D_{\boldsymbol{T}}\mathscr{Y},\, \mathscr{Y})) - \hat{g}_\theta(D_{\boldsymbol{T}}\mathscr{Y},\, D_{\boldsymbol{S}}\mathscr{Y})\}_{(x,0,0)} = -g_\theta(V, W)_x
\end{aligned}$$

and (as X is pseudoharmonic i.e. a smooth solution to (50))

$$\begin{aligned}
\frac{\partial^2}{\partial t \partial s}\{\mathscr{B}(X_{t,s})\}_{t=s=0} &= \int_M \{\|\nabla^H X\|^2 g_\theta(U, X) - g_\theta(V, \Delta_b W)\}\Psi \\
&= -\int_M g_\theta(V,\, \Delta_b W + \|\nabla^H X\|^2 W)\Psi
\end{aligned}$$

and (56) is proved. Finally given an arbitrary smooth 1-parameter variation $\mathscr{X} : M \times I_\delta \to T(M)$ of X through unit vector fields the identity (57) follows from (56) for the particular 2-parameter variation $\mathscr{Y} : M \times I_{\delta/2} \to T(M)$ given by $\mathscr{Y}(x, t, s) = \mathscr{X}(x, t + s)$ for any $x \in M$ and any $t, s \in I_{\delta/2}$. Indeed

$$\frac{d^2}{dt^2}\{\mathscr{B}(X_t)\}_{t=0} = -\int_M g_\theta(V,\, \Delta_b V + \|\nabla^H X\|^2 V)\Psi. \tag{59}$$

On the other hand, for any smooth vector field V on M

$$\begin{aligned}
\Delta_b \|V\|^2 &= \sum_{a=1}^{2n}\{E_a E_a \|V\|^2 - (\nabla_{E_a} E_a)\|V\|^2\} \\
&= 2\sum_a \{E_a(g_\theta(\nabla_{E_a} V,\, V)) - g_\theta(\nabla_{\nabla_{E_a} E_a} V,\, V)\} \\
&= 2\sum_a \{g_\theta(\nabla_{E_a}\nabla_{E_a} V,\, V) + g_\theta(\nabla_{E_a} V,\, \nabla_{E_a} V) - g_\theta(\nabla_{\nabla_{E_a} E_a} V,\, V)\}
\end{aligned}$$

hence

$$\Delta_b \|V\|^2 = 2\{g_\theta(\Delta_b V,\, V) + \|\nabla^H V\|^2\}. \tag{60}$$

Now (57) follows from (59)–(60) and Green's lemma.

References

[1] E. Barletta, On the boundary behavior of the holomorphic sectional curvature of the Bergman metric, Le Matematiche (Catania), **61** (2006), 301–316.

[2] E. Barletta, On the Dirichlet problem for the harmonic vector fields equation, Nonlinear Anal., **67** (2007), 1831–1846.

[3] E. Barletta, S. Dragomir and H. Urakawa, Pseudoharmonic maps from a nondegenerate *CR* manifold into a Riemannian manifold, Indiana Univ. Math. J., **50** (2001), 719–746.

[4] E. Barletta, S. Dragomir and H. Urakawa, Yang-Mills fields on *CR* manifolds, J. Math. Phys., **47** (2006), 1–41.

[5] E. Barletta, S. Dragomir and K. L. Duggal, Foliations in Cauchy-Riemann geometry, Mathematical Surveys and Monographs, **140**, American Mathematical Society, Providence, RI, 2007.

[6] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, **509**, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[7] J. M. Bony, Principe du maximum, inégalité de Harnak et unicité du problème de Cauchy pour les opérateurs elliptiques dégénéré, Ann. Inst. Fourier (Grenoble), **19** (1969), 277–304.

[8] F. G. B. Brito, Total bending of flows with mean curvature correction, Differential Geom. Appl., **12** (2000), 157–163.

[9] S. Campanato, Equazioni ellittiche del secondo ordine e spazi $\mathscr{L}^{2,\lambda}$, Ann. Mat. Pura Appl., **69** (1965), 321–381.

[10] B.-Y. Chen, Geometry of submanifolds and its applications, Science University of Tokyo, 1981.

[11] S. Dragomir, Pseudohermitian immersions between strictly pseudoconvex *CR* manifolds, Amer. J. Math., **117** (1995), 169–202.

[12] S. Dragomir and S. Nishikawa, Foliated *CR* manifolds, J. Math. Soc. Japan, **56** (2004), 1031–1068.

[13] S. Dragomir and D. Perrone, On the geometry of tangent hyperquadric bundles: *CR* and pseudoharmonic vector fields, Ann. Global Anal. Geom., **30** (2006), 211–238.

[14] S. Dragomir and G. Tomassini, Differential geometry and analysis on *CR* manifolds, Progress in Mathematics, (eds. by H. Bass, J. Oesterlé and A. Weinstein), **246**, Birkhäuser, Boston-Basel-Berlin, 2006.

[15] C. Fefferman, The Bergman kernel and biholomorphic equivalence of pseudoconvex domains, Invent. Math., **26** (1974), 1–65.

[16] G. B. Folland, A fundamental solution for a subelliptic operator, Bull. Amer. Math. Soc., **79** (1973), 373–376.

[17] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems, Ann. of Math. Stud., **105**, Princeton Univ. Press, New Jersey, 1983.

[18] O. Gil-Medrano, J. C. González-Dávila and L. Vanhecke, Harmonicity and minimality of oriented distributions, Israel J. Math., **143** (2004), 253–279.

[19] O. Gil-Medrano and A. Hurtado, Spacelike energy of timelike unit vector fields on a Lorentzian manifold, J. Geom. Phys., **51** (2004), 82–100.

[20] C. R. Graham, The Dirichlet problem for the Bergman Laplacian. I-II, Comm. Partial Differential Equations, **8** (1983), 433–476; ibidem, **8** (1983), 563–641.

[21] C. R. Graham, On Sparling's characterization of Fefferman metrics, Amer. J. Math., **109** (1987), 853–874.

[22] C. R. Graham and J. M. Lee, Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains, Duke Math. J., **57** (1988), 697–720.

[23] D.-S. Han and J.-W. Yim, Unit vector fields on spheres, which are harmonic maps, Math. Z., **227** (1998), 83–92.

[24] K. Hasegawa, Harmonic sections of normal bundles for submanifolds and their stability, J. Geom., **83** (2005), 57–64.

[25] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Pure and Applied Mathematics, **80**, Academic Press, Inc., New York-London-Toronto-Sydney-San Francisco, 1978.

[26] A. Higuchi, B. S. Kay and C. M. Wood, The energy of unit vector fields on the 3-sphere, J. Geom. Phys., **37** (2001), 137–155.

[27] S. Hildebrand, H. Kaul and K. Widman, An existence theorem for harmonic mappings of Riemannian manifolds, Acta. Math., **138** (1977), 1–16.

[28] S. Hildebrand and K. Widman, On the Hölder continuity of weak solutions of quasilinear elliptic systems of second order, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **4** (1977), 145–178.

[29] L. Hörmander, Hypoelliptic second-order differential equations, Acta Math., **119** (1967), 147–171.

[30] S.-T. Hu, Homotopy theory, Pure and Applied Mathematics, **VIII**, Academic Press, New York-San Francisco-London, 1959.

[31] T. Ishihara, Harmonic sections in tangent bundles, J. Math. Tokushima Univ., **13** (1979), 23–27.

[32] J. Jost and C.-J. Xu, Subelliptic harmonic maps, Trans. Amer. Math. Soc., **350** (1998), 4633–4649.

[33] J. J. Konderak, On sections of fibre bundles which are harmonic maps, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **42** (1999), 341–352.

[34] A. Korányi and H. M. Reimann, Contact transformations as limits of symplectomorphisms, C. R. Acad. Sci. Paris Sér. I Math., **318** (1994), 1119–1124.

[35] J. M. Lee and R. Melrose, Boundary behaviour of the complex Monge-Ampère equation, Acta Math., **148** (1982), 159–192.

[36] J. M. Lee, The Fefferman metric and pseudohermitian invariants, Trans. Amer. Math. Soc., **296** (1986), 411–429.

[37] J. Nash, The embedding problem for Riemannian manifolds, Ann. of Math. (2), **63** (1956), 20–63.

[38] O. Nouhaud, Applications harmoniques d'une variété Riemannienne dans son fibré tangent, C. R. Acad. Sci. Paris Sér. A–B, **284** (1977), A815–A818.

[39] C. Oniciuc, Harmonic sections in the unitary tangent bundle, Demonstratio Math., **34** (2001), 681–692.

[40] R. Osserman and A. Weinstein (Editors), Geometry of the Laplace operator, Proceedings of Symposia in Pure Mathematics, **36**, American Mathematical Society, Providence, Rhode Island, 1980.

[41] D. I. Papuc, Geometrie diferenţială, Editura Didactică şi Pedagogică, Bucureşti, 1982.

[42] D. Perrone, Harmonic characteristic vector fields on contact metric three-manifolds, Bull. Austral. Math. Soc., **67** (2003), 305–315.

[43] D. Perrone, Contact metric manifolds whose characteristic vector field is a harmonic vector field, Differential Geom. Appl., **20** (2004), 367–378.

[44] D. Perrone, The rough Laplacian and harmonicity of Hopf vector fields, Ann. of Global Anal. Geom., **28** (2005), 91–106.

[45] R. Petit, Harmonic maps and strictly pseudoconvex *CR* manifolds, Comm. Anal. Geom., **10** (2002), 575–610.

[46] B. Solomon, Harmonic maps to spheres, J. Differential Geom., **21** (1985), 151–162.

[47] N. Tanaka, A differential geometric study on strongly pseudo-convex manifolds, Lectures in Mathematics, Department of Mathematics, Kyoto University, Kinokuniya Book Store Co., Ltd., Kyoto, 1975.

[48] H. Urakawa, Yang-Mills connections over compact strongly pseudo-convex *CR* manifolds, Math. Z., **216** (1994), 541–573.

[49] H. Urakawa, Calculus of variations and harmonic maps, Translations of Mathematical Monographs, **132**, Amer. Math. Soc., Providence, Rhode Island, 1993.

[50] S. M. Webster, Pseudohermitian structures on a real hypersurface, J. Differential Geom., **13** (1978), 25–41.

[51] G. Wiegmink, Total bending of vector fields on Riemannian manifolds, Math. Ann., **303** (1995), 325–344.

[52] C. M. Wood, On the energy of a unit vector field, Geom. Dedicata, **64** (1997), 319–330.

[53] C.-J. Xu, Subelliptic variational problems, Bull. Soc. Math. France, **118** (1990), 147–169.

[54] A. Yampolsky, A totally geodesic property of Hopf vector fields, Acta Math. Hungar., **101** (2003), 93–112.

Sorin Dragomir
Università della Basilicata
Dipartimento di Matematica
Contrada Macchia Romana
85100 Potenza, Italy
E-mail: dragomir@unibas.it

Yoshinobu Kamishima
Tokyo Metropolitan University
Department of Mathematics
Minami-Ohsawa, 1-1, Hachioji
Tokyo 192-0397, Japan
E-mail: kami@tmu.ac.jp,
kami@comp.metro-u.ac.jp