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PSEUDOPARALLEL ANTI-INVARIANT SUBMANIFOLDS OF KENMOTSU MANIFOLDS

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Abstract

We consider an anti-invariant, minimal, pseudoparallel and Ricci-generalized pseudoparallel submanifold M of a Kenmotsu space form $\widetilde{M}(c)$, for which ξ is tangent to M.

Keywords: Kenmotsu space form, Anti-invariant submanifold, Pseudoparallel submanifold, Ricci-generalized pseudoparallel submanifold.

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1. Introduction

An n-dimensional submanifold M in an m-dimensional Riemannian manifold $\widetilde{M}$ is *pseudoparallel* [1], if its second fundamental form σ satisfies the following condition

(1.1) $$\overline{R} \cdot \sigma = L_\sigma Q(g, \sigma).$$

Pseudoparallel submanifolds in space forms were studied by A. C. Asperti, G. A. Lobos and F. Mercuri (see [1] and [2]). Also, R. Deszcz, L. Verstraelen and Ş. Yaprak [6] obtained some results on pseudoparallel hypersurfaces in a 4-dimensional space form $N^4(c)$. Moreover, C-totally real pseudoparallel submanifolds of Sasakian space forms were studied by A.Yıldız, C. Murathan, K. Arslan and R. Ezentaş in [12].

On the other hand, in [9], C. Murathan, K. Arslan and R. Ezentaş defined submanifolds satisfying the condition

(1.2) $$\overline{R} \cdot \sigma = L_S Q(S, \sigma).$$

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This kind of submanifold is called *Ricci-generalized pseudoparallel*. In [13], A. Yıldız and C. Murathan studied pseudoparallel and Ricci-generalized pseudoparallel invariant submanifolds of Sasakian space forms. In [10], the present authors considered pseudoparallel and Ricci-generalized pseudoparallel invariant submanifolds of contact metric manifolds.

In the present study, we consider pseudoparallel and Ricci-generalized pseudoparallel, anti-invariant, minimal submanifolds of Kenmotsu space forms. We find a necessary condition for the submanifold to be totally geodesic.

2. Preliminaries

Let $f : M^n \longrightarrow \widetilde{M}^{n+d}$ be an isometric immersion of an n-dimensional Riemannian manifold M into an $(n+d)$-dimensional Riemannian manifold $\widetilde{M}$. We denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M and $\widetilde{M}$, respectively. Then we have the Gauss and Weingarten formulas

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{2.1}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{2.2}$$

where $\nabla^{\perp}$ denotes the normal connection on $T^{\perp}M$ of M, and A_N is the shape operator of M, for $X, Y \in \chi(M)$ and a normal vector field N on M. We call σ the *second fundamental form* of the submanifold M. If $\sigma = 0$ then the submanifold is said to be *totally geodesic*. The second fundamental form σ and A_N are related by

$$g(A_N X, Y) = \widetilde{g}(\sigma(X,Y), N),$$

where g is the induced metric of $\widetilde{g}$ for any vector fields X, Y tangent to M. The mean curvature vector H of M is given by

$$H = \frac{1}{n} Tr(\sigma).$$

The first derivative $\overline{\nabla}\sigma$ of the second fundamental form σ is given by

$$(\overline{\nabla}_X \sigma)(Y,Z) = \nabla_X^{\perp} \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{2.3}$$

where $\overline{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M [4]. If $\overline{\nabla}\sigma = 0$, then f is said to be a *parallel immersion*.

The second covariant derivative $\overline{\nabla}^2\sigma$ of the second fundamental form σ is given by

$$\begin{aligned}(\overline{\nabla}^2\sigma)(Z,W,X,Y) &= (\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z,W) \\ &= \nabla_X^{\perp}((\overline{\nabla}_Y \sigma)(Z,W) - (\overline{\nabla}_Y \sigma)(\nabla_X Z, W) \\ &\qquad - (\overline{\nabla}_X \sigma)(Z, \nabla_Y W) - (\overline{\nabla}_{\nabla_X Y} \sigma)(Z,W).\end{aligned} \tag{2.4}$$

Then we have

$$\begin{aligned}&(\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z,W) - (\overline{\nabla}_Y \overline{\nabla}_X \sigma)(Z,W) \\ &\quad = (\overline{R}(X,Y) \cdot \sigma)(Z,W) \\ &\quad = R^{\perp}(X,Y)\sigma(Z,W) - \sigma(R(X,Y)Z, W) - \sigma(Z, R(X,Y)W),\end{aligned} \tag{2.5}$$

where $\overline{R}$ is the curvature tensor belonging to the connection $\overline{\nabla}$, and

$$R^{\perp}(X,Y) = \left[\nabla^{\perp} X, \nabla^{\perp} Y\right] - \nabla^{\perp}_{[X,Y]},$$

(see [4]).

Now for a $(0,k)$-tensor field T, $k \geq 1$, and a $(0,2)$-tensor field A on (M,g), we define $Q(A,T)$ (see [5]) by

$$Q(A,T)(X_1,\ldots,X_k;X,Y) = -T((X\wedge_A Y)X_1, X_2,\ldots,X_k) - \cdots \\ \cdots - T(X_1,\ldots,X_{k-1},(X\wedge_A Y)X_k), \tag{2.6}$$

where $X \wedge_A Y$ is an endomorphism defined by

$$(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{2.7}$$

Substituting $T=\sigma$ and $A=g$ or $A=S$ in formula (2.6), we obtain $Q(g,\sigma)$ and $Q(S,\sigma)$, respectively. In case $A=g$ we write $X\wedge_g Y = X\wedge Y$ for short.

3. Submanifolds of Kenmotsu manifolds

Let $\widetilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with structure (φ,ξ,η,g), where φ is a tensor field of type $(1,1)$, ξ a vector field, η a 1-form and g the Riemannian metric on $\widetilde{M}$ satisfying

$$\varphi^2 = -I + \eta\otimes\xi, \quad \varphi\xi = 0, \quad \eta(\xi)=1, \quad \eta\circ\varphi = 0,$$
$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$\eta(X) = g(X,\xi), \quad g(\varphi X, Y) = -g(X,\varphi Y),$$

for all vector fields X,Y on $\widetilde{M}$ [3]. An almost contact metric manifold $\widetilde{M}$ is said to be a *Kenmotsu manifold* [7] if the relation

$$(\widetilde{\nabla}_X\varphi)Y = g(\varphi X,Y)\xi - \eta(Y)\varphi X \tag{3.1}$$

holds on $\widetilde{M}$, where $\widetilde{\nabla}$ is the Levi-Civita connection of g. From the above equation, for a Kenmotsu manifold we also have

$$\widetilde{\nabla}_X\xi = X - \eta(X)\xi. \tag{3.2}$$

Moreover, the curvature tensor $\widetilde{R}$ and the Ricci tensor $\widetilde{S}$ of $\widetilde{M}$ satisfy [7]

$$\widetilde{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.3}$$

$$\widetilde{S}(X,\xi) = -2n\eta(X). \tag{3.4}$$

A Kenmotsu manifold is normal (that is, the Nijenhuis tensor of φ equals $-2d\eta\otimes\xi$), but not Sasakian. Moreover, it is also not compact since from the equation (3.2) we get $\mathrm{div}\xi = 2n$. In [7], K. Kenmotsu showed:

(1) That locally a Kenmotsu manifold is a warped product $I\times_f N$ of an interval I and a Kaehler manifold N, with warping function $f(t) = ce^t$, where c is a nonzero constant; and

(2) That a Kenmotsu manifold of constant sectional curvature is a space of constant curvature -1, and so it is locally hyperbolic space.

A plane section in the tangent space $T_x\widetilde{M}$ at $x\in\widetilde{M}$ is called a *φ-section* if it is spanned by a vector X orthogonal to ξ and φX. The sectional curvature $K(X,\varphi X)$ with respect to a φ-section, denoted by the vector X, is called a *φ-sectional curvature*. A Kenmotsu manifold with constant holomorphic φ-sectional curvature c is a *Kenmotsu space form*, and is denoted by $\widetilde{M}(c)$, The *curvature tensor* of a Kenmotsu space form is given by

$$\widetilde{R}(X,Y)Z = \frac{1}{4}(c-3)\{g(Y,Z)X - g(X,Z)Y\} + \frac{1}{4}(c+1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\}. \tag{3.5}$$

Let M be a $(m+1)$-dimensional submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $\widetilde{M}$, with ξ tangent to M. Then we have from Gauss' formula

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X,\xi),$$

which implies from (3.2) that

$$\nabla_X \xi = X - \eta(X)\xi \text{ and } \sigma(X,\xi) = 0, \tag{3.6}$$

for each vector field X tangent to M (see [8]). It is also easy to see that for a submanifold M of a Kenmotsu manifold $\widetilde{M}$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.7}$$

for any vector fields X and Y tangent to M. From the equation (3.7) we get

$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{3.8}$$

for a submanifold M of a Kenmotsu manifold $\widetilde{M}$. Moreover, the Ricci tensor S of M satisfies

$$S(X,\xi) = -m\eta(X). \tag{3.9}$$

We proved the following theorems in [11]:

3.1. Theorem. [11] *Let M be a $(m+1)$-dimensional submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $\widetilde{M}$, with ξ tangent to M. If M is pseudoparallel such that $L_\sigma \neq -1$, then it is totally geodesic.*

3.2. Theorem. [11] *Let M be a $(m+1)$-dimensional submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $\widetilde{M}$, with ξ tangent to M. If M is Ricci-generalized pseudoparallel such that $L_S \neq \frac{1}{m}$, then it is totally geodesic.*

The technique used in the proofs of Theorem 3.1 and Theorem 3.2 is not sufficient to interpret the cases $L_\sigma = -1$ and $L_S = \frac{1}{m}$. These cases are open. For this reason, we give solutions of these cases in Section 4, for anti-invariant, minimal submanifolds of a Kenmotsu space form.

4. Anti-invariant Submanifolds of Kenmotsu Space Forms

Let M be an $(n+1)$-dimensional submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $\widetilde{M}$. A submanifold M of a Kenmotsu manifold $\widetilde{M}$ is called *anti-invariant* if and only if $\varphi(T_xM) \subset T_x^\perp M$ for all $x \in M$ (T_xM and $T_x^\perp M$ are the tangent space and normal space of M at x, respectively).

For an anti-invariant submanifold M of a Kenmotsu space form $\widetilde{M}(c)$, with ξ tangent to M, we have

$$R(X,Y)Z = \frac{1}{4}(c-3)\big\{g(Y,Z)X - g(X,Z)Y\big\} + \frac{1}{4}(c+1)\big\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi\big\} + A_{\sigma(Y,Z)}X - A_{\sigma(X,Z)}Y. \tag{4.1}$$

We denote by S and r the Ricci tensor and scalar curvature of M, respectively. Then we have

$$S(Y,Z) = \frac{1}{4}[n(c-3)-(c+1)]g(Y,Z) - \frac{1}{4}(n-1)(c+1)\eta(Y)\eta(Z) - \sum_i g(\sigma(Y,e_i),\sigma(Z,e_i)) \tag{4.2}$$

and

$$r = \frac{1}{4}[n^2(c-3)-n(c+5)] - \sum_{i,j} g(\sigma(e_i,e_j),\sigma(e_i,e_j)), \tag{4.3}$$

where $\{e_i\}$ is an orthonormal basis of M.

By an easy calculation, we have the following proposition:

4.1. Proposition. *Let M^{n+1} be an anti-invariant, minimal submanifold of a Kenmotsu space form $\widetilde{M}^{2n+1}(c)$. Then we have*

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \left\|\overline{\nabla}\sigma\right\|^2 + \left[\frac{(n+1)(c-3)}{4}\right]\|\sigma\|^2 - \sum_{\alpha,\beta=n+2}^{2n+1}\{[Tr(A_\alpha\circ A_\beta)]^2 + \|[A_\alpha,A_\beta]\|^2\}, \tag{4.4}$$

where $\{e_1,e_2,\dots,e_{n+1}\}$ is an orthonormal basis of M such that $e_{n+1}=\xi$. □

4.2. Theorem. *Let M^{n+1} be an anti-invariant, minimal submanifold of a Kenmotsu space form $\widetilde{M}^{2n+1}(c)$, with ξ tangent to M. If M^{n+1} is pseudoparallel and $\frac{(n+1)(c+1)}{4}\leq 0$ then it is totally geodesic.*

Proof. Suppose that M is an $(n+1)$-dimensional anti-invariant submanifold of the $(2n+1)$-dimensional Kenmotsu space form $\widetilde{M}^{2n+1}(c)$. We choose an orthonormal basis

$$\{e_1,e_2,\dots,e_n,\xi,\varphi e_1=e_1^*,\dots,\varphi e_n=e_n^*\}.$$

Then, for $1\leq i,j\leq n+1$, $n+2\leq\alpha\leq 2n+1$, the components of the second fundamental form σ are given by

$$\sigma_{ij}^\alpha = g(\sigma(e_i,e_j),e_\alpha). \tag{4.5}$$

Similarly, the components of the first and the second covariant derivative of σ are given by

$$\sigma_{ijk}^\alpha = g((\overline{\nabla}_{e_k}\sigma)(e_i,e_j),e_\alpha) = \overline{\nabla}_{e_k}\sigma_{ij}^\alpha \tag{4.6}$$

and

$$\begin{aligned}\sigma_{ijkl}^\alpha &= g((\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}\sigma)(e_i,e_j),e_\alpha)\\ &= \overline{\nabla}_{e_l}\sigma_{ijk}^\alpha\\ &= \overline{\nabla}_{e_l}\overline{\nabla}_{e_k}\sigma_{ij}^\alpha,\end{aligned} \tag{4.7}$$

respectively. Since M is pseudoparallel, then the condition

$$\overline{R}(e_l,e_k)\cdot\sigma = -[(e_l\wedge_g e_k)\cdot\sigma] \tag{4.8}$$

is fulfilled where

$$[(e_l\wedge_g e_k)\cdot\sigma](e_i,e_j) = -\sigma((e_l\wedge_g e_k)e_i,e_j) - \sigma(e_i,(e_l\wedge_g e_k)e_j) \tag{4.9}$$

for $1\leq i,j,k,l\leq n+1$.

Using (2.7) in (4.9), we obtain

$$\begin{aligned}[(e_l \wedge_g e_k)\cdot\sigma](e_i,e_j) = &-g(e_k,e_i)\sigma(e_l,e_j)+g(e_l,e_i)\sigma(e_k,e_j)\\ &-g(e_k,ej)\sigma(e_l,e_i)+g(e_l,e_j)\sigma(e_k,e_i).\end{aligned} \tag{4.10}$$

By virtue of (2.5) we have

$$(\overline{R}(e_l,e_k)\cdot\sigma)(e_i,e_j) = (\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}\sigma)(e_i,e_j) - (\overline{\nabla}_{e_k}\overline{\nabla}_{e_l}\sigma)(e_i,e_j). \tag{4.11}$$

Then using (4.5), (4.7), (4.10) and (4.11), the pseudoparallelity condition (4.8) reduces to

$$\sigma^{\alpha}_{ijkl} = \sigma^{\alpha}_{ijlk} + \{\delta_{ki}\sigma^{\alpha}_{ij} - \delta_{li}\sigma^{\alpha}_{kj} + \delta_{kj}\sigma^{\alpha}_{il} - \delta_{lj}\sigma^{\alpha}_{ki}\}, \tag{4.12}$$

where $g(e_i,e_j)=\delta_{ij}$ and $1\le i,j,k,l\le n+1$, $n+2\le\alpha\le 2n+1$.

The Laplacian $\Delta\sigma^{\alpha}_{ij}$ of σ^{α}_{ij} can be written as

$$\Delta\sigma^{\alpha}_{ij} = \sum_{i,j,k=1}^{n+1}\sigma^{\alpha}_{ijkk}.. \tag{4.13}$$

Then we get

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k,l=1}^{n+1}\sum_{\alpha=n+2}^{2n+1}\sigma^{\alpha}_{ij}\sigma^{\alpha}_{ijkl} + \left\|\overline{\nabla}\sigma\right\|^2, \tag{4.14}$$

where

$$\|\sigma\|^2 = \sum_{i,j,=1}^{n+1}\sum_{\alpha=n+2}^{2n+1}(\sigma^{\alpha}_{ij})^2 \tag{4.15}$$

and

$$\left\|\overline{\nabla}\sigma\right\|^2 = \sum_{i,j,k,l=1}^{n+1}\sum_{\alpha=n+2}^{2n+1}(\sigma^{\alpha}_{ijkl})^2 \tag{4.16}$$

are the square of the length of the second and the third fundamental forms of M, respectively. On the other hand, by the use of (4.5) and (4.7), we have

$$\begin{aligned}\sigma^{\alpha}_{ij}\sigma^{\alpha}_{ijkk} &= g(\sigma(e_i,e_j),e_\alpha)g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}\sigma)(e_i,e_j),e_\alpha)\\ &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}\sigma)(e_i,e_j)g(\sigma(e_i,e_j),e_\alpha),e_\alpha)\\ &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}\sigma)(e_i,e_j),\sigma(e_i,e_j)).\end{aligned} \tag{4.17}$$

On the other hand, by the use of (4.17), equation (4.14) turns into

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n+1} g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}\sigma)(e_i,e_j),\sigma(e_i,e_j)) + \left\|\overline{\nabla}\sigma\right\|^2. \tag{4.18}$$

Substituting (4.17) into (4.18), we have

$$\begin{aligned}\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n+1}&[g((\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}\sigma)(e_k,e_k),\sigma(e_i,e_j))\\ &+\{g(e_i,e_j)g(\sigma(e_k,e_k),\sigma(e_i,e_j)) - g(e_k,e_j)g(\sigma(e_k,e_i),\sigma(e_i,e_j))\\ &\quad + g(e_k,e_i)g(\sigma(e_j,e_k),\sigma(e_i,e_j)) - g(e_k,e_k)g(\sigma(e_i,e_j),\sigma(e_i,e_j))\}]\\ &\quad\quad + \left\|\overline{\nabla}\sigma\right\|^2.\end{aligned} \tag{4.19}$$

Furthermore, by the definitions

$$\|\sigma\|^2 = \sum_{i,j=1}^{n+1} g(\sigma(e_i, e_j), \sigma(e_i, e_j)), \tag{4.20}$$

$$H^\alpha = \sum_{k=1}^{n+1} \sigma_{kk}^\alpha, \tag{4.21}$$

$$\|H\|^2 = \frac{1}{(n+1)^2} \sum_{\alpha=n+2}^{2n+1} (H^\alpha)^2, \tag{4.22}$$

and after some calculations, we find

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j=1}^{n+1} \sum_{\alpha=n+2}^{2n+1} \sigma_{ij}^\alpha (\overline{\nabla}_{ei} \overline{\nabla}_{e_j} H^\alpha) - (n+1)\|\sigma\|^2 + \left\|\overline{\nabla}\sigma\right\|^2.$$

Then, by the use of the minimality condition, the last equation turns into

$$\frac{1}{2}\Delta(\|\sigma\|^2) = -(n+1)\|\sigma\|^2 + \left\|\overline{\nabla}\sigma\right\|^2. \tag{4.23}$$

Comparing the right hand sides of the equations (4.4) and (4.23), we get

$$\left(-(n+1) - \frac{(n+1)(c-3)}{4}\right)\|\sigma\|^2 + \sum_{\alpha,\beta=n+2}^{2n+1} \left\{\left[\mathrm{Tr}(A_\alpha \circ A_\beta)\right]^2 + \|[A_\alpha, A_\beta]\|^2\right\} = 0. \tag{4.24}$$

If $\frac{(n+1)(c+1)}{4} \leq 0$ then $\mathrm{Tr}(A_\alpha \circ A_\beta) = 0$. In particular, $\|A_\alpha\|^2 = \mathrm{Tr}(A_\alpha \circ A_\alpha) = 0$, thus $\sigma = 0$. This finishes the proof of the theorem. □

4.3. Theorem. *Let M^{n+1} be an anti-invariant, minimal submanifold of a Kenmotsu space form $\widetilde{M}^{2n+1}(c)$, with ξ tangent to M. If M^{n+1} is Ricci-generalized pseudoparallel and $\frac{r}{n} - \frac{(n+1)(c-3)}{4} \geq 0$, then it is totally geodesic.*

Proof. If M is Ricci-generalized pseudoparallel, then as in the proof of Theorem 4.2, for $1 \leq i, j \leq n+1$, $n+2 \leq \alpha \leq 2n+1$, we have

$$\begin{aligned}
\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n+1} &[g((\overline{\nabla}_{e_i} \overline{\nabla}_{e_j} \sigma)(e_k, e_k), \sigma(e_i, e_j)) \\
&- \frac{1}{n}\{S(e_i, e_j) g(\sigma(e_k, e_k), \sigma(e_i, e_j)) \\
&\quad - S(e_k, e_j) g(\sigma(e_k, e_i), \sigma(e_i, e_j)) \\
&\quad + S(e_k, e_i) g(\sigma(e_j, e_k), \sigma(e_i, e_j)) \\
&\quad - S(e_k, e_k) g(\sigma(e_i, e_j), \sigma(e_i, e_j))\}] + \left\|\overline{\nabla}\sigma\right\|^2.
\end{aligned} \tag{4.25}$$

Thus, by the use of (4.2), we get

$$\begin{aligned}
\sum_{i,j,k=1}^{n+1} &S(e_i, e_j) g(\sigma(e_k, e_k), \sigma(e_i, e_j)) \\
&= \sum_{i,j,k=1}^{n+1} \sum_{\alpha=n+2}^{2n+1} S(e_i, e_j) g(A_\alpha e_k, e_k) g(A_\alpha e_i, e_j) \\
&= \sum_{i,j,k=1}^{n+1} \sum_{\alpha=n+2}^{2n+1} S(e_i, e_j) \mathrm{Tr}(A_\alpha) g(A_\alpha e_i, e_j) = 0
\end{aligned} \tag{4.26}$$

and

$$
\begin{aligned}
(4.27)\qquad & \sum_{i,j,k=1}^{n+1} S(e_k,e_j)g(\sigma(e_k,e_i),\sigma(e_i,e_j)) \\
&= \sum_{i,j,k=1}^{n+1}\sum_{\alpha=n+2}^{2n+1} S(e_k,e_j)g(A_\alpha e_i,e_k)g(A_\alpha e_i,e_j) \\
&= \sum_{i,j,k=1}^{n+1}\sum_{\alpha=n+2}^{2n+1} S(e_k,e_j)g(A_\alpha e_k,e_i)g(A_\alpha e_j,e_i) \\
&= \sum_{i,j,k=1}^{n+1}\sum_{\alpha=n+2}^{2n+1} S(e_k,e_j)g(A_\alpha e_k,A_\alpha e_j) \\
&= \sum_{i,j,k=1}^{n+1}\sum_{\alpha=n+2}^{2n+1} \frac{1}{4}[n(c-3)-(c+1)]g(e_k,e_j)g(A_\alpha e_k,A_\alpha e_j) \\
&\qquad - \frac{1}{4}(n-1)(c+1)g(A_\alpha e_k,A_\alpha e_j) \\
&\qquad\qquad - \sum_{\alpha=n+2}^{2n+1} g(A_\alpha e_k,A_\alpha e_j)g(A_\alpha e_k,A_\alpha e_j).
\end{aligned}
$$

Moreover, using the equation (4.3), we have

$$(4.28)\qquad \sum_{i,j,k=1}^{n+1} S(e_k,e_k)g(\sigma(e_i,e_j),\sigma(e_i,e_j)) = r\,\|\sigma\|^2 .$$

Then, substituting equations (4.26) - (4.28) in (4.25), we obtain

$$(4.29)\qquad \frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n+1} g((\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}\sigma)(e_k,e_k),\sigma(e_i,e_j)) + \frac{r}{n}\,\|\sigma\|^2 + \left\|\overline{\nabla}\sigma\right\|^2 .$$

Putting $H^\alpha = \sum_{k=1}^{n+1}\sigma^\alpha_{kk}$, the equation (4.29) turns into

$$(4.30)\qquad \frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n+1}\sum_{\alpha=n+2}^{2n+1} \sigma^\alpha_{ij}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}H^\alpha) + \frac{r}{n}\,\|\sigma\|^2 + \left\|\overline{\nabla}\sigma\right\|^2 .$$

Furthermore, making use of the minimality condition, the equation (4.30) can be written as follows

$$(4.31)\qquad \frac{1}{2}\Delta(\|\sigma\|^2) = \frac{r}{n}\,\|\sigma\|^2 + \left\|\overline{\nabla}\sigma\right\|^2 .$$

Consequently, comparing the right hand sides of the equations (4.4) and (4.31), we get

$$\left(\frac{r}{n} - \frac{(n+1)(c-3)}{4}\right)\|\sigma\|^2 + \sum_{\alpha,\beta=n+2}^{2n+1}\left\{\left[\mathrm{Tr}(A_\alpha\circ A_\beta)\right]^2 + \left\|[A_\alpha,A_\beta]\right\|^2\right\} = 0.$$

If $\frac{r}{n} - \frac{(n+1)(c-3)}{4} \geq 0$ then $\mathrm{Tr}(A_\alpha\circ A_\beta) = 0$. In particular, $\|A_\alpha\|^2 = \mathrm{Tr}(A_\alpha\circ A_\alpha) = 0$, thus $\sigma = 0$. Therefore, our theorem is proved. □

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