PASJ: Publ. Astron. Soc. Japan **58**, 987–1013, 2006 December 25

Pulsar Magnetohydrodynamic Winds

Isao OKAMOTO
Division of Theoretical Astrophysics, National Astronomical Observatory, 2-21-1 Osawa, Mitaka, Tokyo 181-8588
Institute for Black Hole Mining, 114-6 Ochikawa, Hino, Tokyo 191-0034
iokamoto@m2.hinocatv.ne.jp
and
Friday B. SIGALO*
Division of Theoretical Astrophysics, National Astronomical Observatory, 2-21-1 Osawa, Mitaka, Tokyo 181-8588
fbsigalo@hotmail.com

(Received 2006 May 27; accepted 2006 September 26)

Abstract

The acceleration and collimation/decollimation of relativistic magnetocentrifugal winds are discussed concerning a cold plasma from a strongly magnetized, rapidly rotating neutron star in a steady axisymmetric state based on ideal magnetohydrodynamics. There exist unipolar inductors associated with the field line angular frequency, α, at the magnetospheric base surface, $S_{\rm B}$, with a huge potential difference between the poles and the equator, which drive electric current through the pulsar magnetosphere. Any "current line" must emanate from one terminal of the unipolar inductor and return to the other, converting the Poynting flux to the kinetic flux of the wind at finite distances. In a plausible field structure satisfying the transfield force-balance equation, the fast surface, $S_{\rm F}$, must exist somewhere between the subasymptotic and asymptotic domains, i.e., at the innermost point along each field line of the asymptotic domain of $\varpi_{\rm A}^2/\varpi^2 \ll 1$, where $\varpi_{\rm A}$ is the Alfvénic axial distance. The criticality condition at $S_{\rm F}$ yields the Lorentz factor, $\gamma_{\rm F} = \mu_\varepsilon^{1/3}$, and the angular momentum flux, β, as the eigenvalues in terms of the field line angular velocity, α, the mass flux per unit flux tube, η, and one of the Bernoulli integrals, μ_δ, which are assumed to be specifiable as the boundary conditions at $S_{\rm B}$. The other Bernoulli integral, μ_ε, is related to μ_δ as $\mu_\varepsilon = \mu_\delta[1 - (\alpha^2\varpi_{\rm A}^2/c^2)]^{-1}$, and both μ_ε and $\varpi_{\rm A}^2$ are eigenvalues to be determined by the criticality condition at $S_{\rm F}$. Ongoing MHD acceleration is possible in the superfast domain. This fact may be helpful in resolving a discrepancy between the wind theory and the Crab-nebula model. It is argued that the "anti-collimation theorem" holds for relativistic winds, based on the curvature of field streamlines determined by the transfield force balance. The "theorem" combines with the "current-closure condition" as a global condition in the wind zone to produce a two-component "quasi-conical" field structure as one of the basic properties of MHD outflows of centrifugal origin in the pulsar magnetosphere.

Key words: magnetohydrodynamics: MHD — stars: winds, acceleration, collimation, jets

1. Introduction

Shortly after the discovery of pulsars, parallel to pulsar electrodynamics pioneered by Goldreich and Julian (1969), the research of pulsar MHD winds was initiated by Michel (1969), who extended Weber and Davis's (1967) theory of the solar wind to relativistic centrifugal winds. Making use of a split-monopolar radial field structure, he investigated cold plasma outflows in a steady axisymmetric state, which yielded the Lorentz factor at infinity as $\gamma_\infty = \mu_\varepsilon^{1/3}$, although acceleration up to $\gamma_\infty = \mu_\varepsilon$ is in principle possible, where μ_ε is one of the Bernoulli integrals along each field line [see equation (44) later]. The remaining energy, corresponding to the difference $\mu_\varepsilon(1 - \mu_\varepsilon^{-2/3}) \approx \mu_\varepsilon$ if $\mu_\varepsilon \gg 1$, was regarded as still being carried by the Poynting flux. Goldreich and Julian (1970) showed that Michel's *minimum-torque* solution is nothing but the *critical* solution with the magnetosonic point at infinity. Their adoption of isothermal pressure revealed that thermal acceleration is possible when the magnetosonic points at finite distances, and if the sound speed tends to zero the magnetosonic point also tends to infinity in the radial field structure. To overcome any inefficiency in the acceleration of cold plasma along *radial* field lines, Kennel, Fujimura, and Okamoto (1983) adopted a polytropic finite-temperature wind, and showed that an arbitrary degree of acceleration is reached. This is because it turned out soon that the Crab-nebula observations indicated that the energy flux is kinetically dominated far from the source (Rees, Gunn 1974; Kennel, Coroniti 1984), but a plasma consisting of e^+–e^- pairs is not of so high temperature to ensure sufficient thermal acceleration.

The discrepancy between the wind model predicting "high σ_∞" and the nebula model showing "low σ_∞" has been referred to as a "long-standing puzzle" (see, e.g., Begelman 1998). This shows the necessity of some breakthrough either in wind theory or in nebula models. By being free of the spell of *radial* field structures, one can accomplish a breakthrough within the framework of a "conventional" picture of ideal MHD (see Okamoto 2002, 2003; cf. Chiueh et al. 1991; Begelman, Li 1994; Begelman 1998). It will for example be

* Present address: Department of Physics, Rivers State University of Science and Technology, Port Harcourt, Nigeria.

shown that one need not abandon the central tenet of the Rees–Gunn model (cf. Begelman 1998).

It has, on the other hand, generally been believed for almost the last two decades since Heyvaerts and Norman (1989) "established" that the MHD outflows have an intrinsic tendency of global collimation toward the rotation axis. The claim for this tendency is referred to as the "hoop-stress paradigm" (Okamoto 1999, hereafter Paper I). It seems that the paradigm was initiated by a self-similar analysis of MHD outflows by Blandford and Payne (1982), then by numerical calculations of MHD equations by Sakurai (1985, 1987), and finally "established" by analytic analyses by Heyvaerts and Norman (1989) in the nonrelativistic case and by Chiueh, Li, and Begelman (1991) in the relativistic case. The paradigm is summarized as follows: "Under some quite general assumptions, collimation of outflow to either parabolic or cylindrical surfaces is inevitable asymptotically" (Blandford 1993). It has already been shown (Paper I) that the "paradigm", i.e., the claim for global collimation of all open field lines, was not deduced from use of the most basic equation for the transfield force balance, which should determine the sign of curvature of each field line, $1/R = \partial\psi/\partial s$, that is, which direction the flow bends, toward the pole or the equator. What can correctly be derived by using the transfield component of the MHD equation of motion (i.e., transfield equation; also see Okamoto 1974) is not the "hoop-stress paradigm", but the "anti-collimation theorem" (Okamoto 2003). In spite of these critiques, the same statement for the global collimation was simply repeated in Heyvaerts and Norman (2003a, b, c), without refuting the "anti-collimation theorem" on a firm physical basis.

As easily conjectured, acceleration and collimation/decollimation are two sides of the coin (i.e., the field-streamline curvature). Extending Heyvaerts and Norman's asymptotic formalism, it can be shown, by becoming free from the spell of *radialness* (zero-curvature), that the fast surface, $S_{\rm F}$, must be situated at the innermost distances of the asymptotic domain of $\varpi^2 \gg \varpi_{\rm A}^2$. Denoting the innermost surface of the asymptotic domain rather vaguely by $S_{\rm a}$, then it can be seen that $S_{\rm F} \approx S_{\rm a}$, both in the nonrelativistic and relativistic winds. The criticality condition at $S_{\rm F}$ yields $\gamma_{\rm F} = \mu_\varepsilon^{1/3}$ for the Lorentz factor, which implies that the flow will be accelerated from the initial value of γ at the magnetospheric base surface, $S_{\rm B}$, to $\gamma_{\rm F} = \mu_\varepsilon^{1/3}$ in the subasymptotic domain of $S_{\rm B} \lesssim S \lesssim S_{\rm a}$. Then, ongoing interactions between the field and the flow take place in the asymptotic, superfast domain of $S_{\rm a} \approx S_{\rm F} \lesssim S \lesssim S_\infty$, to transfer from the Poynting flux to the kinetic flux, so that γ will increase from $\mu_\varepsilon^{1/3}$ at $S_{\rm F}$ to μ_ε at S_∞.

The key quantities, which play a crucial role in the acceleration-collimation/decollimation problem in the asymptotic domain, are the generalized Michel magnetization parameter, σ, and the curvature, $1/R$, because σ plays the role of an independent variable in the asymptotic domain [see equation (73)], and also the field-flow interactions can be ensured only through the inertial term containing $1/R$ in the transfield equation. However, small though it may be, the curvature must be finite, whether positive (collimate) or negative (decollimate), and tends to zero, together with σ, toward infinity along each field line for $s \to \infty$. There is no room for "field regions" with $1/R \approx 0$ in the asymptotic domain (cf. Heyvaerts, Norman 2003a, b, c). The absence of such "field regions" in the superfast domain can be easily shown (see subsection 7.1). The asymptotic, superfast domain must be such a domain that by ongoing interactions through the inertial curvature term with the field, the flow is continually accelerated fully up to $\gamma = \mu_\varepsilon$, followed by a gradual decrease of $1/R$ and σ, with $1/R > 0$ in the polar region (collimate) and $1/R < 0$ in the equatorial region (de-collimate), that is, "quasi-conical" in the sense of "$\varpi/R \to 0$ for $s \to \infty$" (Okamoto 2002, hereafter Paper IV).

It will be shown in this paper that the implausibility of MHD acceleration prevailing widely in the pulsar community is produced by the plausibility of *radialness* of the structure in a region far from the source, and it turns out by showing this that one need not abandon the central tenet of the Rees–Gunn model for the Crab nebula. Similarly, making use of the asymptotic form of the transfield equation, the "current-closure condition" requires the "anti-collimation theorem" to hold for the field-flow structure. In section 2, basic assumptions are introduced, together with clearly defined terminology, and the basic relations for pulsar magnetospheres are derived from ideal MHD in the steady, axisymmetric state. In section 3, several surfaces of crucial importance are defined, such as *critical* surfaces and the surfaces of equipartition of energy and angular momentum between the field and the flow. In section 4 are derived the quadratic equation for the axial distances, ϖ, and the transfield equation.

In section 5, the asymptotic domain is defined, and it is shown that it must be equivalent to the superfast domain — the fast surface, $S_{\rm F}$, must be situated at the innermost distances of the asymptotic domain — and that the equipartition surfaces are farther outside of $S_{\rm F}$. In section 6, the transfield equation in the asymptotic domain is combined with the equations for energy and angular momentum integrals, to clarify the acceleration-collimation/decollimation properties of MHD outflows in general. In section 7, the present results obtained in this paper are contrasted with the previous ones, and in the last section the conclusions are presented.

2. Basic Assumptions, Terminology, and Relations

2.1. Assumptions and Terminology

Making use of ideal MHD with perfect conductivity, we consider the pulsar magnetosphere in a steady axisymmetric state around a rapidly rotating, strongly magnetized neutron star, through which the pulsar wind carries the spin-down energy of the star to infinity. In this paper we confine ourselves to clarifying the basic properties of a magnetized wind carrying energy to infinity, without meeting with the surrounding media, and so constructing a nebula model, like the Crab nebula, is out of the scope of this paper. We assume that plenty of plasma particles are supplied at the magnetospheric base surface, $S_{\rm B}$, and so we do not consider the particle source problem, such as pair-creation processes at the polar cup or somewhere else.

We use the terminology of the "criticality condition" and the "regularity condition", distinguishing these from the "boundary condition" at the stellar or magnetospheric base surface $S_{\rm B}$ (see Okamoto 2006). The "criticality condition" requires the wind solution to pass smoothly through the "critical surfaces", such as the Alfvénic and fast surfaces, $S_{\rm A}$ and $S_{\rm F}$, and the "regularity condition" describes how the solution be "regular" toward the sphere-at-infinity S_∞, in particular on the behavior of the "key quantities" σ and $1/R$. We understand that the criticality problem poses a kind of eigenvalue problem for determining the Alfvénic distances, $\varpi_{\rm A}$, the angular momentum per unit flux tube, β, etc., by the "criticality condition" at $S_{\rm F}$ in terms of α, η, μ_δ, and the magnetic flux, which are given by the "boundary conditions" at $S_{\rm B}$, where α is the angular velocity of each field line, η the mass flux per unit flux tube, and μ_δ one of the Bernoulli integrals. Then, the "regularity condition" on σ and $1/R$ ensures that the flow properties at S_∞ be determined by the "criticality condition" at $S_{\rm F}$, without contradicting the causality principle. We use the words *physical* and *unphysical* for flow solutions in such a way that the *physical* solution behaves physically well all the way from the source to infinity, while the *unphysical* solution does not. Both solutions usually appear as a pair of functions that intersect each other at critical points, e.g., $S_{\rm F}$, like an X.

The fact that the angular frequency α be given at $S_{\rm B}$ as the "boundary condition" in ideal MHD semi-automatically means the existence of some kind of unipolar battery there, which drives the poloidal electric current in the pulsar magnetosphere, crossing field lines and thereby doing MHD work to the mean flow to transfer from the field energy to the flow energy [see Okamoto (2006) for the differences between $S_{\rm B}$ and the black hole horizon surface]. Then the "current-closure condition" must be imposed in the steady state, requiring no current line being snapped on the way from one terminal to the other of the neutron star unipolar battery (see figure 1). It turns out that this condition is closely connected to the acceleration/collimation-decollimation problem of the outflow in the superfast domain.

2.2. Geometrical Relations

Besides the usual cylindrical coordinates (ϖ, ϕ, z) and polar ones (r, θ, ϕ), we use local coordinates (P, s, ψ), where P labels the field lines, s measures the distances along a P-fixed field line, and ψ is the angle between the tangent to it at a given point and the ϖ-axis. Then, the poloidal field is expressed in terms of P by

$$\boldsymbol{B}_{\rm p} = -\frac{\boldsymbol{t}\times\nabla P}{\varpi}. \tag{1}$$

Together with the unit toroidal vector $\boldsymbol{t}$, we use the two other orthogonal unit vectors:

$$\boldsymbol{p} = \boldsymbol{B}_{\rm p}/B_{\rm p}, \quad \boldsymbol{n} = \boldsymbol{p}\times\boldsymbol{t} = -\nabla P/|\nabla P|. \tag{2}$$

Note that the direction of $\boldsymbol{n}$ is toward rather than away, from the axis. We denote the derivatives in the $\boldsymbol{p}$- and $\boldsymbol{n}$-directions as follows:

$$(\boldsymbol{p}\cdot\nabla) = \left(\frac{\partial}{\partial s}\right)_P, \tag{3}$$

$$(\boldsymbol{n}\cdot\nabla) = \left(\frac{\partial}{\partial n}\right)_s = -\frac{1}{|\nabla P|}\nabla P\cdot\nabla = -|\nabla P|\left(\frac{\partial}{\partial P}\right)_s. \tag{4}$$

The following geometrical relations turn out soon to be useful:

$$\left(\frac{\partial\varpi}{\partial s}\right)_P = \cos\psi, \quad \left(\frac{\partial z}{\partial s}\right)_P = \sin\psi. \tag{5}$$

The curvature radius along each field line becomes

$$\frac{1}{R} = \frac{\partial\psi}{\partial s} = \left(\frac{\partial\sin\psi}{\partial\varpi}\right)_P = -\left(\frac{\partial\cos\psi}{\partial z}\right)_P \tag{6}$$

$$= \frac{\nabla^2 P - \nabla P\cdot\nabla\ln\varpi\,|\nabla P|}{|\nabla P|} \tag{7}$$

$$= \frac{1}{B_{\rm p}\varpi}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P}{\partial r}\right) - \frac{1}{2\varpi^2}\frac{\partial}{\partial P}\left(\varpi\frac{\partial P}{\partial r}\right)^2\right] \tag{8}$$

(see appendix 1). The topology of field lines is described by $P(r,\theta) = $ constant. This implies that if $P = P(r,\theta)$ is given by some way or another, then R is calculated along each field line by using equation (7). If one expresses the right-hand side of (7) in spherical coordinates, one has equation (8). It can be seen that if $P = P(\theta)$, then $1/R = 0$, and hence the field structure is obviously radial. Conversely, if $R = R(r,\theta)$ is given, one can in principle solve equation (7) to obtain $P = P(r,\theta)$.

If $R = R(\varpi, P)$ is known, one can follow the change in ψ along a given field line with P by equation (6),

$$\sin\psi = \sin\psi_{\rm a} + \int_{\varpi_{\rm a}}^{\varpi}\frac{\varpi}{R}d\ln\varpi\Big|_P, \tag{9}$$

where $\psi_{\rm a}$ is the value of ψ at some reference level with $\varpi = \varpi_{\rm a}$, and $\int\,|_P$ denotes the integral along a given field line with P. In order that ψ converges to some angle $\infty|_P$ for $\varpi\to\infty|_P$, i.e.,

$$\sin\infty|_P = \sin\psi_{\rm a} + \int_{\varpi_{\rm a}}^{\infty}\frac{\varpi}{R}d\ln\varpi\Big|_P, \tag{10}$$

then ϖ/R must sufficiently rapidly reduce to null, i.e.,

$$\frac{\varpi}{R}\to 0 \quad \text{for}\ \varpi\to\infty, \tag{11}$$

similarly in the nonrelativistic case (see Paper I; Okamoto 2000, 2001, hereafter Paper II, Paper III), because if ϖ/R is, for example, constant or a function of P only, the above integral yields a logarithmic divergence for $|\sin\psi|$. Equation (11) will be referred to as the "regularity condition" at S_∞.

If $R = R(z, P)$ is known, one likewise obtains from equation (6)

$$\cos\psi = \cos\psi_{\rm a} - \int_{z_{\rm a}}^{z}\frac{z}{R}d\ln z\Big|_P. \tag{12}$$

If $R = R(\psi, P)$ is given, one has instead of equation (9) or (12)

$$\varpi = \varpi_{\rm a} + \int_{\psi_{\rm a}}^{\psi}R\cos\psi\,d\psi, \quad z = z_{\rm a} + \int_{\psi_{\rm a}}^{\psi}R\sin\psi\,d\psi. \tag{13}$$

Thus, to depict the field-line topology, it is indispensable to know the change of $R^{-1} = \partial\psi/\partial s$ from the source to infinity along each field-streamline. Without referring to $1/R$, one cannot discuss the collimation/decollimation of the MHD wind.

2.3. *Electromotive Force, Electric Current, and the Current-Closure Condition*

In the coordinate system comoving with the flow with velocity $\boldsymbol{v}$, the electric field vanishes, i.e., $\boldsymbol{E}' = [\boldsymbol{E} + (\boldsymbol{v}/c) \times \boldsymbol{B}]/\sqrt{1 - v^2/c^2} = 0$. The induction equation in the inertial frame then yields

$$\boldsymbol{E}_{\rm p} = -\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} = -\frac{\boldsymbol{v}_{\rm F}}{c} \times \boldsymbol{B} = \frac{\alpha\varpi}{c} B_{\rm p}\boldsymbol{n} = -\frac{\alpha}{c}\nabla P \tag{14}$$

and $\boldsymbol{E}_{\rm t} = 0$ in the steady axisymmetric state, where the relations $\boldsymbol{v} = \kappa\boldsymbol{B} + \boldsymbol{v}_{\rm F}$ and $\boldsymbol{v}_{\rm F} \equiv \alpha\varpi\boldsymbol{t}$ are used [see equations (37) and (40) later], together with equations (1) and (2). The angular velocity of field lines $\alpha(P)$ is usually assumed to be given by the angular velocity of the matter that field lines under consideration are frozen in. This presumes semi-automatically not only the magnetic slingshot effect related to the angular velocity of the field line, $\boldsymbol{v}_{\rm F}$, in the wind zone, but also the existence of a unipolar inductor on the source surface giving rise to an electromotive force (EMF) between any pair of field lines (say P_1 and P_2),

$$\text{EMF} = -\frac{1}{c}\int_{P_1}^{P_2} \alpha(P)\, dP \tag{15}$$

(see Okamoto 2006). Then, every current line must connect the EMF to the Joule-dissipating or MHD-accelerating domain continuously in the steady state.

Let us introduce the "current function", i.e.,

$$I(\varpi, z) = I(s, P) \equiv -\frac{c\varpi B_{\rm t}}{2}. \tag{16}$$

The poloidal electric current becomes in the axisymmetric state

$$\boldsymbol{j}_{\rm p} = -\frac{\boldsymbol{t} \times \nabla I}{2\pi\varpi} \tag{17}$$

[cf. equation (1) for $\boldsymbol{B}_{\rm p}$]. Then, the components perpendicular and parallel to $\boldsymbol{B}_{\rm p}$ become

$$j_\perp = \frac{(\boldsymbol{p}\cdot\nabla I)}{2\pi\varpi} = -\frac{1}{2\pi\varpi}\left(\frac{\partial I}{\partial s}\right)_P, \tag{18}$$

$$j_\parallel = \frac{(\boldsymbol{n}\cdot\nabla I)}{2\pi\varpi} = -\frac{B_{\rm p}}{2\pi}\left(\frac{\partial I}{\partial P}\right)_s. \tag{19}$$

The toroidal component of electric current, in passing, becomes in terms of $1/R$

$$j_{\rm t} = \frac{c}{4\pi}(\nabla \times \boldsymbol{B})_{\rm t} = \frac{cB_{\rm p}}{4\pi}\left(\frac{\partial \ln B_{\rm p}}{\partial n} - \frac{1}{R}\right) \tag{20}$$

[see equation (A4) in appendix 1 for derivation]. Note that $j_{\rm t} \neq 0$ in spite of $\boldsymbol{E}_{\rm t} \equiv 0$ by assuming axial symmetry.

The current function, $I(\varpi, z)$, denotes the total current passing downward through a loop with ϖ = constant. If $I = I_0 \neq 0$ at $\varpi = P = 0$, this means the existence of a *line* current along the axis. A comparison of equation (17) with (1) indicates that, just as lines of P = constant depict field lines, lines with I = constant depict the "current lines" along which the poloidal current flows (see, e.g., figure 1 in Paper III). The collimation-acceleration problem is crucially governed by topological features of the current lines crossing the field lines. There will probably be no large-scale region in a physically reasonable model where $I \propto -\varpi B_{\rm t}$ = constant, and hence $j_\perp = j_\parallel = 0$ (see later section 7).

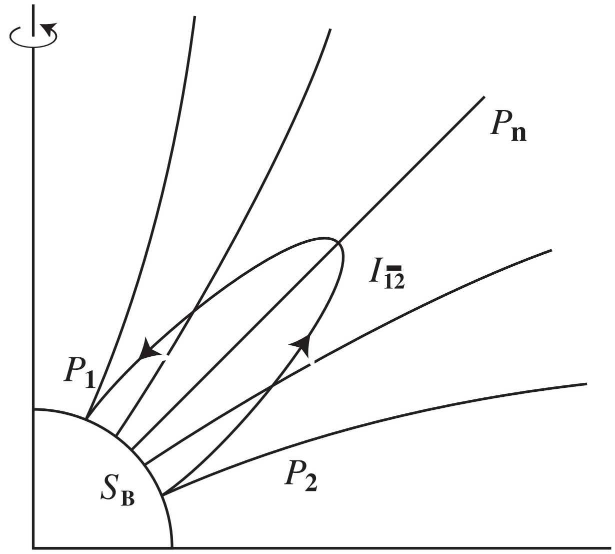

Fig. 1. Specifying the boundary condition for $\alpha = \alpha(P)$ indicates the presence of a kind of unipolar inductor, which a current line (say, $I_{\overline{12}}$) emanates from $P = P_2$ and comes back to $P = P_1$, dissipating electricity to accelerate the flow with $j_\perp > 0$. Note $j_\parallel < 0$ in $P_1 < P < P_{\rm n}$ and $j_\parallel > 0$ in $P_{\rm n} < P < P_2$. See equations (15), (19), and (42).

In the steady axisymmetric state, we assume that the charge inflow into the central source be equal to the outflow from it. Then, imposing the "current-closure condition" expressing charge-neutrality on an arbitrary s-surface (referred to as an s-surface), one has

$$0 \equiv \oint_s \boldsymbol{j}_{\rm p}\cdot d\boldsymbol{A} = -\int \frac{\partial I}{\partial P}\, dP\bigg|_s = I(s, 0) - I(s, \overline{P}), \tag{21}$$

where $d\boldsymbol{A} = 2\pi(dP/B_{\rm p})\boldsymbol{p}$ and $\overline{P}$ is the limiting field line of the wind zone [see equation (4.12) in Paper I]. Unless one introduces a line current at the axis with $P = 0$ and a surface or sheet current at $P = \overline{P}$, equation (21) reduces to

$$I(s, 0) = I(s, \overline{P}) = 0, \tag{22}$$

which indicates the existence of an extremum value of I (say $I_{\rm ex}$) at some field line (say "neutral" field line $P_{\rm n}$), where $j_\parallel \propto (\partial I/\partial P)_s = 0$ from equation (19), and hence $j_\parallel < 0$ in the range of field lines in $0 < P < P_{\rm n}$ and $j_\parallel > 0$ in the range of field lines in $P_{\rm n} < P < \overline{P}$ (see figure 2). These equations (21) and (22) later turn out to be helpful in clarifying the field structure in the asymptotic domain.

Let us now imagine one current line given by $I(s, P)$ = constant (say $I_{\overline{12}}$), which emanates at $P = P_2$ from one terminal of the pulsar unipolar battery on the magnetospheric base $S_{\rm B}$ and returns to the other terminal at $P = P_1$. Note here that one must choose a pair of field lines for P_1 and P_2 so as to satisfy $P_1 < P_{\rm n} < P_2$, and then define $I(s, P) = I(s_{\rm B}, P_2) = I(s_{\rm B}, P_1) = I_{\overline{12}}$ (see figure 1).

The charge density, i.e., Goldreich–Julian density becomes from equations (14) and (20)

$$\varrho_{\rm e} = -\frac{1}{4\pi c}\nabla\cdot(\alpha\nabla P)$$

$$= \frac{\alpha\varpi}{c^2}\left(j_{\rm t} + \frac{cB_{\rm p}}{4\pi}\frac{\partial \ln\alpha\varpi^2}{\partial n}\right) \tag{23}$$

$$= \frac{E_{\rm p}}{4\pi}\left(\frac{\partial}{\partial n}\ln\alpha\, B_{\rm p}\,\varpi^2 - \frac{1}{R}\right), \tag{24}$$

where $E_{\rm p} = (\alpha/c)|\nabla P| = (\alpha\varpi/c)\,B_{\rm p}$ and $j_{\rm t}$ is eliminated by using equation (20). It turns out that the total charge in a volume within the surface with constant s is null, i.e.,

$$0 \equiv \int \varrho_{\rm e}\, dV = -\frac{1}{4\pi c}\int \nabla\cdot(\alpha\nabla P)\, dV\bigg|_s$$

$$= -\frac{1}{4\pi c}\oint \alpha\nabla P\cdot d\boldsymbol{A}\bigg|_s, \tag{25}$$

because $-\nabla P\cdot d\boldsymbol{A} = 2\pi\varpi\, dP(\boldsymbol{n}\cdot\boldsymbol{p}) = 0$. We emphasize that the *local* charge conservation law, $\nabla\cdot\boldsymbol{j}_{\rm p} = 0$, is ensured by equation (17), but equation (22) as the current-closure condition is necessary as a *global* condition, although the total charge within the s-sphere always vanishes.

The curves of $I = I(P)$ with parameter s are shown in figure 2. In order that MHD acceleration takes place along each field line, $I(s, P)$ must be a decreasing function of s, i.e., $j_\perp \propto -(\partial I/\partial s) > 0$ by equation (18), and hence in the P–I diagram with parameter s, the curves of $I = I(s, P)$ go lower and lower with increasing s, and $I(s, P) \to 0$ for $s \to \infty$. We assume for the sake of simplicity that the value of $P_{\rm n}$ where $I(P)$ has an extremum (say $I_{\rm ex}$) does not depend on s.

For the Poynting flux one obtains from (14)

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{\alpha I}{2\pi c}\boldsymbol{B}_{\rm p} + \frac{B_{\rm p}^2}{4\pi}\alpha\,\varpi\,\boldsymbol{t}, \tag{26}$$

and then its divergence becomes from (18)

$$\nabla\cdot\boldsymbol{S} = -\boldsymbol{j}\cdot\boldsymbol{E} = -\frac{\alpha\,\varpi}{c}\,B_{\rm p}\, j_\perp = \frac{\alpha}{2\pi c}(\boldsymbol{B}_{\rm p}\cdot\nabla)\, I. \tag{27}$$

The total Poynting flux passing through an s-surface becomes

$$\oint_s \boldsymbol{S}\cdot d\boldsymbol{A} = \frac{2}{c}\int_0^{\overline{P}} \alpha I\, dP\bigg|_s. \tag{28}$$

If I is a function of P only in some finite domain, consequently with $j_\perp = 0$ but $j_\parallel \neq 0$ from equations (18) and (19), then the total Poynting flux remains constant there, i.e., $\nabla\cdot\boldsymbol{S} = 0$. If that domain extends to infinity, the nonvanishing Poynting flux reaches the "sphere-at-infinity" at $s = \infty$, without being transferred to the kinetic flux. This implies piling-up of electric charges on the "sphere-at-infinity", but because it should not be a garbage dump of charges, one must contrive an "artificial" surface current flowing from the equatorial side to the polar side on the force-free "sphere-at-infinity", endowing it with surface resistivity with $4\pi/c = 377$ ohm (Macdonald, Thorne 1982; Phinney 1983; Okamoto 1992, 2006). A more natural way to dispose this matter is to make the return current flow toward the axis, i.e., $j_\perp > 0$ at finite distances, so that the "current lines" can close crossing field-streamlines in the wind zone. Thus, one has

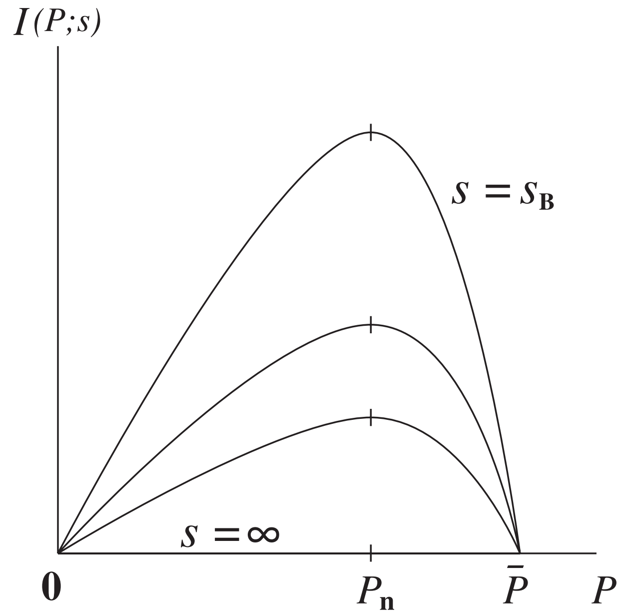

Fig. 2. Schematic picture of $I = I(P; s)$, which satisfies the "current-closure condition" as one of global conditions in the wind zone [see equations (21) and (22)]. For a fixed P, I decreases with increasing s, and $I \to 0$ for $s \to \infty$.

$$I(s, P) = -\frac{c\varpi\, B_{\rm t}}{2} \to 0, \quad \text{for } s \to \infty, \tag{29}$$

which means that the Poynting flux along each field line must tend to null for $s \to \infty$, with the maximum possible acceleration achieved [see equation (46) later]. It will be shown later [see, e.g., equation (195)] that $I \to 0$ in equation (29) is related to $\varpi/R \to 0$ in equation (11) by the transfield equation in the asymptotic domain.

2.4. *MHD Equation of Motion*

The relativistic MHD (RMHD) equation of motion for cold plasma outflow in the steady state reduces to

$$\varrho_{\rm e}\boldsymbol{E} + \frac{1}{c}\boldsymbol{j}\times\boldsymbol{B} = \rho(\boldsymbol{v}\cdot\nabla)\gamma\,\boldsymbol{v}$$

$$= \rho\left[c^2\nabla\gamma - \boldsymbol{v}\times(\nabla\times\gamma\,\boldsymbol{v})\right] \tag{30}$$

(Michel 1969; Gordreich, Julian 1970; Okamoto 1978).

The $\boldsymbol{p}$- and $\boldsymbol{t}$-components of equation (30) become

$$\rho\left(c^2\frac{\partial\gamma}{\partial s} - \frac{v_{\rm t}}{\varpi}\frac{\partial\gamma\,\varpi\, v_{\rm t}}{\partial s}\right) = -\frac{1}{c}\, j_\perp\, B_{\rm t}, \tag{31}$$

$$\rho\, v_{\rm p}\frac{\partial\varpi\,\gamma\, v_{\rm t}}{\partial s} = \frac{1}{c}\, j_\perp\, B_{\rm p}\,\varpi, \tag{32}$$

and the $\boldsymbol{n}$-component is given by

$$\frac{\rho\,\gamma\, v_{\rm p}^2}{R} - \rho\,\gamma\, v_{\rm t}^2\frac{\partial\ln\varpi}{\partial n} = \varrho_{\rm e}\, E_{\rm p} + \frac{1}{c}\left(j_\parallel\, B_{\rm t} - j_{\rm t}\, B_{\rm p}\right) \tag{33}$$

(see appendix 1 for derivation). It can be seen from equation (27) that the right-hand side of equation (32) is equal to $-\nabla\cdot\boldsymbol{S}/\alpha$. It thus turns out that the conversion of electromagnetic energy to kinetic energy is possible only if $\nabla\cdot\boldsymbol{S} < 0$, i.e., $j_\perp > 0$, which implies that $-\varpi\, B_{\rm t} \propto I$ decreases with s

increasing, as already indicated by equation (29). That is, the Lorentz volume force on the mean flow leads to MHD acceleration by $j_\perp > 0$ only.

The transfield component (33) of the equation of motion contains the extra terms of curvature radius R through ϱ_e and j_t. Eliminating $(1/c)\, j_t B_p$ and $\varrho_e E_p$ by using expressions obtained from equations (20) and (24), one reaches the following form for the force balance in the transfield direction:

$$\frac{\rho\gamma v_p^2}{R}\left[1-\frac{B_p^2}{4\pi\rho\gamma v_p^2}\left(1-\frac{\alpha^2\varpi^2}{c^2}\right)\right] = -\frac{\partial}{\partial n}\left[\frac{B_p^2}{8\pi}\left(1-\frac{\alpha^2\varpi^2}{c^2}\right)\right] + \left(\rho\gamma v_t^2+\frac{B_p^2}{4\pi}\frac{\alpha^2\varpi^2}{c^2}\right)\frac{\partial\ln\varpi}{\partial n}+\frac{1}{c}j_\parallel B_t. \tag{34}$$

Equation (34) is equivalent to Chiueh, Li, and Begelman's (1991) equation (11) and also Chiueh, Li, and Begelman's (1998) more understandable equation of the same number (11), but they did not utilize this most important equation explicitly in their discussion of collimation-acceleration (see later section 7).

By using equation (19), the following three terms of the right-hand side of equation (34) are written jointly:

$$\frac{\partial}{\partial n}\left(\frac{B_p^2}{8\pi}\frac{\alpha^2\varpi^2}{c^2}\right)+\frac{B_p^2}{4\pi}\frac{\alpha^2\varpi^2}{c^2}\frac{\partial\ln\varpi}{\partial n}+\frac{1}{c}j_\parallel B_t = -\frac{1}{8\pi\varpi^2}\frac{\partial}{\partial n}\left[(\varpi B_t)^2-\left(\frac{\alpha B_p\varpi^2}{c}\right)^2\right]. \tag{35}$$

Then, substitution of equations (16) and (35) into (34) yields

$$\frac{\rho\gamma v_p^2}{R}\left[1-\frac{B_p^2}{4\pi\rho\gamma v_p^2}\left(1-\frac{\alpha^2\varpi^2}{c^2}\right)\right] = -\frac{\partial}{\partial n}\left(\frac{B_p^2}{8\pi}\right)+\rho\gamma v_t^2\frac{\partial\ln\varpi}{\partial n} - \frac{1}{2\pi\varpi^2c^2}\frac{\partial}{\partial n}\left[I^2-\left(\frac{\alpha B_p\varpi^2}{2}\right)^2\right]. \tag{36}$$

One can use equation (4) to show $\partial\varpi/\partial n = -B_z/B_p$, and then the second term of equation (36) is negative for $B_z > 0$. Thus, in a domain far from S_A, the centrifugal force due to v_t^2 contributes negatively to the curvature of the field lines, i.e., toward the equator.

2.5. *Integral Relations and the Alfvénic Surface*

There holds a kinematical relation in the steady axisymmetric state,

$$\boldsymbol{v}_t = \kappa\boldsymbol{B}_p, \tag{37}$$

which combines with the Poisson equation to yield

$$\rho\kappa = \eta(P). \tag{38}$$

The angular-momentum integral becomes, from equation (32) with the elimination of $j_\perp$ from equation (18),

$$\frac{I}{2\pi c}+\eta\varpi\gamma v_t = -\frac{\beta}{4\pi}. \tag{39}$$

Integration of what is usually called the induction equation yields Ferraro's isorotation law of $\alpha = \alpha(P)$, which leads in the wind zone to

$$v_t = \kappa B_t + \alpha\varpi. \tag{40}$$

It is worth while reiterating that specifying $\alpha = \alpha(P)$ on S_B indicates the existence of such an unipolar inductor there that gives rise to the EMF shown in equation (15). Equations (39) and (40) combine to yield

$$\varpi v_t = \frac{\alpha\varpi^2+\kappa\beta}{1-4\pi\rho\gamma\kappa^2}, \quad I = \frac{-c\beta/2-2\pi\eta\gamma\alpha\varpi^2 c}{1-4\pi\rho\gamma\kappa^2} \tag{41}$$

[also see equations (2.9a, b) and (2.8) in Okamoto 1978; Mestel 1968]. From equations (31), (32), and (40) one obtains

$$\frac{\partial\gamma}{\partial s} = \frac{\alpha\varpi}{c}\frac{j_\perp B_p}{\rho c^2 v_p} = \frac{\boldsymbol{j}\cdot\boldsymbol{E}}{\rho c^2 v_p} = -\frac{\nabla\cdot\boldsymbol{S}}{\rho c^2 v_p}, \tag{42}$$

which combines with equation (18) to yield

$$\gamma\left(1-\frac{\alpha\varpi v_t}{c^2}\right) = \mu_\delta \tag{43}$$

or by using equation (39)

$$\gamma+\frac{\alpha I}{2\pi\eta c^3} = \mu_\varepsilon, \tag{44}$$

where

$$\mu_\varepsilon = \mu_\delta - \frac{\alpha\beta}{4\pi\eta c^2}. \tag{45}$$

Equations (43) and (44) indicate that $1 < \mu_\delta < \gamma \le \mu_\varepsilon$, and in particular the upper limit of γ is given by μ_ε, when $I \to 0$, that is, when the outward Poynting flux tends to zero for $\varpi \to \infty$ [see equations (26) and (29)], i.e.,

$$\gamma_\infty = \mu_\varepsilon. \tag{46}$$

From equations (41) and (43) one obtains

$$\gamma = \mu_\delta\frac{1-4\pi\rho\gamma\kappa^2\left(1-\dfrac{\alpha\beta}{4\pi\eta\mu_\delta c^2}\right)}{1-4\pi\rho\gamma\kappa^2-\dfrac{\alpha^2\varpi^2}{c^2}}, \tag{47}$$

$$\varpi v_t = \frac{\alpha\varpi^2+4\pi\rho\gamma\kappa^2\dfrac{\beta}{4\pi\eta\mu_\delta}}{1-4\pi\rho\gamma\kappa^2\left(1-\dfrac{\alpha\beta}{4\pi\eta\mu_\delta c^2}\right)}, \tag{48}$$

$$I = -\frac{c\beta}{2}\frac{1-\dfrac{\alpha^2\varpi^2}{c^2}\left(1-\dfrac{4\pi\eta\mu_\delta c^2}{\alpha\beta}\right)}{1-4\pi\rho\gamma\kappa^2-\dfrac{\alpha^2\varpi^2}{c^2}}. \tag{49}$$

Imposing the criticality condition at the Alfvénic surface S_A with $\varpi = \varpi_A$, one obtains from equations (47) and (45)

$$-\beta = 4\pi\eta\mu_\varepsilon\alpha\varpi_A^2, \quad \left(\frac{\rho}{\gamma}\right)_A = \frac{4\pi\eta^2}{1-\dfrac{\alpha^2\varpi_A^2}{c^2}}, \tag{50}$$

$$\mu_\varepsilon = \frac{\mu_\delta}{1-\dfrac{\alpha^2\varpi_A^2}{c^2}} \tag{51}$$

[see equation (A34) in appendix 4 for the nonrelativistic limits of μ_δ and μ_ε]. Then, it can be seen that $(\varpi\, v_t)_\infty = \alpha\,\varpi_A^2$ by equations (43) and (51).

When α, η, and μ_δ are given *externally* by the boundary condition at S_B, one of ϖ_A^2, β, and μ_ε must be sought *internally* as the eigenvalue problem by the criticality condition at the fast surface, S_F (see Okamoto 2006), where the other two values are related by relations (45), (50), and (51) (see subsubsection 5.2.2).

2.6. *Variable ζ and Related Quantities*

If one defines the Alfvénic Mach number as

$$M^2 = 4\pi\rho\gamma\kappa^2 = 4\pi\eta^2\frac{\gamma}{\rho}, \tag{52}$$

then its value at S_A, i.e., M_A, becomes

$$M_A^2 = \left(1-\frac{\alpha^2\varpi_A^2}{c^2}\right). \tag{53}$$

Then, similarly in the nonrelativistic case, it is convenient to introduce the following functions using equation (51):

$$\Pi^2 = M_A^2\frac{\rho\varpi^2}{4\pi\eta^2\gamma} = M_A^2\frac{\Phi}{4\pi\eta\, c\, u_p}, \tag{54}$$

$$\zeta \equiv 4\pi\eta\,\mu_\varepsilon\alpha\,\Pi^2 = \frac{\mu_\delta\alpha}{\eta}\frac{\rho\varpi^2}{\gamma} = \frac{\mu_\delta\alpha}{c}\frac{\Phi}{u_p}, \tag{55}$$

$$M^2 = M_A^2\frac{\zeta_A}{\zeta}\frac{\varpi^2}{\varpi_A^2}, \tag{56}$$

where

$$\Phi = B_p\varpi^2 = \varpi\,|\nabla P| \tag{57}$$

and $u_p = \gamma\, v_p/c$ is the poloidal component of the four velocity vector $\boldsymbol{u} = \gamma\boldsymbol{v}/c$. Note that both Π and ζ may be *slowly* decreasing functions of $\varpi|_P$. In the nonrelativistic limit, by equation (A34), ζ and Π^2 reduce to equations (3.1) and (3.2) in Paper I. At S_A from equations (50) and (51)

$$\Pi_A = \varpi_A, \quad \zeta_A = 4\pi\eta\mu_\varepsilon\alpha\,\varpi_A^2 = -\beta. \tag{58}$$

The relation $\zeta_A = -\beta$ indicates that ζ has the same dimension as β, and it is shown later in section 5 that $\zeta \approx -\varpi\, B_t = 2I/c$ in the asymptotic domain, that is, it gives the electric current distribution as well as the angular momentum flux there.

The denominator of γ and I is expressed as follows:

$$1-4\pi\rho\gamma\kappa^2 - \frac{\alpha^2\varpi^2}{c^2} = -\frac{\zeta_A}{\zeta}\frac{\varpi^2}{\varpi_A^2}D, \tag{59}$$

$$D \equiv \left(1-\frac{\alpha^2\varpi_A^2}{c^2}\right) + \frac{\alpha^2\varpi_A^2}{c^2}\left(1-\frac{c^2}{\alpha^2\varpi^2}\right)\frac{\zeta}{\zeta_A}. \tag{60}$$

Note that $D(\varpi) \gtreqless 0$ for $\varpi \gtreqless \varpi_A$, $D(c/\alpha) = M_A^2$ and $D \to M_A^2$ for $\varpi|_P \to \infty$.

Expressing γ, $\varpi\, v_t$, u_t, and I in terms of variables ζ and ϖ, one obtains from equations (47)–(49)

$$\gamma = \frac{\mu_\delta}{D}\left(1-\frac{\varpi_A^2}{\varpi^2}\frac{\zeta}{\zeta_A}\right), \tag{61}$$

$$\varpi\, v_t = \alpha\,\varpi_A^2\frac{1-\dfrac{\zeta}{\zeta_A}}{1-\dfrac{\varpi_A^2}{\varpi^2}\dfrac{\zeta}{\zeta_A}}, \tag{62}$$

$$u_t = \mu_\delta\frac{\alpha\,\varpi_A^2}{c\varpi}\frac{1}{D}\left(1-\frac{\zeta}{\zeta_A}\right), \tag{63}$$

$$\left(\frac{I}{2\pi\eta c}\right) = \mu_\varepsilon\alpha\,\varpi_A^2\frac{\zeta}{\zeta_A}\left(1-\frac{\varpi_A^2}{\varpi^2}\right)\frac{1}{D}. \tag{64}$$

The specific field angular momentum is given by equation (64), while the flow angular momentum per unit mass becomes from equations (61), (63), and (51)

$$\gamma\,\varpi\, v_t = \mu_\varepsilon\alpha\,\varpi_A^2\left(1-\frac{\alpha^2\varpi_A^2}{c^2}\right)\left(1-\frac{\zeta}{\zeta_A}\right)\frac{1}{D}. \tag{65}$$

The total angular momentum per unit flux tube is, of course, equal to $\mu_\varepsilon\alpha\,\varpi_A^2 = -\beta/4\pi\eta$. From equations (61) and (63) one obtains the relation

$$\gamma - \frac{c\, u_t}{\alpha\,\varpi} = \mu_\delta\frac{2I}{c\zeta}. \tag{66}$$

2.7. *Generalized Michel's Magnetization Parameter σ*

At first we define the ratio of the Poynting flux to the kinetic energy flux in the poloidal direction from equations (26), (44), (61), and (64)[1] as

$$w = \frac{|\alpha\varpi\, B_t\, B_p|}{4\pi\gamma\rho\, c^2\, v_p} = \frac{\mu_\varepsilon}{\gamma} - 1$$
$$= \frac{\zeta}{(\zeta_B-\zeta_A)}\frac{\left(1-\dfrac{\varpi_A^2}{\varpi^2}\right)}{\left(1-\dfrac{\varpi_A^2}{\varpi^2}\dfrac{\zeta}{\zeta_A}\right)} = \frac{(2I/c\,\zeta_B)}{1-(2I/c\,\zeta_B)}, \tag{67}$$

where from equations (51) and (58)

$$\zeta_B \equiv \zeta_A + \frac{4\pi\eta\,\mu_\delta c^2}{\alpha} = \frac{4\pi\eta\,\mu_\varepsilon c^2}{\alpha}, \tag{68}$$

$$\frac{\zeta_A}{\zeta_B} = \frac{\alpha^2\varpi_A^2}{c^2} = 1-\frac{\mu_\delta}{\mu_\varepsilon} = 1-M_A^2 \tag{69}$$

[see equation (A36) for the nonrelativistic limits of ζ_A and ζ_B]. Then, from equations (54) and (55) one obtains

$$\frac{\zeta}{\zeta_B} = \frac{\alpha^2\Pi^2}{c^2}. \tag{70}$$

[1] Phinney (1983) and Begelman and Li (1994) denoted the ratio w_∞ at infinity with σ_∞ and the following generalized Michel parameter σ in equation (73) with a.

Similarly, one obtains for the ratio of the field angular momentum to the flow angular momentum from equations (64) and (65)

$$w_{\rm AM} = \frac{(I/2\pi\eta c)}{\gamma\,\varpi\,v_{\rm t}} = \frac{\mu_\varepsilon}{\gamma}\frac{\alpha\,\varpi_{\rm A}^2}{\varpi\,v_{\rm t}} - 1. \tag{71}$$

It is then obvious that $w_{\rm AM} > w$ at each point, since $\varpi\,v_{\rm t} < \alpha\,\varpi_{\rm A}^2$.

Denoting the mass flux per unit flux tube by $f = \rho\,v_{\rm p}\,\varpi^2 = \eta\,\Phi$, we introduce the following two parameters:

$$\varpi_0^2 \equiv \frac{\Phi^2}{4\pi f c} = \frac{\Phi}{4\pi\eta c}, \tag{72}$$

$$\sigma = \frac{\alpha^2\,\varpi_0^2}{c^2} = \frac{\alpha^2\,\Phi}{4\pi\eta c^3} \tag{73}$$

(see Michel 1969 and Okamoto 1978; see also appendix 4 for the nonrelativistic limits of w and σ). Note that σ varies with s through Φ only.

From equations (55), (73), and (68) one can relate σ to ζ and $u_{\rm p}$:

$$\sigma = \frac{\alpha\,\zeta\,u_{\rm p}}{4\pi\eta\,\mu_\delta c^2} = \frac{\zeta}{\zeta_{\rm B} - \zeta_{\rm A}}u_{\rm p}. \tag{74}$$

We use the "regularity condition" for σ toward S_∞, i.e.,

$$\sigma \to 0 \quad \text{for} \;\; \varpi|_P \to \infty, \tag{75}$$

consistently to the "regularity condition" for ϖ/R in equation (11). It is shown later (see subsections 5.1 and 6.2) that in the asymptotic domain, ζ, γ, I, etc., are a two-valued function of σ for the *critical* solution, and along the *physical* branch downstream of $S_{\rm F}$, one has

$$\zeta \to 0, \quad w \to 0, \quad w_{\rm AM} \to 0 \quad \text{for} \;\; \sigma \to 0, \tag{76}$$

in accordance with equations (29) and (46).

3. Several Surfaces of Importance along Field-Streamlines

It is usually thought that in the neighborhood of the star the field is so strong and the curvature is so large that the curvature radiation will be efficient enough to induce the cascade creation of a pair plasma near to, but probably inside of, the light cylinder, and to ensure MHD treatment downstream. It is, however, out of the scope in this paper to make a comprehensive model of supplying a pair plasma into the wind zone. We just postulate some reference level, referred to as the "magnetospheric base" surface, $S_{\rm B}$, well inside the Alfvénic surface, $S_{\rm A}$, and the RMHD outflow starts with charge density $\varrho_{\rm e}$, particle density ρ, velocity $\boldsymbol{v}$, and the Lorentz factor γ, specified together with α, η, and μ_δ at $S_{\rm B}$.

The quantities σ and ζ, defined in equations (74) and (55), play a significant role in RMHD wind theory, similarly in the nonrelativistic case. We anticipate that both σ and ζ are monotonically decreasing functions of $\varpi|_P$, fairly steeply near the source and rather slowly in the asymptotic domain. We pick up some surfaces (or points) of interest along a given open field line (see figure 3):

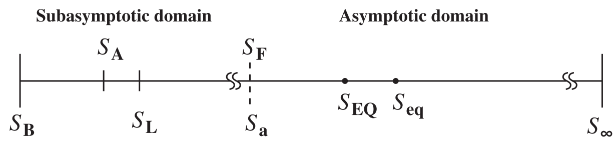

Fig. 3. Schematic distribution of various surfaces of interest along each field line from the magnetospheric base surface, $S_{\rm B}$, to the sphere-at-infinity, S_∞. The wind zone is divided into the two domains, subasymptotic and asymptotic, by the innermost surface, $S_{\rm a}$, satisfying the condition $\varpi_{\rm A}^2/\varpi^2 \ll 1$. It is in the former domain of $S_{\rm B} \le S < S_{\rm a}$ that $S_{\rm A}$ and $S_{\rm L}$ are located. The fast surface, $S_{\rm F}$, must lie between the two domains, i.e., $S_{\rm F} \approx S_{\rm a}$. In the latter domain of $S_{\rm a} \gtrsim S \gtrsim S_\infty$, every field-flow variable is a two-valued function of σ, with the two branches, *physical* and *unphysical*, intersecting each other like X at $S_{\rm F}$ (see figures 4 and 5).

(i) The "magnetospheric base surface", $S_{\rm B}$. If one gives $\alpha(P)$ at $S_{\rm B}$, the EMF over $S_{\rm B}$ is given by equation (15). We may define $S_{\rm B}$ by, e.g., $\zeta = \zeta_{\rm B}$ from equation (68). Note that $\zeta_{\rm B} > \zeta_{\rm A}$ in general and $\zeta_{\rm B} \simeq \zeta_{\rm A}$ in the ultrarelativistic case with $\varpi_{\rm A}^2 \simeq \varpi_{\rm L}^2$. Then the Alfvénic Mach number becomes at $S_{\rm B}$

$$M_{\rm B}^2 = \frac{\alpha^2\,\varpi_{\rm B}^2}{c^2}\left(1 - \frac{\alpha^2\,\varpi_{\rm A}^2}{c^2}\right) < 1, \tag{77}$$

and then from equations (60) and (61),

$$\gamma_{\rm B} = \mu_\delta\left[1 - \frac{\dfrac{\alpha^2\,\varpi_{\rm B}^2}{c^2}\left(1 - \dfrac{\alpha^2\,\varpi_{\rm A}^2}{c^2}\right)}{\left(1 - \dfrac{\alpha^2\,\varpi_{\rm B}^2}{c^2}\right)}\right]^{-1}. \tag{78}$$

Note that $\gamma_{\rm B} > \mu_\delta > 1$. The "magnetospheric base surface", $S_{\rm B}$, as defined above, may not necessarily coincide with the surface of injection of pair plasma into the pulsar magnetosphere as the boundary surface of specifying the input parameters α, η, and μ_δ. This is because, as can be seen in equation (68), $\zeta_{\rm B}$ at $S_{\rm B}$ must be determined by solving the eigenvalue problem due to the criticality condition at $S_{\rm F}$.

In this paper we, however, regard the magnetospheric surface in general as the "surface" from which magnetic fluxes emanate with $\alpha = \alpha(P)$ and on which there exists not only the pulsar unipolar inductor with the EMF given in equation (15), but also the source of copious plasma particles.

(ii) The Alfvénic surface $S_{\rm A}$: We have already obtained equations (50) and (51) by the criticality conditions for γ, $\varpi\,v_{\rm t}$, and I in equations (47)–(49) at $S_{\rm A}$.

Anticipating that ζ is a decreasing function of $\varpi|_P$, we denote the gradient of ζ at $S_{\rm A}$ from equation (55), as follows:

$$\nu_{\rm A} = -\left(\frac{\partial\Pi}{\partial\varpi}\right)_{\rm A} = -\left(\frac{\partial\ln\zeta}{\partial\ln\varpi^2}\right)_{\rm A}. \tag{79}$$

Note that $\nu_{\rm A} > 0$ for the *physical* branch of the solution. Then, by applying l'Hôpital's rule from equations (63), (64), and (61) at $S_{\rm A}$, one obtains in terms of $\nu_{\rm A}$

$$\gamma_{\rm A} = \mu_\varepsilon\frac{M_{\rm A}^2(1+\nu_{\rm A})}{1+\nu_{\rm A}\,M_{\rm A}^2} = \mu_\delta\frac{1+\nu_{\rm A}}{1+\nu_{\rm A}\,M_{\rm A}^2}, \tag{80}$$

$$(\varpi\,v_{\rm t})_{\rm A} = \alpha\,\varpi_{\rm A}^2\frac{\nu_{\rm A}}{1+\nu_{\rm A}}, \tag{81}$$

$$\frac{I_\mathrm{A}}{2\pi\eta c} = \frac{\mu_\varepsilon \alpha \varpi_\mathrm{A}^2}{1 + \nu_\mathrm{A} M_\mathrm{A}^2}, \tag{82}$$

$$u_{\mathrm{tA}} = \mu_\delta \frac{\nu_\mathrm{A} \dfrac{\alpha \varpi_\mathrm{A}}{c}}{1 + \nu_\mathrm{A} M_\mathrm{A}^2}. \tag{83}$$

Substitution of equations (80) and (83) into an identity relation, $\gamma_\mathrm{A}^2 = 1 + u_{\mathrm{pA}}^2 + u_{\mathrm{tA}}^2$, yields

$$\nu_\mathrm{A} = \frac{1}{M_\mathrm{A}^2}\left[\sqrt{\frac{\mu_\varepsilon^2(1 - M_\mathrm{A}^2) M_\mathrm{A}^2}{\mu_\delta^2 M_\mathrm{A}^2 - (1 + u_{\mathrm{pA}}^2)}} - 1\right] \tag{84}$$

[see Heyvaerts and Norman's (1989) equations (48) and (49) and equation (3.14) in Paper I for the nonrelativistic case], where u_{pA} is given from equations (54) and (73) by

$$u_{\mathrm{pA}} = \frac{M_\mathrm{A}^2 \Phi_\mathrm{A}}{4\pi\eta c \varpi_\mathrm{A}^2} = \frac{M_\mathrm{A}^2}{1 - M_\mathrm{A}^2 \sigma_\mathrm{A}}, \quad \sigma_\mathrm{A} = \frac{\alpha^2 \Phi_\mathrm{A}}{4\pi\eta c^3}. \tag{85}$$

It turns out later (see section 5) that by solving the criticality-eigenvalue problem at S_F, one can thus determine the gradient of ζ at S_A, ν_A, as well as the eigenvalues ϖ_A^2, μ_ε, $\zeta_\mathrm{A} = -\beta$, and ζ_B, in terms of the input parameters α, η, and μ_δ for a specified field structure.

It can be seen that by using equations (80)–(82) the ratio of angular momentum of the field to the flow becomes

$$(w_{\mathrm{AM}})_\mathrm{A} = \frac{1}{\nu_\mathrm{A} M_\mathrm{A}^2}. \tag{86}$$

Also, the ratio of the Poynting energy flux to the kinetic energy flux reduces by equations (67)–(69), (79), and (53) to

$$w_\mathrm{A} = \frac{1}{1 + \nu_\mathrm{A}} \frac{\zeta_\mathrm{A}}{\zeta_\mathrm{B} - \zeta_\mathrm{A}} = \frac{1}{1 + \nu_\mathrm{A}} \frac{1 - M_\mathrm{A}^2}{M_\mathrm{A}^2}. \tag{87}$$

For $w_\mathrm{A} < 1$ to hold at S_A, one has the condition that $M_\mathrm{A}^2 > 1/(2 + \nu_\mathrm{A})$ or equivalently $\alpha^2 \varpi_\mathrm{A}^2 / c^2 < (1 + \nu_\mathrm{A})/(2 + \nu_\mathrm{A})$. It will, however, be shown later (see subsection 5.4) that the surfaces of equipartition of energy flux and angular momentum between the field and the flow, i.e., $w = 1$ and $w_{\mathrm{AM}} = 1$, are in the asymptotic domain of $\varpi^2 \gg \varpi_\mathrm{A}^2$. In ultrarelativistic winds with $\zeta_\mathrm{B} \simeq \zeta_\mathrm{A}$ or $M_\mathrm{A}^2 \simeq 0$, one finds $w_\mathrm{A} \gg 1$.

(iii) The light surface S_L: $\alpha \varpi_\mathrm{L}/c = 1$, where $D(\varpi_\mathrm{L}) = M_\mathrm{A}^2$. From equations (56) and (69) one obtains

$$\frac{M_\mathrm{A}^2}{M_\mathrm{L}^2} = \frac{\zeta_\mathrm{L}}{\zeta_\mathrm{B}} < 1. \tag{88}$$

Also, from equations (51) and (61)–(64)

$$\gamma_\mathrm{L} = \mu_\varepsilon \left(1 - \frac{\zeta_\mathrm{L}}{\zeta_\mathrm{B}}\right), \tag{89}$$

$$(\gamma \varpi v_\mathrm{t})_\mathrm{L} = \mu_\varepsilon \alpha \varpi_\mathrm{A}^2 - \mu_\varepsilon \alpha \varpi_\mathrm{L}^2 \frac{\zeta_\mathrm{L}}{\zeta_\mathrm{B}}, \tag{90}$$

$$\frac{I_\mathrm{L}}{2\pi\eta c} = \mu_\varepsilon \alpha \varpi_\mathrm{L}^2 \frac{\zeta_\mathrm{L}}{\zeta_\mathrm{B}}. \tag{91}$$

Further, from equations (67) and (71) one obtains by using relations (53), (68), and (69)

$$w_\mathrm{L} = \frac{\dfrac{\zeta_\mathrm{L}}{\zeta_\mathrm{B}}}{1 - \dfrac{\zeta_\mathrm{L}}{\zeta_\mathrm{B}}}, \quad (w_{\mathrm{AM}})_\mathrm{L} = \frac{\dfrac{\zeta_\mathrm{L}}{\zeta_\mathrm{A}}}{1 - \dfrac{\zeta_\mathrm{L}}{\zeta_\mathrm{A}}}, \tag{92}$$

and if $\zeta_\mathrm{L} < \zeta_\mathrm{A} < \zeta_\mathrm{B} < 2\zeta_\mathrm{L}$, then $w_\mathrm{L} > 1$ and $(w_{\mathrm{AM}})_\mathrm{L} > 1$.

(iv) The "pure Alfvénic surface" (see Okamoto 1978): S_{pA} is defined by $M^2 = 1$, and using equations (53) and (56) one obtains

$$\frac{\zeta_{\mathrm{pA}}}{\zeta_\mathrm{A}} = \frac{\varpi_{\mathrm{pA}}^2}{\varpi_\mathrm{A}^2}\left(1 - \frac{\alpha^2 \varpi_\mathrm{A}^2}{c^2}\right) < 1. \tag{93}$$

From equations (60), (61), (62), (64), and (93), $D(\varpi_{\mathrm{pA}}) = (\alpha^2 \varpi_{\mathrm{pA}}^2/c^2)[1 - (\alpha^2 \varpi_{\mathrm{pA}}^2/c^2)]$, and

$$\gamma_{\mathrm{pA}} = \mu_\varepsilon \frac{\varpi_\mathrm{A}^2}{\varpi_{\mathrm{pA}}^2}, \tag{94}$$

$$(\gamma \varpi v_\mathrm{t})_{\mathrm{pA}} = \mu_\varepsilon \alpha \varpi_\mathrm{A}^2 \left[1 - \left(\frac{c^2}{\alpha^2 \varpi_{\mathrm{pA}}^2}\right)\left(\frac{\varpi_{\mathrm{pA}}^2}{\varpi_\mathrm{A}^2} - 1\right)\right], \tag{95}$$

$$\frac{I_{\mathrm{pA}}}{2\pi\eta c} = \mu_\varepsilon \alpha \varpi_\mathrm{A}^2 \left(\frac{c^2}{\alpha^2 \varpi_{\mathrm{pA}}^2}\right)\left(\frac{\varpi_{\mathrm{pA}}^2}{\varpi_\mathrm{A}^2} - 1\right). \tag{96}$$

It turns out from a comparison of equations (89) and (94) and using equation (88) that

$$\varpi_\mathrm{L} \lesseqqgtr \varpi_{\mathrm{pA}} \Longleftrightarrow \frac{\varpi_\mathrm{A}^2}{\varpi_{\mathrm{pA}}^2} + \frac{M_\mathrm{A}^2}{M_\mathrm{L}^2} \gtreqqless 1. \tag{97}$$

For the nonrelativistic limit of $\varpi_\mathrm{L} \to \infty$, $M_\mathrm{L}^2 \gg M_\mathrm{A}^2$ and $\varpi_{\mathrm{pA}}^2 \to \varpi_\mathrm{A}^2$.

It is needless to say that not only the light surface, but also the pure-Alfvénic surface, are not *critical* surfaces in the relativistic MHD wind theory.

(v) The surface S_a is rather vaguely defined as the innermost surface along each field line of the asymptotic domain of $\varpi_\mathrm{A}^2 \ll \varpi_\mathrm{A}^2 \lesssim \varpi^2 < \infty$. That is, S_a is introduced as a convenient way of clarifying the asymptotic behavior of the physical quantities. Note, however, that there is no sharp boundary of physical meaning between the asymptotic and subasymptotic domains. In this domain of $S \gtrsim S_\mathrm{a}$, the terms of order higher than $\varpi_\mathrm{A}^2/\varpi^2$ may be dropped in every physical quantity, if one remarks that this produces an effect that the independent variable ϖ disappears in the domain, and it is instead the magnetization parameter, $\sigma(\varpi, P)$, that plays a role as a of coordinate variable. It then turns out that every quantity is a two-valued function of σ, that is, there are two branches, *physical* and *unphysical* (see subsection 5.1 later). The two branches must cross each other like an X at the fast surface, S_F, which must exist in the vicinity of S_a (see subsection 5.2).

(vi) The fast magnetosonic surface, S_F: It is certain that there exists no superfast region in the *sub*asymptotic domain of $S < S_\mathrm{a}$, whereas by definition the coordinate variable ϖ_F used to fix that point of ϖ_F disappears in the asymptotic domain with $S \gtrsim S_\mathrm{a}$. It thus turns out that one must retain the terms with the lowest order of $\varpi_\mathrm{A}^2/\varpi^2$ in $S \gtrsim S_\mathrm{a}$, thereby enabling one to fix the location ϖ_F of S_F in terms of the gradient of $\sigma(\varpi, P)$. It is easily conceivable that if σ = constant, $\varpi_{\rm F} \to \infty$. It can be shown that $S_{\rm F}$ must be situated at the innermost distances of $S \gtrsim S_{\rm a}$, that is, the asymptotic domain must almost coincide with the superfast domain, itself, i.e., $S_{\rm F} \approx S_{\rm a}$ (see subsection 5.2).

(vii) The equipartition surfaces, $S_{\rm EQ}$, $S_{\rm eq}$: One can define an equipartition surface, $S_{\rm EQ}$, where the Poynting energy flux equals the kinetic energy flux, and $\gamma_{\rm EQ} = \mu_\varepsilon/2$, by setting $w_{\rm EQ} = 1$ in equation (67). One then obtains from equations (60) and (61)

$$1 - \frac{\varpi_{\rm A}^2}{\varpi_{\rm EQ}^2} = \frac{\left(1 - \dfrac{\alpha^2 \varpi_{\rm A}^2}{c^2}\right)}{\left(2\dfrac{\alpha^2 \varpi_{\rm A}^2}{c^2} - 1\right)} \left(\frac{\zeta_{\rm A}}{\zeta_{\rm EQ}} - 1\right). \tag{98}$$

The equipartition surface of the total angular momentum, $-\beta/4\pi\eta = \mu_\varepsilon \alpha \varpi_{\rm A}^2$, between the field and the flow is given by the condition $w_{\rm AM} = 1$ in equation (71), i.e., $(\varpi u_{\rm t})_{\rm eq} = \mu_\varepsilon \alpha \varpi_{\rm A}^2/2c$, and then by equations (60) and (63)

$$1 - \frac{\varpi_{\rm A}^2}{\varpi_{\rm eq}^2} = \left(1 - \frac{\alpha^2 \varpi_{\rm A}^2}{c^2}\right) \left(\frac{\zeta_{\rm A}}{\zeta_{\rm eq}} - 1\right). \tag{99}$$

It will be shown later (see subsection 5.4) that $S_{\rm a} \approx S_{\rm F} < S_{\rm EQ} < S_{\rm eq}$, that is, it is beyond $S_{\rm F}$ that the equipartition of energy or angular momentum is achieved.

(viii) The "sphere-at-infinity", S_∞: We define S_∞ as a physical sphere at infinity where no Poynting flux reaches. One of the purposes of this paper is to show within the framework of ideal MHD that it is possible, in principle, that all of the Poynting flux from the star is converted to kinetic flux by the ongoing work of the Lorentz force upon the mean flow. This should be ensured by the centripetal cross-field inertial force containing the curvature radius, $1/R$, in balance with the Lorentz force in the asymptotic domain. It is then necessary to take $\varpi/R \to 0$, $I \to 0$, and $\sigma \to 0$ into account as the "regularity condition" at S_∞, already given in equations (11), (29), and (75).

In a real situation, like the Crab pulsar wind, one must of course consider the existence of the nebular surface, $S_{\rm N}$, before reaching infinity, and take into account interactions of the cold wind with the surrounding medium or nebula in the region probably far from $S_{\rm F}$ on the way to S_∞. However, in such various processes as, e.g., the collision of the superfast wind with the medium, the dissipating surface current on the shock surface, etc., will not affect the upstream field-flow structure. These phenomena of great interest are beyond the scope of this paper. Then at S_∞, by the "regularity conditions" given in equations (11) and (75), one has the following results along the *physical* branch:

$$\gamma \to \mu_\varepsilon, \quad \varpi u_{\rm t} \to \mu_\varepsilon \frac{\alpha \varpi_{\rm A}^2}{c}, \quad \frac{2I}{c\zeta} \to \frac{\mu_\varepsilon}{\mu_\delta}, \tag{100}$$

and also $w \to 0$ and $w_{\rm AM} \to 0$, all of which indicate that all of the quantities at S_∞ are determined by the initial values at $S_{\rm B}$ and the eigenvalues at $S_{\rm F}$.

Distinguished from S_∞, we introduce another sphere at infinity for the split-monopolar field structure where the finite Poynting flux reaches as denoted with the pseudo-force-free sphere-at-infinity, $S_{\rm pff\,\infty}$ (see Okamoto 2006). As is well known, if one assumes $\varpi B_{\rm r}$ = constant, then Φ = constant and σ = constant. This implies that without being fully consumed in the flow acceleration, the Poynting flux is carried to "infinity", i.e., $S_{\rm pff\,\infty}$. Thus, it is not a good approximation to assume the split-monopolar field structure with $\varpi B_{\rm r}$ = constant, not only in the neighborhood of the star where the field is so strong that the curvature of field lines is important, but also even in the asymptotic domain where the particle inertia is certainly large, but MHD interactions are still ongoing through the inertial curvature term (see section 6).

As long as the steady state is assumed for axisymmetric MHD winds, the "current-closure condition" must be assumed, that is, no snapping of current lines. Any current line must emanate from one terminal of the unipolar battery given in equation (15) and return to the other terminal, converting the Poynting flux to the kinetic flux by the Lorentz *volume* force. If some current line extends to infinity in some (e.g., force-free) model, the *surface* current must be introduced on the (force-free) sphere-at-infinity, connecting it with the return current line. Any model with current lines extending to infinity, left as it is, will be incomplete in the steady state (see Okamoto 2006).[2]

4. Quadratic Equation for the Axial Distance, the Slope of ζ, and the Transfield Equation

4.1. Quadratic Equation for the Axial Distance

We express $u_{\rm p}$ in terms of ζ and σ from equation (74) as

$$u_{\rm p} = \frac{4\pi\eta\,\mu_\delta c^2 \sigma}{\alpha\zeta} = \frac{\zeta_{\rm B} - \zeta_{\rm A}}{\zeta}\sigma. \tag{101}$$

From equations (60) and (68)

$$D = \left(1 - \frac{\zeta_{\rm A}}{\zeta_{\rm B}}\right) \left[1 + \left(1 - \frac{c^2}{\alpha^2 \varpi^2}\right) \frac{\zeta}{\zeta_{\rm B} - \zeta_{\rm A}}\right]. \tag{102}$$

Thus, from equations (63) and (64) one obtains

$$I = \frac{c\zeta}{2D}\left(1 - \frac{\varpi_{\rm A}^2}{\varpi^2}\right). \tag{103}$$

We also make full use of an identity relation,

$$\gamma^2 = 1 + u_{\rm p}^2 + u_{\rm t}^2. \tag{104}$$

Then, substituting equations (61) and (63) into equation (104), one obtains

$$(1 + u_{\rm p}^2) D^2 = \mu_\delta^2 \left[\left(1 - \frac{\varpi_{\rm A}^2}{\varpi^2}\frac{\zeta}{\zeta_{\rm A}}\right)^2 - \frac{\alpha^2 \varpi_{\rm A}^2}{c^2}\frac{\varpi_{\rm A}^2}{\varpi^2}\left(1 - \frac{\zeta}{\zeta_{\rm A}}\right)^2\right]. \tag{105}$$

Then from equations (102) and (105) one obtains the Bernoulli equation for a relativistic outflow in an algebraic form,

$$A\frac{\varpi^4}{\varpi_{\rm A}^4} + B\frac{\varpi^2}{\varpi_{\rm A}^2} + C = 0, \tag{106}$$

(see Michel 1969; Okamoto 1978), where

[2] This paragraph is added to reply to the referee's request.

$$A = \mu_\varepsilon^2 - (1 + u_\mathrm{p}^2)\left(1 + \frac{\zeta}{\zeta_\mathrm{B} - \zeta_\mathrm{A}}\right)^2, \tag{107}$$

$$B = -\mu_\varepsilon^2 \left[\frac{2\zeta}{\zeta_\mathrm{A}} + \frac{\zeta_\mathrm{A}^2}{\zeta_\mathrm{B}^2}\left(1 - \frac{\zeta}{\zeta_\mathrm{A}}\right)^2\right] + 2(1 + u_\mathrm{p}^2)\frac{\zeta_\mathrm{B}}{\zeta_\mathrm{B} - \zeta_\mathrm{A}}\frac{\zeta}{\zeta_\mathrm{A}}\left(1 + \frac{\zeta}{\zeta_\mathrm{B} - \zeta_\mathrm{A}}\right), \tag{108}$$

$$C = \left[\mu_\varepsilon^2 - (1 + u_\mathrm{p}^2)\left(\frac{\zeta_\mathrm{B}}{\zeta_\mathrm{B} - \zeta_\mathrm{A}}\right)^2\right]\left(\frac{\zeta}{\zeta_\mathrm{A}}\right)^2, \tag{109}$$

and for a given σ, u_p is related to ζ by equation (101). The discriminant of equation (106) becomes

$$B^2 - 4AC = 4\mu_\varepsilon^2 \frac{\zeta_\mathrm{A}^2}{\zeta_\mathrm{B}^2}\left(1 - \frac{\zeta}{\zeta_\mathrm{A}}\right)^2 \times \left\{\mu_\varepsilon^2\left[\frac{\zeta}{\zeta_\mathrm{A}} + \frac{\zeta_\mathrm{A}^2}{4\zeta_\mathrm{B}^2}\left(1 - \frac{\zeta}{\zeta_\mathrm{A}}\right)^2\right] - (1 + u_\mathrm{p}^2)\frac{\zeta_\mathrm{B}\,\zeta}{\zeta_\mathrm{A}(\zeta_\mathrm{B} - \zeta_\mathrm{A})}\left(1 - \frac{\zeta}{\zeta_\mathrm{A}}\right)\right\}. \tag{110}$$

As is usually done in the cold-wind approximation, we have neglected the gas pressure as well as gravity. Therefore, contrary to the nonrelativistic case, one can explicitly set up a quadratic equation for $\varpi^2/\varpi_\mathrm{A}^2$. But notice that coefficients A, B, and C are functions of not only P through μ_ε, ζ_B, and ζ_A, but also of ζ and σ through u_p in equation (101), and hence these coefficients are *weakly* dependent on ϖ through σ for a fixed field line.

Solving the quadratic equation (106), one can obtain the two values of $(\varpi/\varpi_\mathrm{A})^2$ in terms of ζ or u_p. One may then conversely have $\zeta = \zeta(\varpi^2)$ or $u_\mathrm{p} = u_\mathrm{p}(\varpi^2)$ for each field line. If $\sigma =$ constant is assumed, as is so often done, e.g., like a split-monopolar field, then A, B, and C are not dependent on ϖ, even weakly.

However, in real situations the situation is much more complicated. We may suppose that the field structure, e.g., $\sigma = \sigma(\varpi, P)$ is given in a first step of iterations, so that μ_ε, ζ_A, and ζ_B can be obtained as eigenvalues by solving the "criticality problem" at S_F for each field line. Then, substituting u_p from equation (101) into A, B, and C in equations (107)–(109), which are dependent *weakly* on ϖ through σ, one needs to solve the quadratic equation for $\varpi^2/\varpi_\mathrm{A}^2$ in terms of ζ by suitable iteration, and finally obtains $\zeta = \zeta(\varpi; P)$ and then $u_\mathrm{p} = u_\mathrm{p}(\varpi; P)$. It turns out (see subsubsection 5.2.2) that this *weakly* dependent part of ζ on ϖ *per se* is most essential for locating S_F near S_a, and thereby achieving MHD acceleration in the asymptotic domain of $S \gtrsim S_\mathrm{a} \approx S_\mathrm{F}$.

4.2. Slope of ζ

In order to solve the "eigenvalue problem", one requires the flow to pass smoothly through the critical surfaces at finite distances. For this purpose, one needs to calculate the slope of physical quantities, e.g., ζ as follows:

$$\frac{\partial \ln \zeta}{\partial \varpi} = \frac{\mathcal{N}}{\mathcal{D}}, \tag{111}$$

$$\mathcal{D} \equiv u_\mathrm{p}^2 D - \mu_\delta^2 \frac{\zeta}{\zeta_\mathrm{B}}\left(\frac{2I}{c\zeta}\right)^2 = u_\mathrm{p}^2 D - \gamma^2 \frac{\zeta}{\zeta_\mathrm{B}}\left(\frac{1 - \dfrac{\varpi_\mathrm{A}^2}{\varpi^2}}{1 - \dfrac{c^2}{\alpha^2\varpi^2}\dfrac{\zeta}{\zeta_\mathrm{B}}}\right)^2, \tag{112}$$

$$\mathcal{N} \equiv \left(u_\mathrm{p}^2 \frac{\partial \ln \sigma}{\partial \varpi} - \frac{u_\mathrm{t}^2}{\varpi}\right) D + 2\mu_\delta \frac{c\,u_\mathrm{t}}{\alpha\varpi^2}\frac{\zeta}{\zeta_\mathrm{B}}\left(\frac{2I}{c\zeta}\right) \tag{113}$$

(see appendix 2 for derivation). At S_A, using equations (82) and (83), one obtains from equations (112) and (113)

$$\mathcal{N}(\varpi_\mathrm{A}) = -\frac{2\nu_\mathrm{A}}{\varpi_\mathrm{A}}\mathcal{D}(\varpi_\mathrm{A}) = \frac{2\nu_\mathrm{A}}{\varpi_\mathrm{A}} \frac{\mu_\delta^2 \dfrac{\alpha^2\varpi_\mathrm{A}^2}{c^2}}{\left[1 + \nu_\mathrm{A}\left(1 - \dfrac{\alpha^2\varpi_\mathrm{A}^2}{c^2}\right)\right]^2}. \tag{114}$$

Substituting equation (114) into (111) at S_A, one can easily reproduce the definition of ν_A in equation (79). From equations (101), (102), and (103) one can also show that

$$\mathcal{D} = -u_\mathrm{p}^2 \frac{\varpi_\mathrm{A}^2}{\varpi^2}\frac{\zeta}{\zeta_\mathrm{A}}\left(1 - 4\pi\rho\gamma\kappa^2 - \frac{\alpha^2\varpi^2}{c^2} + \frac{B_\mathrm{t}^2}{B_\mathrm{p}^2}\right) \tag{115}$$

[see equations (2.18a, b) and (2.19a, b) in Okamoto (1978)]. The condition $\mathcal{D}(\varpi) = \mathcal{N}(\varpi) = 0$ yields the criticality condition at the fast magnetosonic surface, S_F, through which the flow should smoothly pass.

4.3. Transfield Equation

The curvature, $\partial\psi/\partial s = 1/R$, along each field line is a crucial quantity to judge whether the flow does collimate or decollimate, similarly in the nonrelativistic case. The equation determining $1/R$ is the transfield component of the RMHD equation of motion, which is expressed in terms of the balance of various forces in the cross-field direction, as shown in equation (36). This becomes from equations (19) and (52)

$$\left(1 - M^2 - \frac{\alpha^2\varpi^2}{c^2}\right)\frac{1}{R} = \frac{\partial \ln B_\mathrm{p}}{\partial n} - M^2 \frac{u_\mathrm{t}^2}{u_\mathrm{p}^2}\frac{\partial \ln \varpi}{\partial n} + \frac{B_\mathrm{t}^2}{B_\mathrm{p}^2}\frac{\partial \ln \varpi B_\mathrm{t}}{\partial n} - \frac{\alpha^2\varpi^2}{c^2}\frac{\partial \ln B_\mathrm{p}\alpha\varpi^2}{\partial n}. \tag{116}$$

It is not difficult to derive equation (116) from Chiueh, Li, and Begelman's (1991) equation (11). The quantities related to $1/R$ are j_t and ϱ_e, and from equations (20) and (24) one obtains

$$\left(1 - M^2 - \frac{\alpha^2\varpi^2}{c^2}\right)\left(\frac{4\pi j_{\rm t}}{cB_{\rm p}}\right)$$
$$= -M^2\frac{\partial \ln B_{\rm p}}{\partial n} + M^2\frac{u_{\rm t}^2}{u_{\rm p}^2}\frac{\partial \ln\varpi}{\partial n} - \frac{B_{\rm t}^2}{B_{\rm p}^2}\frac{\partial \ln\varpi B_{\rm t}}{\partial n}$$
$$+ \frac{\alpha^2\varpi^2}{c^2}\frac{\partial \ln\alpha\varpi^2}{\partial n} \tag{117}$$

and

$$\left(1 - M^2 - \frac{\alpha^2\varpi^2}{c^2}\right)\left(\frac{4\pi \varrho_{\rm e}}{E_{\rm p}}\right)$$
$$= -M^2\frac{\partial \ln B_{\rm p}}{\partial n} + M^2\frac{u_{\rm t}^2}{u_{\rm p}^2}\frac{\partial \ln\varpi}{\partial n} - \frac{B_{\rm t}^2}{B_{\rm p}^2}\frac{\partial \ln\varpi B_{\rm t}}{\partial n}$$
$$+ (1 - M^2)\frac{\partial \ln\alpha\varpi^2}{\partial n}. \tag{118}$$

At $S_{\rm A}$, the bracket of the left-hand side of equation (116) vanishes, and to avoid a kink, i.e., $R = 0$, one must impose the right-hand side to vanish as well. To determine $R_{\rm A}$, $(\varrho_{\rm e})_{\rm A}$, and $j_{\rm tA}$, one must use l'Hôpital's rule for equations (116)–(118).

For later convenience, we rewrite the square bracket of equation (36), which yields the last two terms of equation (116). Utilizing equations (55), (61), (63), (103), and (104), it can be shown that

$$I^2 - \left(\frac{\alpha\Phi}{2}\right)^2 = \frac{c^2}{4}\varpi^2(B_{\rm t}^2 - |\boldsymbol{E}|^2)$$
$$= \frac{c^2}{4}\frac{\zeta^2}{\mu_\delta^2}\left(1 - \frac{\mu_\delta^2}{D^2}\frac{\varpi_{\rm A}^2}{\varpi^2}\left(1 - \frac{\zeta}{\zeta_{\rm A}}\right)\right.$$
$$\left.\times\left\{2 - \frac{\alpha^2\varpi_{\rm A}^2}{c^2}\left[1 - \frac{\zeta}{\zeta_{\rm A}} + \frac{c^2}{\alpha^2\varpi^2}\left(1 + \frac{\zeta}{\zeta_{\rm A}}\right)\right]\right\}\right), \tag{119}$$

which must of course be positive , i.e., $\varpi^2(B_{\rm t}^2 - |\boldsymbol{E}|^2) > 0$. Note that in the asymptotic domain of $\varpi^2 \gg \varpi_{\rm A}^2$,

$$I^2 - \left(\frac{\alpha\Phi}{2}\right)^2 \approx \frac{c^2}{4}\frac{\zeta^2}{\mu_\delta^2}, \tag{120}$$

and $(2I/c\zeta)_\infty^2 = (\alpha\Phi/c\zeta)_\infty^2 + 1/\mu_\delta^2 = \mu_\varepsilon^2/\mu_\delta^2$ for $\varpi \to \infty|_P$ [see equation (100)].

5. The Asymptotic Domain and the Fast Surface $S_{\rm a}$

We denote the innermost surface of the asymptotic domain of $\varpi^2 \gg \varpi_{\rm A}^2$ rather vaguely with $S_{\rm a}$ and $\varpi = \varpi_{\rm a}(P)$; the suffix "a" is attached to quantities in this domain, like $D_{\rm a}$, $\mathcal{D}_{\rm a}$, etc.[3] A careful and subtle treatment is necessary for locating the fast surface, $S_{\rm F}$, in the cold-wind approximation in the neighborhood of $S_{\rm a}$, because $S_{\rm a} \approx S_{\rm F}$, like in the case of nonrelativistic winds (see Paper III).

5.1. *Flow Properties in the Asymptotic Domain*

5.1.1. *Solution for $w = w(\sigma)$*

In the asymptotic domain with $\varpi_{\rm A}^2/\varpi^2 \ll 1$ we use the nondimensional variable w instead of ζ, where both quantities are related by equations (67) and (69) as

$$w = \frac{\zeta}{\zeta_{\rm B} - \zeta_{\rm A}} = \frac{\zeta}{M_{\rm A}^2\zeta_{\rm B}}. \tag{121}$$

Then, one obtains from equations (61), (63), (102), and (103)

$$\gamma = \frac{\mu_\delta}{D_{\rm a}} = \frac{\mu_\varepsilon}{1 + w}, \tag{122}$$

$$\varpi\, u_{\rm t} = \mu_\varepsilon\frac{\alpha\,\varpi_{\rm A}^2}{c}\frac{1}{1 + w}\left[1 - \left(\frac{\zeta_{\rm B}}{\zeta_{\rm A}} - 1\right)w\right], \tag{123}$$

$$I = \frac{c}{2}\frac{\zeta}{D_{\rm a}} = \frac{c\,\zeta_{\rm B}}{2}\frac{w}{1 + w}, \tag{124}$$

where from equations (102) and (69) one obtains

$$D_{\rm a} = \left(1 - \frac{\zeta_{\rm A}}{\zeta_{\rm B}}\right)(1 + w) = M_{\rm A}^2(1 + w). \tag{125}$$

The ratio of the field angular momentum to the flow angular momentum becomes from equations (71) and (121)

$$w_{\rm AM} = \frac{\zeta_{\rm B}}{\zeta_{\rm A}}\frac{w}{1 - \left(\dfrac{\zeta_{\rm B}}{\zeta_{\rm A}} - 1\right)w}. \tag{126}$$

The quadratic equation (106) reduces simply to $A \approx 0$ in the asymptotic domain and then from equations (107) and (122)

$$u_{\rm p} \approx \sqrt{\gamma^2 - 1} = \sqrt{\frac{\mu_\varepsilon^2}{(1 + w)^2} - 1}. \tag{127}$$

This form can also be derived from equations (122) and (104) with the condition $u_{\rm p}^2 \approx \gamma^2 - 1 \gg u_{\rm t}^2$. Then, from equations (74) and (127)

$$\sigma = w\,u_{\rm p} = w\sqrt{\frac{\mu_\varepsilon^2}{(1 + w)^2} - 1} \tag{128}$$

[see equation (3) in Paper IV], which can also be derived from equations (73) and (120) for $\varpi^2 \gg \varpi_{\rm A}^2$. The nonrelativistic version of equation (128) is given by equation (A39) in appendix 4.

It is worth remarking here that the coordinate variables ϖ and z disappear in the above expressions for the asymptotic behavior of the variables. It is Φ or σ that plays the role of an independent variable in the asymptotic domain. Keeping this fact in mind, let us clarify the flow properties in the asymptotic domain.

By solving $w = w(\sigma; \mu_\varepsilon)$ as a function of $\sigma(\varpi; P)$ from equation (128), we can determine all other variables (ζ, γ, etc.) in terms of $w(\sigma)$ by equations (121)–(124). Differentiation of w with σ yields

$$\frac{dw}{d\sigma} = \frac{\sigma}{w}\frac{(1 + w)}{F_{\rm a}(1 + w)}, \tag{129}$$

where

[3] Contrary to Chiueh, Li, and Begelman (1991), Eichler (1993), and Tomimatsu (1994), who defined the asymptotic domain as in $\varpi \gg \varpi_{\rm L}$, we utilize the domain as defined above, because the "asymptotic domain" should not lose its physical meanings even in the nonrelativistic limit of $c \to \infty$.

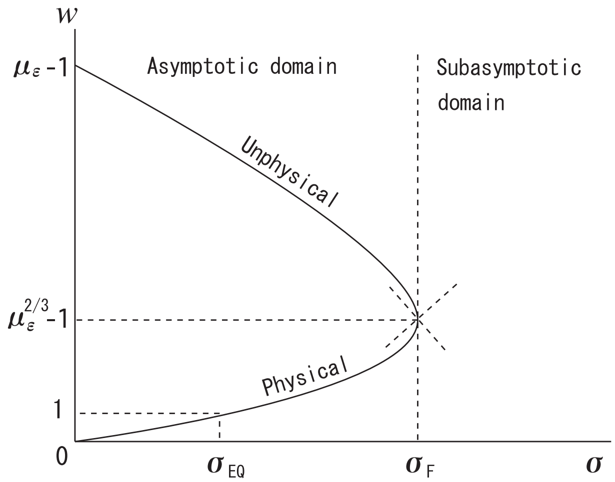

Fig. 4. Schematic figure of $w(\sigma) = \zeta/(\zeta_\mathrm{B} - \zeta_\mathrm{A})$, as given by equation (128) in the asymptotic domain. The two branches join with a vertical slope at the critical point. To connect the solution causally with that in the subasymptotic domain, one must transform the critical point to an X-type point, taking the *weak* dependence of, e.g., ζ on $\varpi_\mathrm{A}^2/\varpi^2$ into account. See figure 2 in Paper III for the nonrelativistic case.

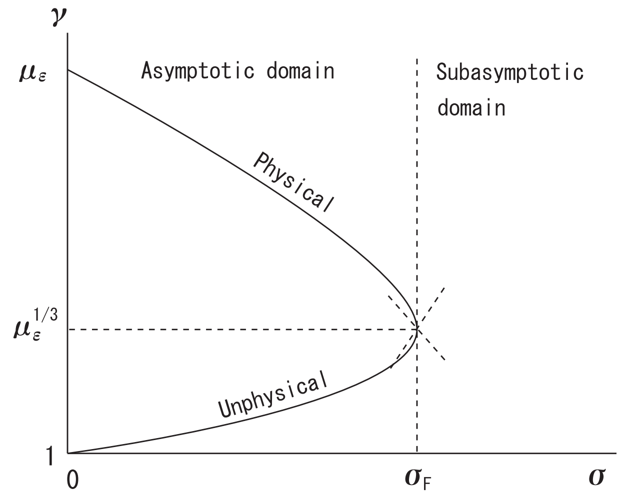

Fig. 5. Schematic picture of $\gamma = \gamma(\sigma)$, as given by equation (131). The upper branch stands for the *physical* solution with $\gamma_\infty = \mu_\varepsilon$, and the lower stands for the *unphysical* solution with $\gamma \to 1$. Note that both branches join at the fast magnetosonic point with an infinitely steep gradient. To connect the two solutions to those in $\sigma > \sigma_\mathrm{F}$, one must modify the solutions to possess an X-type geometry in the vicinity of S_F. See figure 1 in Paper IV.

$$F_\mathrm{a}(x) = \frac{\mu_\varepsilon^2}{x^2} - x. \tag{130}$$

Note that $F_\mathrm{a}(x) \gtreqless 0$ for $x \lesseqgtr \mu_\varepsilon^{2/3}$. As shown in figure 4, the algebraic equation (128) for $w = w(\sigma)$ has two branches. The upper branch expresses the *unphysical* solution, yielding $w = \mu_\varepsilon - 1$, $\gamma = 1$, and $u_\mathrm{p} = 0$ at $\sigma = 0$, whereas the lower one gives the *physical* branch with $w = 0$, $\gamma = \mu_\varepsilon$, and $u_\mathrm{p} = \sqrt{\mu_\varepsilon^2 - 1}$ at $\sigma = 0$. The two branches join at $\sigma = \sigma_\mathrm{F} = w_\mathrm{F}^{3/2}$ and $w = w_\mathrm{F} \equiv \mu_\varepsilon^{2/3} - 1$. It must be remarked here that there is no other point at which the two branches cross each other in the region of $\sigma < \sigma_\mathrm{F}$, and this implies that $\sigma = \sigma_\mathrm{F} = w_\mathrm{F}^{3/2} = (\mu_\varepsilon^{2/3} - 1)^{3/2}$ corresponds to the fast surface, S_F, and the domain of $0 \le \sigma \le \sigma_\mathrm{F}$ is nothing but the superfast domain. Simultaneously we must note that the solution of $w = w(\sigma)$ possesses a vertical tangent, $(dw/d\sigma)_\mathrm{F} = \pm\infty$ at $\sigma = \sigma_\mathrm{F}$. This implies that the two branches cannot constitute an X-type critical point, and that there is no solution in the region $\sigma > \sigma_\mathrm{F}$. To connect the *physical* solution for $\sigma < \sigma_\mathrm{F}$ smoothly with the solution upstream for $\sigma > \sigma_\mathrm{F}$, one must consider the criticality condition at S_F rigorously with the *weak* dependence of the variables on ϖ in the neighborhood of S_a (see subsection 5.2).

5.1.2. *Solution for $\gamma = \gamma(\sigma)$*

Concerning the relation between γ and σ, equations (122) and (128) similarly give

$$\sigma = \left(\frac{\mu_\varepsilon}{\gamma} - 1\right)\sqrt{\gamma^2 - 1}, \quad \frac{d\gamma}{d\sigma} = \frac{\sqrt{\gamma^2 - 1}}{F_\mathrm{a}(\gamma)}. \tag{131}$$

One can then depict such a picture as figure 4 with an infinite gradient at $\sigma = \sigma_\mathrm{F}$ (see figure 5, and figure 1 in Paper IV). It is seen that $d\gamma/d\sigma < 0$ for the *physical* branch with $\mu_\varepsilon \ge \gamma \ge \mu_\varepsilon^{1/3}$. It is easy to obtain equation (4.3a) in Paper III, by taking the nonrelativistic limit of equation (131).

Because Φ, and hence σ, are decreasing functions of s or $\varpi|_P$, one can expect ongoing acceleration down to infinity, with the extremum values of $\gamma \to \mu_\varepsilon$ and $\varpi\, v_\mathrm{t} \to \alpha\, \varpi_\mathrm{A}^2$, i.e., $w = w_\mathrm{AM} = 0$. This implies complete transfer of the field energy to the flow energy. This is ensured by $j_\perp > 0$ in the asymptotic domain, because one obtains from equations (18), (121), (124), and (128)

$$j_\perp = -\frac{c\,\zeta_\mathrm{B}}{4\pi\varpi}\frac{1}{(1+w)^2}\frac{\partial w}{\partial s} \propto -\frac{\partial\sigma}{\partial s} > 0. \tag{132}$$

Remember that we are presuming that ζ is a slowly decreasing function of $\varpi|_P$ or s. This is a natural reflection of the presumption that Φ or σ is a slowly decreasing function of $\varpi|_P$ or s, so that all quantities in the asymptotic domain change slowly with ϖ, like w and γ.

5.2. *The Criticality Condition at S_F*

The fast surface S_F is formally defined as a surface consisting of the magnetosonic critical points through which the flow passes smoothly without becoming singular; this surface is given by the conditions $\mathcal{D} = \mathcal{N} = 0$ in equation (111). Similarly in nonrelativistic cases, the fast surface will show up somewhere between the subasymptotic domain where those terms on order of $\varpi_\mathrm{A}^2/\varpi^2$ are significant, and the asymptotic domain where they are negligible with σ playing the role of an independent variable, instead of ϖ, that is, $S_\mathrm{F} \approx S_\mathrm{a}$.

It is shown above that all of the physical quantities, such as w, ζ, γ, I, etc., are expressible as a two-valued function of σ in the asymptotic domain [see subsection 6.2 for $I = I(\sigma)$], but have no values in the subasymptotic domain with $\sigma > \sigma_\mathrm{F}$, where all quantities are dependent on not only σ, but also the coordinates ϖ and z explicitly. In order to connect the above solutions to those upstream in $\sigma > \sigma_\mathrm{F}$, avoiding divergence at $\sigma = \sigma_\mathrm{F}$, one must naturally take the $\varpi_\mathrm{A}^2/\varpi^2$ terms so far neglected explicitly into account in the neighborhood of S_a in equation (111).

In the asymptotic domain, $\mathcal{D}$ in equation (112) reduces from equations (122), (125), and (127) to

$$\mathcal{D}_{\rm a} = u_{\rm p}^2 D_{\rm a} - \gamma^2 \frac{\zeta}{\zeta_{\rm B}} = \left(1 - \frac{\zeta_{\rm A}}{\zeta_{\rm B}}\right) F_{\rm a}\left(\frac{\mu_\varepsilon}{\gamma}\right), \tag{133}$$

where $F_{\rm a}(x)$ is given by equation (130). On the other hand, $\mathcal{N}$ becomes by equations (69), (113), (121), and (123)–(125)

$$\frac{\varpi \mathcal{N}_{\rm a}}{D_{\rm a}} = u_{\rm p}^2 \frac{\partial \ln \sigma}{\partial \ln \varpi} - u_{\rm t}^2 + 2\mu_\delta \frac{c\, u_{\rm t}}{\alpha \varpi}\frac{1}{D_{\rm a}^2}\frac{\zeta}{\zeta_{\rm B}}$$
$$= u_{\rm p}^2 \frac{\partial \ln \sigma}{\partial \ln \varpi} + \mu_\varepsilon^2 \frac{\varpi_{\rm A}^2}{\varpi^2} G_{\rm a}, \tag{134}$$

where

$$G_{\rm a}(w) = \frac{w}{(1+w)^3}\left[1 - \left(\frac{\zeta_{\rm B}}{\zeta_{\rm A}} - 1\right) w\right] \times \left[(w+3) - \frac{\zeta_{\rm A}}{\zeta_{\rm B}}\frac{(w+1)^2}{w}\right]. \tag{135}$$

In the above derivation of $\mathcal{D}_{\rm a}$, we may well neglect the term $u_{\rm t}^2$ in γ in equation (112), and hence all terms of order $O(\varpi^{-2})$. But for $\mathcal{N}_{\rm a}$ to vanish somewhere near $S_{\rm a}$, one must take into account the terms with an order of $O(\varpi^{-2})$ through $u_{\rm t}$ in equation (134). This is of crucial importance to locate the $S_{\rm F}$ at correct distances, $\varpi_{\rm F}$, and to connect the asymptotic solutions causally to the solutions upstream, by making the points into X-type *critical* points.

5.2.1. The eigenvalues

By imposing $\mathcal{D}_{\rm a} = 0$ or $F_{\rm a}(\mu_\varepsilon/\gamma) = 0$ in equations (133), (122), and (130), as already shown in subsection 5.1, we obtain

$$\gamma_{\rm F} = \mu_\varepsilon^{1/3}, \tag{136}$$

$$\frac{\zeta_{\rm F}}{\zeta_{\rm B}} = \left(1 - \frac{\zeta_{\rm A}}{\zeta_{\rm B}}\right)(\mu_\varepsilon^{2/3} - 1), \tag{137}$$

$$w_{\rm F} = \gamma_{\rm F}^2 - 1 = \mu_\varepsilon^{2/3} - 1. \tag{138}$$

Then from equations (125), (137), (51), and (68)

$$\zeta_{\rm F} = \frac{4\pi\eta\,\mu_\delta\, c^2}{\alpha}(\gamma_{\rm F}^2 - 1), \tag{139}$$

$$u_{\rm tF} = \frac{c}{\alpha\,\varpi_{\rm F}}(\mu_\varepsilon^{1/3} - \mu_\delta), \tag{140}$$

$$I_{\rm F} = \frac{c\,\zeta_{\rm B}}{2}\left(1 - \frac{1}{\gamma_{\rm F}^2}\right) \tag{141}$$

[see equation (157) for $\varpi_{\rm F}$]. The angular momenta of the field and the flow at $S_{\rm F}$ become

$$\frac{I_{\rm F}}{2\pi\eta c} = \frac{\mu_\varepsilon c^2}{\alpha}\left(1 - \frac{1}{\gamma_{\rm F}^2}\right), \tag{142}$$

$$(\gamma\,\varpi\,v_{\rm t})_{\rm F} = \mu_\varepsilon\,\alpha\,\varpi_{\rm A}^2 - \frac{\mu_\varepsilon c^2}{\alpha}\left(1 - \frac{1}{\gamma_{\rm F}^2}\right). \tag{143}$$

From equations (73), (74), (121), (127), and (138), we obtain the magnetization parameter at $S_{\rm F}$,

$$\sigma_{\rm F} = \frac{\alpha^2\Phi_{\rm F}}{4\pi\eta c^3} = \frac{\zeta_{\rm F}\,u_{\rm pF}}{\zeta_{\rm B} - \zeta_{\rm A}} = (\gamma_{\rm F}^2 - 1)^{3/2}, \tag{144}$$

or conversely

$$\gamma_{\rm F}^2 = 1 + \sigma_{\rm F}^{2/3}. \tag{145}$$

Also, from equation (138)

$$w_{\rm F} = \sigma_{\rm F}^{2/3}. \tag{146}$$

From equations (142) and (143), or from equations (71) and (137) one obtains for the ratio $w_{\rm AM}$ at $S_{\rm F}$

$$(w_{\rm AM})_{\rm F} = \frac{\mu_\varepsilon^{1/3}(\mu_\varepsilon^{2/3} - 1)}{\mu_\varepsilon^{1/3} - \mu_\delta} = \frac{\sigma_{\rm F}^{2/3}}{1 - \dfrac{\mu_\delta}{\sqrt{1 + \sigma_{\rm F}^{2/3}}}}. \tag{147}$$

The location of $S_{\rm A}$ is thus determined using equations (136), (145), and (73) as

$$\frac{\alpha^2\varpi_{\rm A}^2}{c^2} = 1 - \frac{\mu_\delta}{(1 + \sigma_{\rm F}^{2/3})^{3/2}} = 1 - \frac{\mu_\delta}{\left[1 + \left(\dfrac{\alpha^2\Phi_{\rm F}}{4\pi\eta c^3}\right)^{2/3}\right]^{3/2}} \tag{148}$$

[see appendix 4 for the nonrelativistic limits of $\zeta_{\rm F}$, $u_{\rm tF}$, and $I_{\rm F}$ in equations (139)–(141) and $\varpi_{\rm A}$ in equation (148)].

Then, the eigenvalue $-\beta = \zeta_{\rm A}$ becomes from equations (58), (69), and (148)

$$-\beta = \zeta_{\rm A} = \frac{4\pi\eta c^2}{\alpha}\left\{\left[1 + \left(\frac{\alpha^2\Phi_{\rm F}}{4\pi\eta c^3}\right)^{2/3}\right]^{3/2} - \mu_\delta\right\}, \tag{149}$$

which reduces to $(\alpha/c)\,\Phi_{\rm F}$ for $\eta \to 0$ in the force-free limit and to equation (4.13) in Paper III for the limit of $c \to \infty$ [see equation (10.1a) in Okamoto (1978); see also Okamoto (2006)].

Following Michel (1969), we normalize the total angular momentum per unit flux tube by $\alpha\,\varpi_0^2$ at $S_{\rm F}$ from equations (72) and (73),

$$\lambda_{\rm F} \equiv \frac{\mu_\varepsilon\,\alpha\,\varpi_{\rm A}^2}{\alpha\,\varpi_0^2} = \frac{\mu_\delta}{\sigma_{\rm F}}\frac{\dfrac{\alpha^2\varpi_{\rm A}^2}{c^2}}{1 - \dfrac{\alpha^2\varpi_{\rm A}^2}{c^2}}, \tag{150}$$

or conversely

$$\frac{\alpha^2\varpi_{\rm A}^2}{c^2} = \frac{\sigma_{\rm F}\lambda_{\rm F}}{\mu_\delta + \sigma_{\rm F}\lambda_{\rm F}}. \tag{151}$$

One then obtains from equations (51) and (145)

$$\gamma_{\rm F}^3 = (1 + \sigma_{\rm F}^{2/3})^{3/2} = \mu_\varepsilon = \frac{\mu_\delta}{1 - \dfrac{\alpha^2\varpi_{\rm A}^2}{c^2}} = \mu_\delta + \sigma_{\rm F}\lambda_{\rm F}. \tag{152}$$

Equation (148) gives $\varpi_{\rm A}^2$ explicitly in terms of α, η, μ_δ, and $\Phi_{\rm F}$. Thus, $\varpi_{\rm A}^2$ can be determined by fixing $\Phi_{\rm F}$ at $S_{\rm F}$, as described in the next subsubsection. Then, substituting equation (151) into (150) gives

$$\lambda_{\rm F} = \frac{(1 + \sigma_{\rm F}^{2/3})^{3/2} - \mu_\delta}{\sigma_{\rm F}}, \tag{153}$$

and also from equations (139) and (152)

$$\frac{\zeta_{\rm F}}{\zeta_{\rm B}} = \frac{\mu_\delta\, \sigma_{\rm F}^{2/3}}{(1+\sigma_{\rm F}^{2/3})^{3/2}}. \tag{154}$$

For the ultrarelativistic case of $\sigma_{\rm F} \gg 1$ or the force-free limit of $\eta \to 0$ with $\alpha^2\varpi_0^2/c^2 \gg 1$, then $\lambda_{\rm F} \simeq 1$, $\mu_\varepsilon \simeq \sigma_{\rm F} \gg 1$, $\alpha^2\varpi_{\rm A}^2/c^2 \simeq 1$, $\zeta_{\rm A}/\zeta_{\rm B} \simeq 1-\mu_\delta/\sigma_{\rm F}$ and $\zeta_{\rm F}/\zeta_{\rm B} \simeq \mu_\delta/\sigma_{\rm F}^{1/3}$. Thus, $(w_{\rm AM})_{\rm F} \approx w_{\rm F} \gg 1$. Negligible angular momentum has been transferred from the field to the flow down to $S_{\rm F}$.

5.2.2. *The location of the fast surface S_F*

To determine the location of $S_{\rm F}$, itself, in the present asymptotic formalism, we must make use of equation $\mathcal{N}_{\rm a} = 0$ with the terms of order $\varpi_{\rm A}^2/\varpi^2$ retained. From equations (134)–(138), one then obtains the most crucial condition,

$$\left(\frac{\mathcal{N}_{\rm a}}{D_{\rm a}}\right)_{\rm F} = (\gamma_{\rm F}^2-1)\left(\frac{\partial\ln\sigma}{\partial\ln\varpi}\right)_{\rm F} + \mu_\varepsilon^2\frac{\varpi_{\rm A}^2}{\varpi_{\rm F}^2}G_{\rm a}(w_{\rm F}) = 0, \tag{155}$$

where

$$G_{\rm a}(w_{\rm F}) = \frac{(\gamma_{\rm F}^2-1)}{\gamma_{\rm F}^6}\left[1-\left(1-\frac{\zeta_{\rm A}}{\zeta_{\rm B}}\right)\gamma_{\rm F}^2\right] \times\left[(\gamma_{\rm F}^2+2)-\frac{\zeta_{\rm A}}{\zeta_{\rm B}}\frac{\gamma_{\rm F}^4}{\gamma_{\rm F}^2-1}\right]. \tag{156}$$

Thus, by using equations (156) and (136), the gradient of $\ln\sigma$ at $S_{\rm F}$ is given by equation (155) as

$$\frac{\alpha^2\varpi_{\rm F}^2}{c^2} = \gamma_{\rm F}^2(\gamma_{\rm F}^2-1)\left(\frac{\dfrac{\alpha^2\varpi_{\rm A}^2}{c^2}}{1-\dfrac{1}{\gamma_{\rm F}^2}}-1\right) \times\left[\frac{2}{\gamma_{\rm F}^2}-\left(\frac{\dfrac{\alpha^2\varpi_{\rm A}^2}{c^2}}{1-\dfrac{1}{\gamma_{\rm F}^2}}-1\right)\right]\left|\frac{\partial\ln\varpi}{\partial\ln\sigma}\right|_{\rm F} \tag{157}$$

(see appendix 4 for the nonrelativistic limit of $\varpi_{\rm F}^2$). This is the critical condition by which one can locate the fast surface, $S_{\rm F}$, where the steady axisymmetric outflow must smoothly pass through in the *prescribed* field structure, and therefore to determine the eigenvalues of the flow parameters in terms of input values of α, η, and μ_δ given at the source surface as the boundary conditions. That is, by inserting $\gamma_{\rm F}$ and $\varpi_{\rm A}^2/\varpi_{\rm L}^2$ from equations (145) and (148) into (157), we obtain one equation to determine the eigenvalues $\sigma_{\rm F}$ and $\varpi_{\rm F}$ self-consistently from the *prescribed* field structure with $\sigma = \sigma(\varpi; P)$.

For the present formalism to be valid, not only condition $(\gamma\,\varpi\,v_{\rm t})_{\rm F} > 0$ holds in equation (143), but also the square brackets on the right-hand side of equation (157) must be positive; that is, one obtains

$$\left(1-\frac{1}{\gamma_{\rm F}^2}\right) < \frac{\alpha^2\varpi_{\rm A}^2}{c^2} < \left(1-\frac{1}{\gamma_{\rm F}^2}\right)\left(1+\frac{2}{\gamma_{\rm F}^2}\right). \tag{158}$$

Equation (158) is one of the necessary conditions for the presence of $S_{\rm F}$. If $\gamma_{\rm F}^2 > 2$, the upper limit of $\alpha^2\varpi_{\rm A}^2/c^2$ is given by unity.

Begelman and Li (1994) numerically calculated $\delta_{\rm F} \equiv (d\ln\sigma/d\ln\varpi)_{\rm F}$ in terms of $t_{\rm F} \equiv (\alpha\,\varpi_{\rm F}/c)^{-1}$ for three different values of w = 10, 10^2, and 10^5. Their figure 2(a) seems to well depict the relation analytically given in equation (157). Takahashi and Shibata (1998) adopted $\Phi \propto \varpi^{-0.4}$, i.e., $(\partial\ln\sigma/\partial\ln\varpi)_P = -0.4$, to show that a transfast MHD wind is possible (see also subsection 6.5).

Thus, $(\varpi_{\rm F}^2/\varpi_{\rm L}^2)\,|\partial\ln\sigma/\partial\ln\varpi|_{\rm F}$ is finite at $S_{\rm F}$, by equation (157). If one assumes the split-monopolar structure, as has often been done so far (see, e.g., Michel 1969; Goldreich, Julian 1970), then $(\partial\ln\sigma/\partial\varpi)_P = 0$ everywhere, and hence one must take $\varpi_{\rm F}^2 \to \infty$; that is, one must regard $S_{\rm F}$ as being at infinity. There is, however, no physical reason to have to *a priori* suppose that $(\partial\ln\sigma/\partial\ln\varpi)_{\rm F} = 0$ and $\varpi_{\rm F}^2 \to \infty$. In Okamoto (1978), we mistakenly took $\varpi_{\rm F}^2 \to \infty$, in spite of presuming a "general" field with $(\partial\ln\sigma/\partial\ln\varpi)_{\rm F} \neq 0$. No radial field model can accommodate $S_{\rm F}$ at finite distances for cold winds.

5.3. *The Gradients of ζ, w, etc., at S_F*

The gradient of each quantity in all of the wind domains of $S_{\rm B} \lesssim S \lesssim S_\infty$ can in principle be related to

$$\nu \equiv -\frac{\partial\ln\zeta}{\partial\ln\varpi^2} = -\frac{\varpi}{2}\frac{\mathcal{N}}{\mathcal{D}}, \tag{159}$$

except for the critical surfaces, where both $\mathcal{D}$ and $\mathcal{N}$ in equations (112) and (113) vanish, and one must use l'Hôpital's rule. As shown so far, independent coordinate variables ϖ and z in the asymptotic domain of $S \gtrsim S_{\rm a}$ appear through the quantity σ only, under the assumption of magnetic fluxes continuously extending from $S_{\rm B}$ to S_∞. It is thus possible to apply the asymptotic formalism, not only to fix the location of $S_{\rm F}$, but to fix the gradient $\nu_{\rm F}$ at $S_{\rm F}$, as follows.

From the criticality condition $\mathcal{N}_{\rm a} = 0$ at $S_{\rm F}$, the equation (157) has been obtained between the location of $S_{\rm F}$, $\varpi = \varpi_{\rm F}$, and the gradient of σ there. For both the *physical* and *unphysical* solutions of $w = w(\sigma)$ in $\sigma < \sigma_{\rm F}$ to pass smoothly through $S_{\rm F}$ to join those in $\sigma > \sigma_{\rm F}$, one must determine the finite gradients of $(\partial\ln w/\partial\ln\sigma)_{\rm F}$, by applying l'Hôpital's rule to equation (111); that is, one must use at $S_{\rm F}$

$$\frac{\partial\ln\zeta}{\partial\varpi}\frac{\partial\mathcal{D}_{\rm a}}{\partial\varpi} = \frac{\partial\mathcal{N}_{\rm a}}{\partial\varpi}, \tag{160}$$

where $\mathcal{D}_{\rm a}$ and $\mathcal{N}_{\rm a}$ are given in equations (133)–(135). After some manipulation (see appendix 3), one arrives at a quadratic equation for $(\partial\ln w/\partial\ln\sigma)_{\rm F}$ or $(\partial\ln\zeta/\partial\ln\sigma)_{\rm F}$, which reads

$$\left(\frac{\partial\ln\zeta}{\partial\ln\sigma}\right)_{\rm F}^2 - 2K_1\left(\frac{\partial\ln\zeta}{\partial\ln\sigma}\right)_{\rm F} + K_2 = 0, \tag{161}$$

$$K_1 \equiv \frac{1}{2}+\frac{\gamma_{\rm F}^2}{3}-\frac{1}{6}\frac{\dfrac{\zeta_{\rm A}}{\zeta_{\rm B}}\gamma_{\rm F}^2}{1-\left(1-\dfrac{\zeta_{\rm A}}{\zeta_{\rm B}}\right)\gamma_{\rm F}^2} + \frac{\gamma_{\rm F}^2}{6}\frac{1+2\left(1-\dfrac{\zeta_{\rm A}}{\zeta_{\rm B}}\right)\gamma_{\rm F}^2}{\gamma_{\rm F}^2+2-\dfrac{\zeta_{\rm A}}{\zeta_{\rm B}}\dfrac{\gamma_{\rm F}^4}{\gamma_{\rm F}^2-1}}, \tag{162}$$

$$K_2 \equiv \gamma_F^2 \left[\frac{2}{3} + \frac{1}{\varpi} \left(\frac{\partial \varpi}{\partial \ln \sigma} \right) + \frac{1}{3} \left(\frac{\partial \varpi}{\partial \ln \sigma} \right)^2 \left(\frac{\partial^2 \ln \sigma}{\partial \varpi^2} \right) \right]_F , \tag{163}$$

where we note that γ_F, $\zeta_A/\zeta_B = \alpha^2 \varpi_A^2 / c^2$, $(\partial\varpi/\partial\ln\sigma)_F$, and $(\partial^2 \ln\sigma/\partial\varpi^2)_F$ are already known. It seems likely that the absolute value of the second term in the square bracket for K_2 is larger than the other two terms, and hence $K_2 < 0$. We may thus expect $K_1^2 - K_2 > 0$, ensuring that there are two real roots in equation (161) (see subsection 6.5 for a ultrarelativistic wind).

Solving the quadratic equation (161), one obtains two values of the slope, $\nu_F \equiv -(\partial\ln\zeta/\partial\ln\varpi^2)_F = -(\partial\ln\zeta/\partial\ln\sigma)_F \times (\partial\ln\sigma/\partial\ln\varpi^2)_F$, with a positive sign for the *physical* branch and a negative one for the *unphysical* solution [cf. ν_A in equation (84)]. The sign for the slope of γ or u_p is opposite to that of ζ between these two branches.

5.4. *The Equipartition Surfaces, S_{EQ} and S_{eq}*

One of our main concerns in MHD outflows is MHD acceleration, that is, how efficiently the magnetic energy and angular momentum are transferred to those of the flow. One of the parameters used to measure the efficiency is the ratio w or w_{AM}. Thus, the locations of S_{EQ} where $w = 1$, and S_{eq} where $w_{AM} = 1$ have physical meanings. From equations (146) and (147), one obtains $(w_{AM})_F > w_F = \sigma_F^{2/3} = \gamma_\infty^{2/3} - 1$, which indicates $w_F >$ or $\gg 1$ for $\gamma_F >$ or $\gg \sqrt{2}$; that is, for example, the Poynting flux is still much larger than the kinetic flux at S_F. Thus, the two surfaces S_{EQ} and S_{eq} are located far beyond S_F in the asymptotic domain.

From equations (98), (67), and (68), for $\varpi_{EQ}^2 \gg \varpi_A^2$, one obtains

$$\zeta_{EQ} = \zeta_B - \zeta_A = \zeta_B \frac{\mu_\delta}{\mu_\varepsilon} = \frac{4\pi\,\eta\,\mu_\delta\, c^2}{\alpha}. \tag{164}$$

The ratio of ζ_{EQ} to ζ_F becomes from equations (139) and (164)

$$\frac{\zeta_{EQ}}{\zeta_F} = \frac{1}{\mu_\varepsilon^{2/3} - 1} = \frac{1}{\gamma_\infty^{2/3} - 1}, \tag{165}$$

and hence $\zeta_{EQ} <$ or $\ll \zeta_F$. Also, by putting $w_{EQ} = 1$, from equations (122), (123), (124), (126), and (128)

$$\gamma_{EQ} = \frac{\mu_\varepsilon}{2}, \tag{166}$$

$$(w_{AM})_{EQ} = \frac{1}{2\dfrac{\zeta_A}{\zeta_B} - 1} = \frac{1}{1 - 2\dfrac{\mu_\delta}{\mu_\varepsilon}} > 1, \tag{167}$$

$$\sigma_{EQ} = \sqrt{\frac{\mu_\varepsilon^2}{4} - 1}. \tag{168}$$

The ratio $(w_{AM})_{EQ}$ is still larger than unity at S_{EQ}. Thus, one sees that $\gamma_{EQ} = \gamma_F^3/2 >$ or $\gg \gamma_F$ from equation (136), which means $S_{EQ} >$ or $\gg S_F$.

From equation (71) for $w_{AM} = 1$ one obtains for $\varpi_{eq}^2 \gg \varpi_A^2$

$$\zeta_{eq} = \zeta_A \frac{\left(1 - \dfrac{\zeta_A}{\zeta_B}\right)}{\left(2 - \dfrac{\zeta_A}{\zeta_B}\right)} = \zeta_{EQ} \frac{\left(1 - \dfrac{\mu_\delta}{\mu_\varepsilon}\right)}{\left(1 + \dfrac{\mu_\delta}{\mu_\varepsilon}\right)}. \tag{169}$$

Then, by substituting $\zeta = \zeta_{eq}$ from equation (169) into (121), one obtains

$$w_{eq} = \frac{\dfrac{\zeta_A}{\zeta_B}}{2 - \dfrac{\zeta_A}{\zeta_B}} = 1 - \frac{2\mu_\delta}{\mu_\varepsilon + \mu_\delta} < 1, \tag{170}$$

and then from equations (122) and (128)

$$\gamma_{eq} = \frac{1}{2}(\mu_\varepsilon + \mu_\delta), \tag{171}$$

$$\sigma_{eq} = \frac{\mu_\varepsilon - \mu_\delta}{2} \sqrt{1 - \left(\frac{2}{\mu_\varepsilon + \mu_\delta} \right)^2}. \tag{172}$$

Thus, $\zeta_{EQ} > \zeta_{eq}$, which means that $\varpi_{EQ} < \varpi_{eq}$; in other words, the equipartition of energy is accomplished earlier than that of angular momentum.

The actual axial distances, ϖ_{EQ}, and ϖ_{eq}, must be sought not from equations (67) and (71), but from equations (168) and (170) for a specified or solved function, $\sigma = \sigma(\varpi, P)$, in this asymptotic formalism.

6. Transfield Force Balance in the Asymptotic Domain

It is argued in the previous section that the flow properties in the asymptotic, superfast domain can be described in terms of the magnetization parameter, $\sigma(\varpi, P)$. One can estimate various quantities as a two-valued function of σ. For example, the ratio of the Poynting flux to the kinetic energy flux, w, has two branches, *physical* and *unphysical*. Along the *physical* branch, w decreases nearly linearly with σ, like $w \simeq \sigma/\sqrt{\mu_\varepsilon^2 - 1}$ for $\sigma \to 0$, by equation (128). To connect the solutions in $\sigma \le \sigma_F$ with those upstream in $\sigma > \sigma_F$, one must rigorously treat the criticality condition with ϖ_A^2/ϖ^2, which are taken into account in equation for $\mathcal{N}_a = 0$, at S_F, because S_F is itself situated between the asymptotic domain and the subasymptotic domain. To determine $\sigma(\varpi, P)$, itself, one must solve the equation describing the transfield force balance, i.e., the transfield equation (33) or (36), consistently with these flow properties, yielding a decreasing $\sigma(\varpi)$ for $\varpi \to \infty|_P$.

6.1. *Properties of Transfield Force Balance*

At first we derive the asymptotic forms for R, j_t and ϱ_e from equations (116)–(118) and (120). The following relations from equations (52) and (124)–(127) are useful in the asymptotic domain:

$$M^2 = \frac{\zeta_B - \zeta_A}{\zeta} \frac{\alpha^2 \varpi^2}{c^2}, \tag{173}$$

$$M^2 + \frac{\alpha^2 \varpi^2}{c^2} \approx \frac{\alpha^2 \varpi^2}{c^2} \frac{\zeta_B}{\zeta} D_a, \tag{174}$$

$$\rho\gamma v_p^2 = \frac{\Phi^2}{4\pi\varpi^4} M^2, \tag{175}$$

$$\frac{B_t}{B_p} \approx -\frac{\alpha\varpi}{c} \frac{c}{\alpha\Phi} \frac{\zeta}{D_a}, \tag{176}$$

$$\frac{\alpha^2\Phi^2}{c^2} \approx \frac{\zeta^2}{D_a^2} - \frac{\zeta^2}{\mu_\delta^2} = u_p^2 \frac{\zeta^2}{\mu_\delta^2}. \tag{177}$$

We can then easily obtain from equations (116)–(118)

$$\frac{\varpi}{R} \approx \frac{c^2}{2\alpha^2\Phi}\frac{1}{D_\mathrm{a}}\frac{\zeta}{\zeta_\mathrm{B}}\frac{\partial}{\partial P}\frac{\zeta^2}{\mu_\delta^2}, \tag{178}$$

$$\frac{1}{c}j_\mathrm{t}B_\mathrm{p} \approx -\frac{\Phi}{8\pi\varpi^3}\frac{c^2}{\alpha^2\varpi^2}\left[\frac{1}{D_\mathrm{a}}\frac{\zeta}{\zeta_\mathrm{B}}\frac{\partial}{\partial P}\frac{\zeta^2}{\mu_\delta^2} + \frac{\alpha^2\Phi^2}{c^2}\frac{\partial \ln B_\mathrm{p}^2}{\partial P}\right], \tag{179}$$

$$\varrho_\mathrm{e}E_\mathrm{p} \approx \frac{\Phi}{8\pi\varpi^3}\left[\frac{(1-\zeta_\mathrm{A}/\zeta_\mathrm{B})}{D_\mathrm{a}}\frac{\partial}{\partial P}\frac{\zeta^2}{\mu_\delta^2} - \frac{\partial}{\partial P}\frac{\zeta^2}{D_\mathrm{a}^2}\right] \tag{180}$$

[see equation (A45) in appendix 4 for the nonrelativistic limit of equation (178)]. Also, from equations (19) and (176)

$$\frac{1}{c}j_\parallel B_\mathrm{t} = -\frac{\Phi}{8\pi\varpi^3}\frac{\partial}{\partial P}(\varpi B_\mathrm{t})^2 \approx -\frac{\Phi}{8\pi\varpi^3}\frac{\partial}{\partial P}\frac{\zeta^2}{D_\mathrm{a}^2}. \tag{181}$$

Substitution of $\partial(\zeta^2/\mu_\delta^2)/\partial P$ from equation (178) and $\partial(\zeta^2/D_\mathrm{a}^2)/\partial P$ from equations (181) into (180) reproduces the original form of the force balance in the transfield direction; that is,

$$\frac{\rho\gamma v_\mathrm{p}^2}{R} \approx \varrho_\mathrm{e}E_\mathrm{p} + \frac{1}{c}j_\parallel B_\mathrm{t}, \tag{182}$$

which naturally coincides with the asymptotic expression (33), because $\rho\gamma v_\mathrm{t}^2|\partial\ln\varpi/\partial n|$ and $|j_\mathrm{t}B_\mathrm{p}/c| \ll |j_\parallel B_\mathrm{t}/c|$ and $|\varrho_\mathrm{e}E_\mathrm{p}|$ by a factor of $\alpha^2\varpi^2/c^2$. Thus, similarly, equation (180) for ϱ_e expresses the original force balance, itself. On the other hand, dropping the term $B_\mathrm{p}^2/4\pi\rho\gamma v_\mathrm{p}^2$ of the left-hand side as well as the first two terms of the right-hand side in equation (36), gives

$$\frac{\rho\gamma v_\mathrm{p}^2}{R}\left[1 + \frac{1}{M^2}\frac{\alpha^2\varpi^2}{c^2}\right] \approx \frac{1}{c}j_\parallel B_\mathrm{t} + \frac{\alpha^2\varpi^2}{c^2}\frac{B_\mathrm{p}^2}{4\pi}\frac{\partial}{\partial n}\ln\alpha B_\mathrm{p}\varpi^2. \tag{183}$$

Making use of equations (3), (4), (52), (55), and (124), one can easily show that the two expressions in (182) and (183) are equivalent to each other, and also one can derive equation (178) from (183).

The curvature of a field line at S_F becomes, when using equations (136)–(139) and (144),

$$\frac{1}{R_\mathrm{F}} = \frac{c}{\alpha\varpi_\mathrm{F}}\frac{\sqrt{\gamma_\mathrm{F}^2-1}}{\gamma_\mathrm{F}^2}\frac{\partial}{\partial P}\left(\frac{\zeta}{\mu_\delta}\right)_\mathrm{F}, \tag{184}$$

which indicates that because the fast surface must locate at finite distances, the curvature radius is also finite there, and therefore the field structure cannot be even locally radial. This is consistent with the condition of ongoing acceleration, for which field lines carrying energy/angular momentum must be curved with finite curvature.

Finally, we give two more expressions for the transfield force balance in the form of second-order partial differential equation for P. Utilizing relations (4), (7), (68), (72), and (73), it is not difficult to derive

$$\nabla^2 P - \nabla P\cdot\nabla\ln\varpi|\nabla P| = \frac{1}{\left(1+\dfrac{\zeta}{\zeta_\mathrm{B}-\zeta_\mathrm{A}}\right)}\frac{\sigma}{u_\mathrm{p}^3}\nabla P\cdot\nabla\ln\frac{\alpha\Phi}{c\,u_\mathrm{p}}, \tag{185}$$

where u_p is given in terms of ζ in equations (127) and (121). The left-hand side of equation (185) is equal to $|\nabla P|/R$, and the right-hand side is proportional to the magnetization parameter, σ. If expression (8) is used for $1/R$, the transfield equation reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial P}{\partial r}\right) - \frac{1}{2\sin^2\theta}\frac{\partial}{\partial P}\left(\varpi\frac{\partial P}{\partial r}\right)^2 = \frac{1}{4\pi\eta\alpha}\frac{1}{3\sin^2\theta}\frac{\partial}{\partial P}\left(\frac{\alpha\Phi}{cu_\mathrm{p}}\right)^3 = \frac{1}{4\pi\eta\alpha}\frac{1}{3\sin^2\theta}\frac{\partial}{\partial P}\left(\frac{\zeta}{\mu_\delta}\right)^3. \tag{186}$$

If $P = P(\theta)$, obviously $1/R = 0$ and $\alpha\Phi/cu_\mathrm{p} = \zeta/\mu_\delta =$ constant.

It is worth emphasizing that the expressions given in equations (178), (182), (183), (185), (186), and (193) are all equivalent to each other for describing the field-flow topology in the asymptotic domain. This means that even if one does not necessarily seek a perfect solution to the second-order partial differential equation (185), or (186), one may be able to elucidate physically important properties of the MHD acceleration-collimation/decollimation of outflows in the asymptotic domain, as shown in the next subsections.

6.2. *The Current Function and MHD Acceleration*

Let us here consider the role of the current function, I, in the asymptotic domain. All of the physical quantities of interest can be made to be related to I. For example, w and ζ become by equations (121) and (124) as follows:

$$w = \frac{\dfrac{2I}{c\zeta_\mathrm{B}}}{1-\dfrac{2I}{c\zeta_\mathrm{B}}}, \tag{187}$$

$$\zeta = (\zeta_\mathrm{B}-\zeta_\mathrm{A})\frac{\dfrac{2I}{c\zeta_\mathrm{B}}}{1-\dfrac{2I}{c\zeta_\mathrm{B}}}. \tag{188}$$

Then, from equations (128) and (187), a relation is obtained between σ and I,

$$\sigma = \mu_\varepsilon\frac{\dfrac{2I}{c\zeta_\mathrm{B}}}{1-\dfrac{2I}{c\zeta_\mathrm{B}}}\sqrt{\left(1-\frac{2I}{c\zeta_\mathrm{B}}\right)^2 - \frac{1}{\mu_\varepsilon^2}}. \tag{189}$$

This expression describes the current distribution in the asymptotic domain as a function of the "independent variables" σ and P (through ζ_B and μ_ε). For example, a curve of $I(\sigma, P) =$ constant depicts the topology of a "current line", along which the poloidal current $\boldsymbol{j}_\mathrm{p}$ flows in the domain of $S \gtrsim S_\mathrm{a}$. Also, this gives the behavior of I as a function of σ for a fixed P, which is more or less similar to that shown in figures 4 and 5.

Needless to say that, just like in an ordinary DC circuit, every current line in the asymptotic domain must continually and causally be connected, beyond $S_{\rm F} \approx S_{\rm a}$, passing through the Alfvénic surface, $S_{\rm A}$, to the unipolar battery on the magnetospheric-base surface, $S_{\rm B}$, defined in equation (14). However, as already shown, the solution of equation (189) yields infinitely steep gradients at $I = I_{\rm F}$, where $I_{\rm F}$ is the value of I at $S_{\rm F}$, given by equation (141). In reality, differentiation with σ yields

$$\left(\frac{\partial I}{\partial \sigma}\right)_P = \frac{I}{\sigma}\frac{\left(1-\frac{2I}{c\zeta_{\rm B}}\right)\left[\left(1-\frac{2I}{c\zeta_{\rm B}}\right)^2-\frac{1}{\mu_\varepsilon^2}\right]}{\left(1-\frac{2I}{c\zeta_{\rm B}}\right)^3-\frac{1}{\mu_\varepsilon^2}}, \tag{190}$$

which indicates $(\partial I/\partial\sigma)_{\rm F} = \pm\infty$ at $I = I_{\rm F}$ and $\sigma = \sigma_{\rm F}$. As done in subsection 5.3, to connect the solutions of equation (189) in the domain of $S \gtrsim S_{\rm a}$ causally with those in the domain of $S < S_{\rm a}$, one must give finite gradients to the solutions, to make an X-type crossing at $S_{\rm F}$, and to relate to the gradient $\nu_{\rm F} = -(\partial \ln\zeta/\partial \ln\varpi^2)_{\rm F}$, as determined by the solutions of equation (161). Using equation (188), one can relate the gradient of $(\partial \ln I/\partial \ln\sigma)_{\rm F}$ to $\nu_{\rm F}$ as

$$\left(\frac{\partial \ln I}{\partial \ln\sigma}\right)_{\rm F} = \frac{1}{\mu_\varepsilon^{2/3}}\left(\frac{\partial \ln\zeta}{\partial \ln\sigma}\right)_{\rm F} = -\frac{\nu_{\rm F}}{\mu_\varepsilon^{2/3}}\left(\frac{\partial \ln\varpi^2}{\partial \ln\sigma}\right)_{\rm F}. \tag{191}$$

It thus turns out that equations (189)–(191), together with those in section 5, describe the behavior of the current function, $I = I(\sigma)$, along each field line in the domain from $S_{\rm F} \approx S_{\rm a}$ to S_∞, when Φ or σ is given as a slowly decreasing function of s or $\varpi|_P$.

Let us next examine how the acceleration of the flow is coupled with the field/current topology. Substitution of $\alpha\Phi/c$ from equation (177) into equation (178) yields for $k \equiv \varpi/R$ along each field line in terms of w and ζ

$$k \equiv \frac{\varpi}{R} = \frac{c}{\alpha}\frac{w}{\sqrt{\mu_\varepsilon^2-(1+w)^2}}\frac{\partial}{\partial P}\left(\frac{\zeta_{\rm B}\, w}{\mu_\varepsilon}\right) \tag{192}$$

$$= \frac{\frac{c}{\alpha}\frac{\zeta}{(\zeta_{\rm B}-\zeta_{\rm A})}}{\sqrt{\mu_\varepsilon^2-\left(1+\frac{\zeta}{\zeta_{\rm B}-\zeta_{\rm A}}\right)^2}}\frac{\partial}{\partial P}\left(\frac{\zeta}{\mu_\delta}\right). \tag{193}$$

Also, expressing in terms of γ and I using of equations (121), (122), and (124), equation (193) becomes

$$k = \frac{c}{\alpha}\frac{\left(1-\frac{\gamma}{\mu_\varepsilon}\right)}{\sqrt{\gamma^2-1}}\frac{\partial}{\partial P}\left[\frac{\zeta_{\rm B}}{\mu_\varepsilon}\left(\frac{\mu_\varepsilon}{\gamma}-1\right)\right] \tag{194}$$

$$= \frac{c}{\mu_\varepsilon\alpha}\frac{\left(\frac{2I}{c\zeta_{\rm B}}\right)}{\sqrt{\left(1-\frac{2I}{c\zeta_{\rm B}}\right)^2-\frac{1}{\mu_\varepsilon^2}}}\frac{\partial}{\partial P}\left(\frac{4\pi\eta c^2}{\alpha}\frac{\frac{2I}{c\,\zeta_{\rm B}}}{1-\frac{2I}{c\,\zeta_{\rm B}}}\right). \tag{195}$$

Note that $k \equiv \varpi/R$ is in general a function of not only P, but also s, through ζ, γ, or I and is hence closely related to the magnetization parameter, σ.

It can be seen in equations (131), (189), (194), and (195) that $\varpi/R \to 0$ for $\gamma \to \gamma_\infty \equiv \mu_\varepsilon$, $I \to 0$, and $\sigma \to 0$, or conversely $\gamma \to \mu_\varepsilon$, $I \to 0$, and $\sigma \to 0$ for $\varpi/R \to 0$. This means that as the Poynting flux decreases outwardly together with I, w, and ζ and the kinetic flux increases with γ, the flow becomes more and more ballistic, and then field-streamlines become more and more straightened, i.e., $\varpi/R \to 0$. This is because, as a consequence of ongoing field-flow interactions in the asymptotic domain, the magnetic energy is converted to particle energy, which in turn makes particles more and more ballistic, and hence field-streamlines are elongated (Okamoto 2003).

It is easy to show for the nonrelativistic limit of $c \to \infty$ that equation (193) reduces to equations (6.11) and (6.12) in Paper I and (3.9) in Paper II [see appendix 4 for the nonrelativistic limit of equation (194)]. Also, $\varrho_{\rm e}$ in equation (180) becomes negligible, because $\zeta_{\rm A}/\zeta_{\rm B} \to 0$, $D_{\rm a} \to 1$, and $\mu_\delta \to 1$ in the limit of $c \to \infty$ and the two terms cancel each other. Thus, there is a continuous connection from the relativistic case to the nonrelativistic case, or *vice versa*. Thus, if it is implausible that the transfield component of the Lorentz force, $j_\parallel B_{\rm t}/c$, vanishes *almost exactly* in the nonrelativistic case, it will be so even in the relativistic case that the magnetic pressure force is canceled *almost exactly* by a cross-field electric force (cf. Chiueh et al. 1998; see section 7). That is, we shall find no reasonable reason to require $\varrho_{\rm e} E_{\rm p}$ to cancel out $(1/c)\, j_\parallel B_{\rm t}$ almost exactly in equation (182). If one demands an almost exact cancellation, i.e., $\varpi/R \approx 0$ in equation (178), then one obtains $\zeta/\mu_\delta \approx$ constant, which is nothing but a relativistic generalization of the "solvability condition for the conical structure at infinity" (Heyvaerts, Norman 1989; Chiueh et al. 1991). An almost exact cancellation leads to $1/R = 0$, and thereby to unreasonable prohibition of MHD interactions between the field and the flow, leading to no acceleration and no collimation/decollimation.

6.3. *Collimation/Decollimation*

MHD acceleration is a cause as well as a result of a straightening of the field lines, i.e., $1/R \to 0$, mainly in the superfast domain. This means that even there, the curvature of the field lines possesses a finite value, positive or negative, however small it may be becoming. In the range of field lines with $1/R = \partial\psi/\partial s > 0$, field-streamlines collimate toward the axis, while in the range of $1/R < 0$, field-streamlines decollimate toward the equator. Then, assuming for the sake of simplicity that $k = \varpi/R$ does not change sign along each field line, let us refer to the field line separating the range with $\varpi/R = k > 0$ from that with $\varpi/R = k < 0$ as the separatrix field line $P_{\rm x}$. Then, $P_{\rm x}$ is defined by the condition $k = 0$ in equation (193) or (195), i.e.,

$$\frac{\partial \ln I}{\partial P} = \frac{\mu_\varepsilon'}{\mu_\varepsilon} + \frac{2I}{c\,\zeta_{\rm B}}\frac{d}{dP}\ln\frac{\zeta_{\rm B}}{\mu_\varepsilon}, \quad \text{or} \quad \frac{\partial \ln\zeta}{\partial P} = \frac{\mu_\delta'}{\mu_\delta}. \tag{196}$$

On the other hand, we have so far used the current-closure condition within the wind zone in the steady state. Then, in the asymptotic domain, from equations (22) and (124), one obtains $I(s, 0) = I(s, \overline{P}) = 0$ at an arbitrary surface with constant s:

$$\frac{2I}{c} = \frac{\zeta}{D_{\rm a}(P,\zeta)} = 0, \quad \text{at } P = 0,\ \overline{P}. \tag{197}$$

Hence, we must presume the existence of a "neutral" field line denoted by $P = P_{\rm n}$, which is given by the solution of equation $j_{\|} = 0$, i.e., by equations (19), (121), and (124):

$$\frac{\partial \ln I}{\partial P} = 0, \quad \text{or} \quad \frac{\partial \ln \zeta}{\partial P} = -\frac{\zeta_{\rm A}' \zeta_{\rm B} - (\zeta_{\rm A} - \zeta)\zeta_{\rm B}'}{\zeta_{\rm B}(\zeta_{\rm B} - \zeta_{\rm A})}. \tag{198}$$

Then, $j_{\|} < 0$ in the range of $0 < P < P_{\rm n}$ and $j_{\|} > 0$ in the range of $P_{\rm n} < P < \overline{P}$. It is conceivable that $P_{\rm n} = P_{\rm n}(s)$.

The line $P_{\rm x}$ is obviously bifurcated from $P_{\rm n}$, owing to a relativistic effect due to the $\varrho_{\rm e} E_{\rm p}$ term. A more detailed investigation will be necessary on the field/current line topology in the neighborhood of $P_{\rm n}$ and $P_{\rm x}$. In the limit of $c^2 \to \infty$, $P_{\rm x} \to P_{\rm n}$. Similarly in the nonrelativistic case, one can however conclude that the "two-component" structure is one of the basic properties of stationary, axisymmetric MHD outflows with the current-closure condition fulfilled in the wind zone, with convex field lines near the axis ($1/R > 0$) and concave ones ($1/R < 0$) near the equator, separated by some "critical" field line(s) probably in between $P_{\rm n}$ and $P_{\rm x}$.

6.4. *A Simple Model*

Acceleration and collimation/decollimation are the consequences of long-range interactions between the field and the flow, which require finite, positive, or negative, values of $k = \varpi/R$, however small it may be, in the asymptotic, superfast domain. And yet, full acceleration of $\gamma_\infty = \mu_\varepsilon$ and $(\varpi v_{\rm t})_\infty = \alpha \varpi_{\rm A}^2$ is both a cause and a result of $k = \varpi/R \to 0$ for $\sigma \to 0$ as $\varpi \to \infty|_P$. Then, combining equations (193) and (9) for $\varpi/R = k$ and $\sin\psi$, it turns out that ζ in equation (55) must be a slowly decreasing function of $\ln\varpi$ along each field line. This indicates that the asymptotic domain of $\varpi^2 \gtrsim \varpi_{\rm a}^2 \gg \varpi_{\rm A}^2$, where the field-flow interactions are due to a "long-range" force, is much larger than the subasymptotic domain of $\varpi^2 \lesssim \varpi_{\rm a}^2$. This feature of MHD acceleration is consistent with the large-scaleness of observed jet phenomena, where MHD processes are probably deeply involved.

Here, by introducing a simple assumption of variable separation with respect to P and s in the asymptotic domain,

$$\zeta = \zeta_{\rm a}(P)\frac{\ln s_{\rm a}}{\ln s}, \tag{199}$$

we work out the behavior of ψ with s. Substitution of equation (199) into equation (193) yields

$$k = \frac{A}{\ln s\sqrt{(\mu_\varepsilon^2 - 1)(\ln s)^2 - 2B\ln s - B^2}}, \tag{200}$$

$$A \equiv B\frac{c}{\alpha}\frac{d}{dP}\frac{\zeta_{\rm a}\ln s_{\rm a}}{\mu_\delta}, \quad B \equiv \frac{\zeta_{\rm a}\ln s_{\rm a}}{\zeta_{\rm B} - \zeta_{\rm A}}, \tag{201}$$

where both of A and B are functions of P only. Then, from equation (9) one obtains

$$\sin\psi = \sin\psi_{\rm a} + A\int_{\varpi_{\rm a}}^{\varpi} \frac{d\varpi/\varpi}{\ln s\sqrt{(\mu_\varepsilon^2 - 1)(\ln s)^2 - 2B\ln s - B^2}}\Bigg|_P. \tag{202}$$

It goes without saying that the absolute values of the integrated values must be smaller than unity in $\varpi_{\rm a} \lesssim \varpi \le \infty$, so that $|\sin\psi(\varpi)| \le 1$ holds.

Putting $d\varpi/\varpi \approx ds/s$ in equation (202), one can easily carry out integration as follows:

$$\sin\psi = \sin\psi_{\rm a} + A\Lambda(s, P), \tag{203}$$

$$\begin{aligned}\Lambda &\equiv \left[\sin^{-1}\frac{1}{\mu_\varepsilon}\left(1 + \frac{B}{\ln s_{\rm a}}\right) - \sin^{-1}\frac{1}{\mu_\varepsilon}\left(1 + \frac{B}{\ln s}\right)\right] \\ &= \left[\sin^{-1}\frac{1}{\mu_\varepsilon}\left(1 + \frac{\zeta_{\rm a}}{\zeta_{\rm B} - \zeta_{\rm A}}\right) - \sin^{-1}\frac{1}{\mu_\varepsilon}\left(1 + \frac{\zeta}{\zeta_{\rm B} - \zeta_{\rm A}}\right)\right] \\ &= \left[\sin^{-1}\frac{D_{\rm a}(\zeta_{\rm a})}{\mu_\delta} - \sin^{-1}\frac{D_{\rm a}(\zeta)}{\mu_\delta}\right],\end{aligned} \tag{204}$$

where we note $D_{\rm a}/\mu_\delta < 1$. For $s \to \infty$, $D_{\rm a}(\zeta) \to D_{\rm a}(0) = 1 - \zeta_{\rm A}/\zeta_{\rm B}$, and then

$$\sin\infty|_P = \sin\psi_{\rm a} + A\left[\sin^{-1}\frac{D_{\rm a}(\zeta_{\rm a})}{\mu_\delta} - \sin^{-1}\frac{1}{\mu_\varepsilon}\right]. \tag{205}$$

It can be seen that whether ψ increases from $\psi_{\rm a}$ or not depends on the sign of $A(P)$ in equation (201), that is, the sign of $d\zeta_{\rm a}\ln s_{\rm a}/dP \propto \partial(D_{\rm a}/\mu_\delta)/dP$ or naturally the sign of $1/R$. As shown in the previous subsection, the "critical" field line, $P_{\rm c}$, is different from the "neutral" field line, $P_{\rm n}$. It will be of no doubt that as long as the current-closure condition is fulfilled within the wind zone, both of the field lines exist near the midst of the wind zone, unless, e.g., the equatorial sheet current is considered.

6.5. *Crab-like Ultrarelativistic Winds*

We now apply the results from sections 5 and 6 to an ultrarelativistic wind, like a Crab pulsar wind with $\gamma_\infty \sim 10^6$ and $w_\infty \sim 10^{-2}$. From equation (46) one obtains $\mu_\varepsilon = \gamma_\infty \sim 10^6$ for one of the eigenvalues for a "critical" solution, and then from equation (69) one obtains

$$\frac{\zeta_{\rm A}}{\zeta_{\rm B}} = \frac{\alpha^2\varpi_{\rm A}^2}{c^2} = 1 - \frac{\mu_\delta}{\mu_\varepsilon} \sim 1 - 10^{-6}\mu_\delta, \tag{206}$$

i.e., $\varpi_{\rm A} \approx (1 - \mu_\delta/2\mu_\varepsilon)\varpi_{\rm L}$. From equations (136)–(138) and (144) at $S_{\rm F}$

$$\gamma_{\rm F} = \mu_\varepsilon^{1/3} \sim 10^2, \tag{207}$$

$$\frac{\zeta_{\rm F}}{\zeta_{\rm B}} = \frac{\mu_\delta}{\mu_\varepsilon}(\mu_\varepsilon^{2/3} - 1) \sim \frac{\mu_\delta}{\mu_\varepsilon^{1/3}} \sim 10^{-2}\mu_\delta, \tag{208}$$

$$w_{\rm F} = \mu_\varepsilon^{2/3} - 1 \sim \mu_\varepsilon^{2/3} \sim 10^4, \tag{209}$$

$$\sigma_{\rm F} = (\mu_\varepsilon^{2/3} - 1)^{3/2} \sim \mu_\varepsilon \sim 10^6. \tag{210}$$

At $S_{\rm EQ}$ with $w_{\rm EQ} = 1$ from equations (164) and (168)

$$\frac{\zeta_{\rm EQ}}{\zeta_{\rm B}} = 1 - \frac{\zeta_{\rm A}}{\zeta_{\rm B}} = \frac{\mu_\delta}{\mu_\varepsilon} \sim 10^{-6}\mu_\delta, \tag{211}$$

$$\sigma_{\rm EQ} = \sqrt{\frac{\mu_\varepsilon^2}{4} - 1} \sim \frac{\mu_\varepsilon}{2} \sim 5\times 10^5. \tag{212}$$

At $S_{\rm eq}$ with $w_{\rm AM} = 1$ from equations (169), (170), and (172)

$$\frac{\zeta_{\rm eq}}{\zeta_{\rm B}} \sim \frac{\mu_\delta}{\mu_\varepsilon}\left(1 - 2\frac{\mu_\delta}{\mu_\varepsilon}\right), \quad w_{\rm eq} = 1 - 2\frac{\mu_\delta}{\mu_\varepsilon}, \tag{213}$$

$$\sigma_{\rm eq} \sim \frac{\mu_\varepsilon}{2}\left(1-\frac{\mu_\delta}{\mu_\varepsilon}\right). \tag{214}$$

It can thus be seen that as the flow travels from $S_{\rm F}$ to $S_{\rm EQ}$ and then to $S_{\rm eq}$ in the asymptotic domain, w decreases from 10^4 to unity at $S_{\rm EQ}$ and to $(1-2\mu_\delta/\mu_\varepsilon)$ at $S_{\rm eq}$, and then $w \to 0$ for $S \to S_\infty$. σ rather slowly decreases from μ_ε at $S_{\rm F}$ to $(1/2)\,\mu_\varepsilon$ at $S_{\rm EQ}$ and $\sigma \to 0$, while γ increases from 10^2 at $S_{\rm F}$ to 5×10^5 at $S_{\rm EQ}$, and then $\gamma \to 10^6$ for $S \to S_\infty$. From unity at $S_{\rm B}$, $\zeta/\zeta_{\rm B}$ decreases to $(1-10^{-6}\mu_\delta)$ at $S_{\rm A}$, then to $10^{-2}\mu_\delta$ at $S_{\rm F}$, to $10^{-6}\mu_\delta$ at $S_{\rm EQ}$ and finally to null at S_∞.

With respect to the location of $S_{\rm F}$, from equation (157), one obtains

$$\frac{\alpha^2\varpi_{\rm F}^2}{c^2} \sim \left(1-\frac{\mu_\delta^2}{\mu_\varepsilon^{2/3}}\right)\frac{1}{\delta_{\rm F}}, \tag{215}$$

where $\delta_{\rm F} = |(\partial\ln\sigma/\partial\ln\varpi)_{\rm F}|$. If $\mu_\delta^2/\mu_\varepsilon^{2/3} \approx 10^{-4}\mu_\delta^2 \ll 1$, then $\alpha\,\varpi_{\rm F}/c \sim 1/\sqrt{\delta_{\rm F}}$. If one adopts a power law like $\Phi \propto \varpi^{-\delta}$ in the neighborhood of $S_{\rm a} \approx S_{\rm F}$, $\varpi_{\rm F} \sim (1/\sqrt{\delta})(c/\alpha)$ is obtained. Begelman and Li (1994) calculated a $\delta_{\rm F}$ versus $c/\alpha\,\varpi_{\rm F}$ relation for $\sigma_{\rm F}$ = 10, 10^2, 10^5. Equation (215) or (157) seems to well reproduce their numerical results (see their figure 2). For $\delta_{\rm F} \sim 0.4$ (Takahasi, Shibata 1998), then $\varpi_{\rm F} \sim 1.6\,(c/\alpha)$. This seems to yield a too-close location of $S_{\rm F}$ to $S_{\rm L}$.

The coefficients in the expressions for the slope of $w = w(\sigma)$ become from equations (162), (163), and (206)–(210)

$$K_1 \sim \frac{1}{2}+\frac{\mu_\varepsilon^{2/3}}{4},$$
$$K_2 \sim -\mu_\varepsilon^{2/3}\left[\frac{1}{\delta_{\rm F}}-\frac{2}{3}-\frac{\varpi_{\rm F}^2}{3\delta_{\rm F}^2}\left(\frac{\partial^2\ln\sigma}{\partial\varpi^2}\right)_{\rm F}\right]. \tag{216}$$

Thus, one obtains from equation (161)

$$\left(\frac{\partial\ln\zeta}{\partial\ln\sigma}\right)_{\rm F} = K_1 \pm \sqrt{K_1^2-K_2}. \tag{217}$$

The location of $S_{\rm F}$ may be situated in a delicate place near the border line between the asymptotic domain, where variables change with a logarithmic scale, and the subasymptotic domain, where terms with $\varpi_{\rm A}^2/\varpi^2$ are still significant. Therefore, accurate estimations of $\delta_{\rm F}$ and the second derivative of σ will be needed to locate $S_{\rm F}$ exactly.

As is well known, the Crab neutron star is an oblique rotator, and the energy conversion due to the dissipation of a series of current sheets associated with nonaxisymmetric, nonsteady wind may be important (e.g., Lyubarsky, Kirk 2001). The purpose of this paper is, as obvious already, to demonstrate that there is no difficulty in MHD acceleration for axisymmetric magnetocentrifugal winds.

7. Comparison with Previous Theories

To fully understand our results in the previous sections, it would be helpful to clarify the similarities and dissimilarities of the present theory with others. For later convenience, we firstly show one "artificial solution" in the asymptotic domain, which may appear to be plausible, but physically impossible. This solution was indeed treated as if realizable in the literature so far, as shown below. Secondly we discuss some other solutions found in the literature.

7.1. *Models with Zero Curvature in the Domain of $S \gtrsim S_a$*

7.1.1. *The model with $\zeta(P,s)/\mu_\delta(P) = g_1(s)$*

At first let us consider such a solution that leads to zero curvature of stream-fieldlines in the asymptotic domain, which demands variable separation of $\zeta(P,s) = \mu_\delta(P)\,g_1(s)$ in equation (178), where $g_1(s)$ is an arbitrary function of s at this moment. One then has

$$\frac{1}{R} = k = 0, \tag{218}$$

that is, the field lines must be straight, perhaps conical, i.e., $\psi = \psi_{\rm a}$ in equation (9). From equations (121)–(125) one obtains

$$w = \frac{\mu_\varepsilon}{\zeta_{\rm B}}g_1, \quad \gamma = \frac{\mu_\varepsilon}{\left(1+\dfrac{\mu_\varepsilon}{\zeta_{\rm B}}g_1\right)}, \tag{219}$$

$$\varpi\,u_{\rm t} = \frac{\mu_\varepsilon\,\alpha\,\varpi_{\rm A}^2}{c}\frac{\left(1-\dfrac{\mu_\delta}{\zeta_{\rm A}}g_1\right)}{\left(1+\dfrac{\mu_\varepsilon}{\zeta_{\rm B}}g_1\right)}, \tag{220}$$

$$I = \frac{c}{2}\frac{\mu_\varepsilon g_1}{1+\dfrac{\mu_\varepsilon}{\zeta_{\rm B}}g_1}, \tag{221}$$

and then from equations (18) and (19)

$$j_\perp = -\frac{c}{4\pi\varpi}\frac{\mu_\varepsilon g_1'}{\left(1+\dfrac{\mu_\varepsilon}{\zeta_{\rm B}}g_1\right)^2}, \tag{222}$$

$$j_\parallel = \frac{c\Phi}{4\pi\varpi^2}\frac{\mu_\varepsilon g_1}{\left(1+\dfrac{\mu_\varepsilon}{\zeta_{\rm B}}g_1\right)^2}\left(\frac{\mu_\varepsilon'}{\mu_\varepsilon}+\frac{\mu_\varepsilon}{\zeta_{\rm B}}\frac{\zeta_{\rm B}'}{\zeta_{\rm B}}g_1\right), \tag{223}$$

where $g_1' = dg_1/ds$ and

$$\Phi = \frac{4\pi\eta c^3}{\alpha^2}\sigma = \frac{cg_1}{\alpha}\sqrt{\frac{\mu_\varepsilon^2}{\left(1+\dfrac{\mu_\varepsilon}{\zeta_{\rm B}}g_1\right)^2}-1}. \tag{224}$$

Then, by equations (180) and (182)

$$\varrho_{\rm e} = -\frac{(1/c)\,j_\parallel B_{\rm t}}{E_{\rm p}} = -\frac{c\,g_1(s)^2}{8\pi\alpha\varpi^2}\frac{\partial}{\partial P}\left(\frac{\mu_\delta^2}{D_{\rm a}^2}\right)$$
$$= -\frac{c}{4\pi\alpha\varpi^2}\frac{\mu_\varepsilon^2 g_1^2}{\left(1+\dfrac{\mu_\varepsilon}{\zeta_{\rm B}}g_1\right)^3}\left(\frac{\mu_\varepsilon'}{\mu_\varepsilon}+\frac{\mu_\varepsilon}{\zeta_{\rm B}}\frac{\zeta_{\rm B}'}{\zeta_{\rm B}}g_1\right). \tag{225}$$

It can thus be seen that this *artificial* choice of $\zeta(P,s)/\mu_\delta(P) = g_1(s)$ produces a decoupling of the field with the flow. If g_1 = constant, and hence $j_\perp = 0$ by equation (222), all quantities including w, γ, I, etc., are a function of P only, and therefore no changes take place along each field line in the region of g_1 = constant. On the other hand, if $g_1' < 0$ and $g_1 \to 0$

for $s \to \infty$, then w, I, and Φ diminish to null, and $\gamma \to \mu_\varepsilon$, $\varpi\, u_t \to \mu_\varepsilon\, \alpha\, \varpi_A^2/c$ with $j_\perp > 0$. A strange thing is that in spite of no field-flow coupling due to $1/R = 0$, the field energy decreases, while the flow energy increases toward infinity. It is not certain how to determine function $g_1(s)$ causally in terms of the upstream quantities at S_F. This is the reason why this *choice* is physically impossible.

7.1.2. *The spell of radial-field models*

It may be concluded that a long-standing puzzle, i.e., disagreement between MHD wind theory and the Crab nebula model, is brought about by the assumption of a radial field model. The split-monopolar field model extensively utilized in pulsar wind theory has $\Phi = B_p\, \varpi^2$ = constant and σ = constant throughout the wind region, independent of s along each field line (i.e., $g_1' = 0$ in subsubsection 7.1.1), and hence no cross-field current, $j_\perp = 0$ and no MHD acceleration, i.e., $\partial\gamma/\partial s = 0$ for a cold wind. The disagreement is the result invited by an abuse of the *radialness*, and yet its plausibility in the asymptotic domain has prevented one from discovering the seed of the puzzle.

It seems that the plausibility of the *radialness* in the asymptotic domain has recently invited another claim for the existence of "field regions" with $1/R \approx 0$ (Heyvaerts, Norman 2003a, b, c). The logic needed to justify such regions seems to be as follows. Far from the source, plasma particles will be well accelerated to behave ballistically so that field-streamlines will be straightened enough already, i.e., $1/R \approx 0$, which combines with the transfield equation (182), to yield $\varrho_e\, E_p + (1/c)\, j_\parallel\, B_t \approx 0$, a kind of force-free state (referred to as the *pseudo*-force-free state; see Okamoto 2003). This is obviously a strange conclusion, because the kinematically-dominated state leads to the opposite magnetically dominated state. Certainly factor $1/R$ will be sufficiently small in the superfast domain, but however small it may be, it appears in the inertial term of the transfield equation, as a combination of the factor $\gamma\, \rho\, v_p^2$, which is conversely sufficiently large. The transfield force balance requires the product of both, i.e., $\gamma\, \rho\, v_p^2/R$ to be equal to the Lorentz force. Neglecting the presence of a large factor, $\gamma\, \rho\, v_p^2$, one cannot divide the most fundamental equation, i.e., the transfield equation, into the two, i.e., $1/R \approx 0$ and $\varrho_e\, E_p + (1/c)\, j_\parallel\, B_t \approx 0$, corresponding the two opposite extreme physical states.

7.2. *"Anti-Collimation Theorem"*

The "hoop-stress paradigm" has governed the field of magnetized winds and jets for these two decades. In spite of outflows from rotating central objects, of which the prime motive force is mainly the centrifugal force, even though strongly magnetized, the claim that *all* of field-streamlines collimate *globally* to the rotation axis in the asymptotic domain seems to be "unnatural" (Okamoto 1999). And yet, the paradigm seems to insist that this property is due to axisymmetry of the system (see, e.g., Spruit 1996) or to anisotropy introduced by the magnetic field (Blandford 2002). Whether the flow collimates or decollimates should be determined on the basis of the sign of curvature of its field-streamlines, which in turn must be fixed by the cross-field force balance. That is to say, the paradigm is neither based upon the sign of curvature of field-streamlines nor, hence, upon the consequence of the transfield force balance in the steady state (cf. Heyvaerts, Norman 1989; Chiueh et al. 1991). The case in which R was utilized so far was entirely limited to irrelevantly showing logarithmic collimation of the outflow (Chiueh et al. 1991, 1998; Eichler 1993; Tomimatsu 1994; Begelman, Li 1994; Bogovalov, Tsinganos 1999; also see subsubsection 7.3.3 later).

It is then of crucial significance to derive the correct expression of the transfield force balance containing the curvature of field lines. Not surprisingly, Chiueh, Li, and Begelman (1991) have already given a correct form of the transfield equation. If they had correctly interpreted theirs, they would have reached the same conclusion as that of this paper as well as Papers I–III. Eichler (1993) also had the same chance to reach the correct conclusion on collimation, because he cited Chiueh, Li, and Begelman's transfield equation. But both Chiueh, Li, and Begelman (1991) and Eichler (1993) and also Tomimatsu (1994) have remained in the same line of research as Heyvaerts and Norman (1989), consequently allowing the paradigm to govern the field of MHD outflows in the 1990s and even in this century (Heyvaerts, Norman 2003a, b, c). The influence of the paradigm was exerted even on the force-free pulsar magnetosphere models. We have already made a "critical" review of force-free models for cylindrical jets (see Okamoto 1997).

Similar to Papers I–III for nonrelativistic winds, and hence, dissimilar to Chiueh, Li, and Begelman and Eichler for relativistic winds, we interpret the transfield equation containing R as it indicates, that is, as the expression for determining R as a result of the transfield force balance. It is thus the "anticollimation theorem", but not the "hoop-stress paradigm" that has a firm physical basis in MHD outflows, whether relativistic or nonrelativistic.

7.3. *Some Comments on Chiueh, Li and Begelman's Works*

In spite of treating the same topics as in this paper, Chiueh, Li, and Begelman's conclusions are quite different from the present ones. It will be important to clarify why and where significant disagreements take place.

7.3.1. *Begelman and Li's (1994) analysis*

Begelman and Li (1994) studied the conditions that lead to converting most of the Poynting flux into the kinetic energy flux in cold relativistic MHD winds. They ascribed extreme inefficiency of plasma acceleration along a precisely radial flow to near cancellation of the toroidal magnetic pressure and tension forces. They noticed that from their energy integral (2), for a given total energy, μ_ε, and angular momentum, $l \equiv \mu_\varepsilon\, \alpha\, \varpi_A^2$, the dimensionless flow variables at any location, $x = \alpha\, \varpi/c$, along a flux tube depend only on the local value of $\sigma(x)$ [their $a(x)$]. Since in the asymptotic domain of $x^2 \gg x_A^2$ (i.e., $\varpi^2 \gg \varpi_A^2$), coordinate x or ϖ disappears and hence the flow variables depend only on σ literally. Their "simplified" equations (5) and (7) are completely equivalent to equation (128) in this paper, and if one puts $M_{f\infty}^2 = 1$ in their equation (6), one arrives at $\tau_\infty^3\, a_\infty^2 = 1$ for the criticality condition at S_F, which equals $w_F = \sigma_F^{2/3}$, as can be seen in equation (146). Then, for ultrarelativistic flows with $\mu_\varepsilon \gg 1$, from equation (128) one obtains $w \approx \sigma/\mu_\varepsilon$ for the supermagnetosonic solution in the *physical* branch and $w \approx \mu_\varepsilon$ for the submagnetosonic solution in the *unphysical* branch (see figure 4). Since $\sigma_F \approx \mu_\varepsilon$ by equations (138) and (146), one has $w \approx \sigma/\sigma_F$ in the *physical* branch.

With respect to the location of S_F, Begelman and Li remarked that only a negative value of $\delta_f \equiv (\partial \ln \Phi / \partial \ln \varpi)_{PF} = (\partial \ln \sigma / \partial \ln \varpi)_{PF}$ can move S_F to a finite radius, which means that the divergence of the flux tube must be faster, rather than slower, than in a radial wind. As can be seen in equation (136), the kinetic energy at S_F is equal to the cubic root of the total energy, i.e., $\gamma_F = \mu_\varepsilon^{1/3}$, and yet $w_F = \mu_\varepsilon^{2/3} - 1$. They stated that "this differs drastically from the near equipartition between kinetic and magnetic energies that occurs at S_F in the nonrelativistic flows", because $v_{pF}^2/2 = \varepsilon/3$ and $w_F = 2$ [see equation (A40) and appendix 4]. As can be seen in equations (A40), (A43), and (A44), the nonrelativistic limits of the present results coincide with those given in Paper III for nonrelativistic magnetized centrifugal winds, as naturally should be so. But $w_F = \mu_\varepsilon^{2/3} - 1 \gg 1$ for relativistic cases, compared with $w_F = 2$ for the nonrelativistic case, takes place not because a large electric field in relativistic flows may cancel the magnetic pressure to large extent, but just because significant acceleration must occur beyond S_F in the relativistic flow. In any case, we confirm their conclusion that *nearly all energy could be converted eventually into kinetic form* ($w_\infty \ll 1$)*, provided that the field lines have such a geometry that the quantity* σ *decreases significantly from its value at the fast point,* σ_F*, to its asymptotic value* σ_∞ ($\sigma_\infty/\sigma_F \ll 1$) *once the fast point is passed.*

Then, led by the "hoop-stress paradigm", that is, the idea of the "general" tendency of collimation toward the rotation axis, Begelman and Li next attempted to determine the condition under which such solutions of the relativistic Grad-Shafranov equation can develop a vanishingly small ratio of the Poynting flux to the kinetic flux. The asymptotic solutions they sought are related to "current-free paraboloidal asymptotes", which they refer to as "force-free" in the sense that the *cross-field* force balance is determined by the electromagnetic stresses, with inertial forces playing a negligible role, though not "force-free" in the poloidal direction, since the flows continue to accelerate ($j_\perp > 0$; see the case of $g_1' < 0$ in subsubsection 7.1.1). They "generalize" these solutions to include flows that have a finite total flux, confined by an external medium with the pressure decreasing to zero at infinity. They stated that these asymptotic solutions are uniquely determined by the pressure boundary condition and the distribution of a particular flux function. They also presumed the existence of a non-force-free region where the inertia is still important on field lines that have not yet been collimated parallel to the rotation axis, illustrating how solutions in the non-force-free region join smoothly onto the corresponding force-free asymptotes. To show these presumptions, they utilize their asymptotic form [see their equation (11)] of the transfield equation with the same number (11) in Chiueh, Li, and Begelman (1991). These are coincident with equations (183) and (34), respectively.

We must point out that, in spite of utilizing the equivalent expressions describing the cross-field force balance, the results we have drawn here are quite opposite to those of Begelman and Li. For both nonrelativistic winds in Papers I–III and the relativistic winds described here, we have demonstrated that one form of the transfield equation demands $\sigma(\varpi) \to 0$ or $\zeta(P, \varpi) \to 0$ for $\varpi \to \infty|_P$, and therefore $w \to 0$ [see equations (10) and (11)]. As argued in Paper III, the transfield equation should be utilized not to show logarithmic collimation, but to avoid logarithmic divergence. This indicates ongoing interactions of the field with the flow, which ensures ongoing acceleration in the whole asymptotic domain of $\varpi_F \approx \varpi_a \lesssim \varpi \lesssim \infty$ as well as in the subasymptotic domain of $\varpi \lesssim \varpi_F$. There is no physical reason to support the existence of the "force-free" region both in the cross-field as well as the poloidal direction in the asymptotic domain. Smooth flow of the poloidal electric current and the current-closure condition indicate that the two-component structure is the case in the relativistic as well as nonrelativistic winds. For another form of the transfield equation, one can show that the second-order partial differential equation for P is expressible in terms of $\xi \equiv \ln r$ and θ for coordinates, so that the asymptotic variables change rather more slowly with $\ln r$ than r. We favor Begelman and Li's non-force-free region extended to $\varpi \to \infty|_P$, with the two-componentness taken into account, but do not agree with their conclusion that *it is very difficult for an ultrarelativistic wind, such as the Crab pulsar wind with* $\gamma \sim 10^6$*, to reach the kinetically dominated asymptotic state discussed here.*

7.3.2. Begelman's (1998) analysis

To resolve any discrepancy between the pulsar wind theory yielding $w_\infty \gg 1$ and the Rees–Gunn Crab model predicting $w_\infty \ll 1$, Begelman (1998) proposed that the existence of a concentric toroidal field outside the pulsar wind's termination shock is physically implausible. Abandoning the central tenet of the Rees–Gunn model for the Crab nebula, he seems to be favoring the result from pulsar wind theory based upon *near* cancellation of the magnetic force by the electric force, that is, domination of the Poynting flux on the kinetic energy flux even far outside. The present analysis of the transfield equation indicates that the condition for field lines to reach infinity is $I \to 0$ for $\sigma \to 0$ with $\varpi \to \infty|_P$, as can be seen in equation (29), that is, the Poynting flux vanishing toward infinity, $w \to 0$. This is a natural basic property of RMHD winds, which is similar to that of nonrelativistic winds (see Papers I–III). This basic property of RMHD winds supports the central tenet of the Rees–Gunn model for the Crab nebula.

The current-closure condition as a global condition should be imposed within the wind zone in the steady state, and thus the field-flow structure has a two-component nature; that is, the structure consists of polar flow with $j_\| < 0$ and equatorial wind with $j_\| > 0$. Begelman's stability analysis of a concentric toroidal field structure due to pinch and kink modes would itself be interesting, because it may have a significant application to the polar rather collimated jet-like outflow. Thus, instabilities may certainly destroy the concentric toroidal field, but this interesting topic is out of the scope of this paper. The point is that one needs not abandon the central tenet of the Rees–Gunn model, just to resolve the long-standing puzzle, and of course more detailed work is certainly needed.

7.3.3. Chiueh, Li, and Begelman's (1998) analysis

Chiueh, Li, and Begelman's (1998) presented what they called a "critical" examination of the ideal MHD model for the stationary Crab pulsar wind with $\gamma_\infty \sim 10^6$ and

$w_\infty \sim 10^{-2}$–10^{-3}. They concluded that transitions to a low w_∞-configuration cannot occur gradually in regions well beyond $S_{\rm L}$, and pined down the only situation where a stationary ideal MHD low-w wind may exist, requiring almost the entire acceleration to take place in the immediate neighborhood of $S_{\rm L}$. They demanded drastic modifications to the conventional picture of the pulsar dipole magnetosphere, in that the outer magnetosphere must be dominated by the toroidal fields and that the pulsar wind is carried by only a small fraction of the magnetospheric field lines emerging from the star. To derive these results, they again utilized the same set of ideal-MHD equations as given in this paper, although our conclusions are quite opposite to theirs, as briefly discussed in the following.

From the criticality condition at $S_{\rm F}$, they obtained $\mu_\varepsilon \approx \sigma_{\rm F}$ in their equation (10), which is naturally coincident with the result from equations (136) and (144) for $\gamma_{\rm F}^2 = \gamma_\infty^{2/3} \gg 1$. As shown in equation (146), $w_{\rm F} = \sigma_{\rm F}^{2/3} \approx \gamma_\infty^{2/3} \gg 1$, and therefore the Poynting flux still dominates the kinetic energy flux at $S_{\rm F}$. Certainly the only way to attain a low value of w_∞ is for the flux tubes to diverge significantly outside $S_{\rm F}$, thus lowering Φ or σ. Assuming that the variation of magnetic field in the cross-field direction occurs on a scale of order ϖ, and that the spatial derivative can be estimated to be $1/\varpi$, they derived

$$\left|\frac{\varpi}{R}\right| \lesssim \frac{1}{1+\tau}\left(\frac{1}{\gamma^2} + \frac{\beta}{(\alpha\varpi/c)^2}\right) + \frac{\beta}{(\alpha\varpi/c)^2} \qquad (226)$$

from their equation (11), where β is a constant of $O(1)$ and $\tau = M^2/(\alpha\,\varpi/c)^2$. Their expression (11) for the rate at which a flux tube diverges is exactly the same as, e.g., equation (36) here. They utilized this equation for estimating the flux surface collimation rate. We note that the change in the collimation angle ψ over a distance $\Delta\varpi$ is

$$\Delta\psi \approx \begin{cases} \dfrac{\ln(\alpha\,\Delta\varpi/c)}{\gamma^2(1+\tau)}, & \dfrac{\alpha\varpi}{c} \gg \gamma\sqrt{1+\tau} \gg 1, \\[2ex] \dfrac{(\alpha\,\Delta\varpi/c)}{(\alpha\,\varpi/c)^3}, & 1 \ll \dfrac{\alpha\varpi}{c} \ll \gamma\sqrt{1+\tau}. \end{cases} \qquad (227)$$

They stated that the upper part of equation (227) is a generalization of the logarithmic collimation in the works of Chiueh, Li, and Begelman (1991), Eichler (1993), Begelman and Li (1994), and Tomimatsu (1994), and the lower rate may remain valid over many decades of the local wind radius for an extremely large γ. What they deduced from equation (227) is as follows. The fact that $\varpi/R \ll 1$ implies that $\Phi = B_{\rm p}\,\varpi^2$ changes only by a small factor along the flux tube surfaces, and the change of the kinetic energy must be negligible compared with the total energy. Thus, the poloidal field lines in a smoothly varying ultrarelativistic MHD flow must be nearly straight far beyond $S_{\rm L}$; in this region efficient conversion of the Poynting flux to the kinetic energy flux is impossible. According to Chiueh, Li, and Begelman, the fundamental reason for this behavior is that the magnetic pressure in the cross-field direction is canceled almost exactly by a cross-field electric field in the region with both γ and $\alpha\,\varpi/c$ being large. "A diverging channel configuration needed for sufficient acceleration can never be set up".

As shown in equation (182) for the Grad-Shafronov equation in the asymptotic domain, the degree of the "cancellation" of the magnetic pressure by a cross-field electric field [i.e., $\varrho_{\rm e}\,E_{\rm p} + (1/c)\,j_{\|}\,B_{\rm t}$] is equal to the "rate" (i.e., ϖ/R) at which a flux tube diverges, multiplied by a large factor, $\rho\,\gamma\,v_{\rm p}^2$. Needless to say, what is meant by this transfield force balance is nothing but expressing that the inertial centripetal force upon the poloidal motion along a curved field line must be in balance with the transfield component of the Lorentz force. Then, if one concludes for some reason or other that both $\varrho_{\rm e}\,E_{\rm p} + (1/c)\,j_{\|}\,B_{\rm t} \approx 0$ and $\varpi/R \approx 0$ hold separately in some, or the whole, of the asymptotic domain, this means that the two, opposite extreme states, i.e., inertially dominated and magnetically dominated, must coexist there. These mutually contradicting states are a product of artificially dividing one equation for the cross-field force balance into the two conditions.

Instead the procedure taken in this paper is to treat the Grad-Shafronov equation faithfully as it indicates in the asymptotic domain. This leads to the relation between the "rate" (i.e., ϖ/R) and each of such quantities as w, ζ, γ, I, etc., through the Lorentz force, $\varrho_{\rm e}\,E_{\rm p} + (1/c)\,j_{\|}\,B_{\rm t}$, as given in equations (192)–(195). These quantities in turn are a two-valued function of σ, which passes smoothly through the fast X-type critical point at $S_{\rm F} \approx S_{\rm a}$, as given in equations (128), (131), and (189). Taking the *physical* branch of each solution satisfying the "regularity condition" near S_∞, one can naturally accomplish full MHD acceleration, i.e., $\gamma \to \mu_\varepsilon$ and $\gamma\,\varpi\,v_{\rm t} \to \mu_\varepsilon\,\alpha\,\varpi_{\rm A}^2$ with $w \to 0$, $w_{\rm AM} \to 0$, $\zeta \to 0$, and $I \to 0$ for $\sigma \to 0$. This means that the "regularity condition", $\varpi/R \to 0$, is fulfilled, but this must happen rapidly enough to assure $|\sin\psi_\infty| \le 1$, as indicated in equation (11).

8. Conclusions

It is shown in this paper that nondissipative ideal MHD is sound and robust enough to be applied to relativistic outflows and related phenomena. If we treat the MHD equations correctly, and deduce results from them under appropriate conditions, one can feasibly understand the basic structure of a pulsar magnetosphere with two-componentness, where the expected full MHD acceleration takes place. It is however only by handling physical equations and conditions properly that one can obtain theoretical results consistent with the observational facts. Even in the 21st century, the ideal MHD will still play the role of a good paradigm for exploring the phenomena of magnetized centrifugal outflows in the nonrelativistic, relativistic, and general-relativistic regimes with a possible exception of dissipative processes in some localized regions. The followings are the basic properties for pulsar MHD winds clarified in this paper:

1) The induction equation, the most basic in MHD, integrates in the steady axisymmetric state to yield Ferraro's isorotation law for each field line, and the requirement that $\alpha = \alpha(P)$ be given as the boundary condition at the stellar surface or magnetospheric base must imply the existence of a unipolar inductor or battery there.

2) The concept of a "current line" is as important as that of a "field line". Contrary to field lines in the wind zone, each current line defined by $I = I(s, P)$ must close like in a kind

of DC circuit, emanating from one terminal of the battery and returning to the other terminal, without snapping on the way. This is referred to as the "current-closure condition", as one of the global conditions, which is closely related to the acceleration-collimation/decollimation of field-streamlines.

3) A dipole-like distribution of magnetic fluxes at S_B is implicitly assumed so far, with antisymmetry between the upper and lower hemispheres. Then, in the upper hemisphere the current-closure condition means that the domain of ingoing current, $j_\parallel < 0$, is separated by the "neutral" field line, P_n, ($j_\parallel = 0$), from the domain of the outgoing current by $j_\parallel > 0$, with the current line continuously crossing field lines $j_\perp > 0$, thereby causing MHD acceleration, i.e., $\partial\gamma/\partial s \propto j_\perp \propto -\partial I/\partial s > 0$ in the wind zone.

4) The asymptotic formalism for the wind zone is a useful tool for clarifying the basic properties in the asymptotic domain where the main acceleration and collimation/decollimation of the flow take place. The domain is defined as that where the factors with orders more than ϖ_A^2/ϖ^2 are negligible, and consequently the coordinate variables ϖ and z appear only through the flux function, $\Psi = B_p\varpi^2$, or the generalized magnetization parameter, σ. Every field-flow quantity is a two-valued function of σ, and the two branches intersect each other at the innermost distances (or surface S_a) of the asymptotic domain.

5) The asymptotic domain must also be the superfast domain, and the fast surface, S_F, lies in the neighborhood of the innermost surface of the asymptotic domain, i.e., $S_F \approx S_a$. The criticality condition at S_F yields the eigenvalues for ϖ_A^2, μ_ε, γ_F, etc., in terms of the input values of α, η, μ_δ, and the magnetic flux at S_B. In order to fix S_F, itself, at the innermost distances, the lowest-order terms of ϖ_A^2/ϖ^2 must be retained to relate ϖ_F to the gradient of σ along each field line.

6) The main MHD acceleration takes place in the asymptotic domain, because $\gamma_F = \mu_\varepsilon^{1/3}$ at S_F and $\gamma_\infty = \mu_\varepsilon$ at S_∞, which are much larger than γ_F for an ultrarelativistic wind [cf. $(1/2)\,v_{pF}^2 = (1/3)\,\varepsilon$ and $(1/2)\,v_{p\infty}^2 = \varepsilon$ for the nonrelativistic case]. The large-scaleness of astrophysical jets, in general, if these are associated with MHD processes, will be a manifestation of the vastness of the superfast, asymptotic domain.

7) In the subasymptotic domain of $S_B \le S \lesssim S_a$, the coordinate ϖ is naturally live. Wind with $\gamma_B \approx \mu_\delta$ at S_B must smoothly pass through the critical surface, S_A, with $\gamma_A \approx \mu_\delta(1 + \nu_A)$, to flow beyond S_a. The eigenvalue for ϖ_A must be determined by the criticality condition fairly downstream at S_F, just beyond the subasymptotic domain, whereas the values at S_∞, such as γ_∞, $u_{t\infty}$, etc., are already fixed by the criticality condition far upstream at S_F. The boundary surface, S_a, is introduced for mathematical convenience and hence not a physical surface, but the physical properties are fundamentally different across it.

8) In order that magnetized outflows can be an efficient carriers of energy/angular momentum from the central source in the steady state, the current-line topology as well as the field-line topology must obviously be reasonable across surfaces S_a and S_F. The field lines will reach the sphere-at-infinity, S_∞, with diminishing magnetic fluxes, i.e., $\Phi \to 0$ or $\sigma \to 0$, while every current line must close at finite distances mainly in the superfast domain with $j_\perp > 0$. It is difficult to find a way to connect the "quasi-force-free" region where $I = \text{constant}$, $j_\perp = 1/R = 0$ causally to any region where the poloidal current flows.

9) One of the crucially important quantities is the field-line curvature, $1/R$, which has often been mistakenly regarded as being negligible in the asymptotic domain. However small it may be, when multiplied by a quite large factor, $\gamma\,\rho\,v_p^2$, the product as an inertial force must be in balance with the relativistic Lorentz force. Otherwise, one would reach an incorrect result that "the inertially-dominated state is equivalent to the magnetically dominated state". By using the transfield equation correctly, it can be shown that as $\sigma \to 0$, then $I \to 0$, $\gamma \to \mu_\varepsilon, \ldots$, and $\varpi/R \to 0$, or *vice versa*. It is the sign of ϖ/R, determined by the transfield force balance, whether the flow collimates with $\varpi/R > 0$ and $\partial\psi/\partial s > 0$, or the flow decollimates with $\varpi/R < 0$ and $\partial\psi/\partial s < 0$.

Appendix 1. Derivation of the Three Components of the RMHD Equation of Motion

(1-I) We now derive equation (7) and (20) for R and j_t. In terms of angle ψ between $\boldsymbol{B}_p$ and the ϖ-axis, the two unit vectors become

$$\boldsymbol{p} = (\cos\psi, 0, \sin\psi), \quad \boldsymbol{n} = (-\sin\psi, 0, \cos\psi), \tag{A1}$$

and the two components of $\boldsymbol{B}_p$ become from equation (1)

$$B_\varpi = B_p\cos\psi = -\frac{1}{\varpi}\frac{\partial P}{\partial z}, \quad B_z = B_p\sin\psi = \frac{1}{\varpi}\frac{\partial P}{\partial\varpi}. \tag{A2}$$

Then,

$$\frac{4\pi}{c}j_t = \frac{\partial B_\varpi}{\partial z} - \frac{\partial B_z}{\partial\varpi}$$

$$= -\frac{1}{\varpi}\left(\nabla^2 P - \nabla P\cdot\nabla\ln\varpi^2\right) \tag{A3}$$

$$= -\frac{1}{\varpi}\left[(\nabla P\cdot\nabla)\ln\frac{|\nabla P|}{\varpi} + \frac{|\nabla P|}{R}\right], \tag{A4}$$

where equation (6) for $1/R$ is used. Thus, one obtains equation (20) for j_t by using equation (3) and $B_p = |\nabla P|/\varpi$. Equation (7) for $1/R$ is derived by solving equations (A3) and (A4). One more direct way of verifying equation (7) is as follows: Noting $\nabla P = -|\nabla P|\boldsymbol{n}$ from equation (2), one has

$$\nabla^2 P = \frac{1}{\varpi}\left[\frac{\partial}{\partial\varpi}(\varpi|\nabla P|\sin\psi) - \frac{\partial}{\partial z}(\varpi|\nabla P|\cos\psi)\right]$$

$$= -\frac{1}{\varpi}(\boldsymbol{n}\cdot\nabla)\varpi|\nabla P| + |\nabla P|(\boldsymbol{p}\cdot\nabla)\psi$$

$$= (\nabla P\cdot\nabla)\ln\varpi\,|\nabla P| + \frac{|\nabla P|}{R} \tag{A5}$$

and then one can arrive at the expression for $1/R$.

(1-II) We give the three components expressed in equations (31)–(33) from the vectorial form of the RMHD equation of motion in equation (30). In terms of ψ, the two components of $\boldsymbol{v}_t$ are given by

$$v_\varpi = v_{\rm p}\cos\psi, \quad v_z = v_{\rm p}\sin\psi. \tag{A6}$$

Then,

$$\nabla\times\gamma\,\boldsymbol{v} = -\frac{\boldsymbol{t}}{\varpi}\times\nabla(\gamma\,\varpi\,v_{\rm t}) + (\nabla\times\gamma\,\boldsymbol{v})_{\rm t}\boldsymbol{t}, \tag{A7}$$

$$\begin{aligned}(\nabla\times\gamma\,\boldsymbol{v})_{\rm t} &= \frac{\partial}{\partial z}(\gamma\,v_\varpi) - \frac{\partial}{\partial\varpi}(\gamma\,v_z)\\ &= (\boldsymbol{n}\cdot\nabla)\gamma\,v_{\rm p} - \gamma\,v_{\rm p}(\boldsymbol{p}\cdot\nabla)\psi\\ &= \frac{\partial\gamma\,v_{\rm p}}{\partial n} - \frac{\gamma\,v_{\rm p}}{R},\end{aligned} \tag{A8}$$

where equations (A6) and (3) are utilized. Thus, one has

$$\begin{aligned}\boldsymbol{v}\times(\nabla\times\gamma\,\boldsymbol{v}) &= \frac{v_{\rm t}}{\varpi}\nabla(\gamma\,\varpi\,v_{\rm t})\\ &\quad - \frac{v_{\rm p}}{\varpi}\frac{\partial\gamma\,\varpi\,v_{\rm t}}{\partial s}\boldsymbol{t} + \left(\frac{\partial\gamma\,v_{\rm p}}{\partial n} - \frac{\gamma\,v_{\rm p}}{R}\right)\boldsymbol{n}.\end{aligned} \tag{A9}$$

Then, the RHD of equation (30) becomes

$$\begin{aligned}&\rho[c^2\nabla\gamma - \boldsymbol{v}\times(\nabla\times\gamma\,\boldsymbol{v})]\\ &= \rho\left(c^2\frac{\partial\gamma}{\partial s} - \frac{v_{\rm t}}{\varpi}\frac{\partial\gamma\,\varpi\,v_{\rm t}}{\partial s}\right)\boldsymbol{p}\\ &\quad + \frac{\rho\,v_{\rm p}}{\varpi}\frac{\partial\gamma\,\varpi\,v_{\rm t}}{\partial s}\boldsymbol{t} + \left(-\rho\,\gamma\,v_{\rm t}^2\frac{\partial\ln\varpi}{\partial n} + \frac{\rho\,\gamma\,v_{\rm p}^2}{R}\right)\boldsymbol{n}.\end{aligned} \tag{A10}$$

The Lorentz force can also be given in terms of three components ($j_{||}$, $j_{\rm t}$, and $j_\perp$) as

$$\frac{1}{c}\boldsymbol{j}\times\boldsymbol{B} = \frac{1}{c}\left[-j_\perp\,B_{\rm t}\,\boldsymbol{p} + j_\perp\,B_{\rm p}\,\boldsymbol{t} + (j_{||}\,B_{\rm t} - j_{\rm t}\,B_{\rm p})\boldsymbol{n}\right] \tag{A11}$$

[see equation (1.2) in Paper I]. Noting $\varrho_{\rm e}\boldsymbol{E} = \varrho_{\rm e}E_{\rm p}\boldsymbol{n}$, upon substituting equations (A10) and (A11) into (30), one obtains equations (31)–(33) for the $\boldsymbol{p}$-, $\boldsymbol{t}$-, and $\boldsymbol{n}$-components of the RMHD equations of motion.

Appendix 2. Derivation of the Slope of ζ

Here we derive the expressions for $\mathcal{D}$ and $\mathcal{N}$ given in equations (112) and (113), which appear in equation (111) for the slope of ζ. From equations (39), (44), (64), and (65), one obtains

$$\gamma = \mu_\varepsilon - \mu_\varepsilon\frac{\zeta}{\zeta_{\rm B}}\left(\frac{2I}{c\zeta}\right), \tag{A12}$$

$$u_{\rm t} = \frac{c}{\alpha\varpi}\left[\frac{\alpha^2\varpi_{\rm A}^2}{c^2} - \mu_\varepsilon\frac{\zeta}{\zeta_{\rm B}}\left(\frac{2I}{c\zeta}\right)\right], \tag{A13}$$

$$= \mu_\delta\frac{\alpha\,\varpi_{\rm A}^2}{c\varpi}\frac{1}{D}\left(1 - \frac{\zeta}{\zeta_{\rm A}}\right), \tag{A14}$$

$$\frac{2I}{c\zeta} = \frac{1}{D}\left(1 - \frac{\varpi_{\rm A}^2}{\varpi^2}\right). \tag{A15}$$

Differentiation of γ in equations (104) and (A12) and $u_{\rm t}$ in equation (A13) yield

$$\gamma\frac{\partial\gamma}{\partial\varpi} = u_{\rm p}^2\frac{\partial\ln u_{\rm p}}{\partial\varpi} + u_{\rm t}^2\frac{\partial\ln u_{\rm t}}{\partial\varpi}, \tag{A16}$$

$$\frac{\partial\gamma}{\partial\varpi} = -\mu_\varepsilon\frac{\zeta}{\zeta_{\rm B}}\left(\frac{2I}{c\zeta}\right)\frac{\partial}{\partial\varpi}\ln\left[\zeta\cdot\left(\frac{2I}{c\zeta}\right)\right], \tag{A17}$$

$$\frac{\partial\ln u_{\rm t}}{\partial\varpi} = -\frac{1}{\varpi} - \frac{c\mu_\varepsilon}{\alpha\,\varpi\,u_{\rm t}}\frac{\zeta}{\zeta_{\rm B}}\left(\frac{2I}{c\zeta}\right)\frac{\partial}{\partial\varpi}\ln\left[\zeta\cdot\left(\frac{2I}{c\zeta}\right)\right]. \tag{A18}$$

Also, from equation (74)

$$\frac{\partial\ln u_{\rm p}}{\partial\varpi} = -\frac{\partial\ln\zeta}{\partial\varpi} + \frac{\partial\ln\sigma}{\partial\varpi}. \tag{A19}$$

Substitution of equations (A17), (A18), and (A19) into equation (A16) yields

$$\begin{aligned}&u_{\rm p}^2\frac{\partial\ln\zeta}{\partial\varpi} - \left(\gamma - \frac{c\,u_{\rm t}}{\alpha\varpi}\right)\mu_\varepsilon\frac{\zeta}{\zeta_{\rm B}}\left(\frac{2I}{c\zeta}\right)\frac{\partial}{\partial\varpi}\ln\left(\zeta\cdot\frac{2I}{c\zeta}\right)\\ &\quad = u_{\rm p}^2\frac{\partial\ln\sigma}{\partial\varpi} - \frac{u_{\rm t}^2}{\varpi}.\end{aligned} \tag{A20}$$

Making use of equations (61), (A14), and (A15), one has the relation

$$\gamma - \frac{c\,u_{\rm t}}{\alpha\varpi} = \mu_\delta\frac{2I}{c\zeta}, \tag{A21}$$

and therefore

$$\begin{aligned}&u_{\rm p}^2\frac{\partial\ln\zeta}{\partial\varpi} - \mu_\delta\,\mu_\varepsilon\frac{\zeta}{\zeta_{\rm B}}\left(\frac{2I}{c\zeta}\right)^2\frac{\partial}{\partial\varpi}\ln\left(\zeta\cdot\frac{2I}{c\zeta}\right)\\ &\quad = u_{\rm p}^2\frac{\partial\ln\sigma}{\partial\varpi} - \frac{u_{\rm t}^2}{\varpi}.\end{aligned} \tag{A22}$$

Next, differentiation of equations (A15) and (60) yields

$$\frac{\partial}{\partial\varpi}\ln\left(\frac{2I}{c\zeta}\right) = \frac{2}{\varpi}\frac{\dfrac{\varpi_{\rm A}^2}{\varpi^2}}{\left(1 - \dfrac{\varpi_{\rm A}^2}{\varpi^2}\right)} - \frac{1}{D}\frac{\partial D}{\partial\varpi}, \tag{A23}$$

$$\frac{\partial D}{\partial\varpi} = \frac{\zeta}{\zeta_{\rm B}}\left(1 - \frac{c^2}{\alpha^2\varpi^2}\right)\left(\frac{\partial\ln\zeta}{\partial\varpi} + \frac{2}{\varpi}\frac{\dfrac{c^2}{\alpha^2\varpi^2}}{1 - \dfrac{c^2}{\alpha^2\varpi^2}}\right). \tag{A24}$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial\varpi}\ln\left(\zeta\cdot\frac{2I}{c\zeta}\right) &= \left[1 - \frac{1}{D}\frac{\zeta}{\zeta_{\rm B}}\left(1 - \frac{c^2}{\alpha^2\varpi^2}\right)\right]\frac{\partial\ln\zeta}{\partial\varpi}\\ &\quad + \frac{1}{D}\frac{2}{\varpi}\frac{\varpi_{\rm A}^2}{\varpi^2}\left(1 - \frac{\alpha^2\varpi_{\rm A}^2}{c^2}\right)\frac{1 - \dfrac{\zeta}{\zeta_{\rm A}}}{1 - \dfrac{\varpi_{\rm A}^2}{\varpi^2}}\\ &= \frac{1}{D}\left(1 - \frac{\alpha^2\varpi_{\rm A}^2}{c^2}\right)\left[\frac{\partial\ln\zeta}{\partial\varpi} + \frac{2}{\varpi}\frac{\varpi_{\rm A}^2}{\varpi^2}\frac{1 - \dfrac{\zeta}{\zeta_{\rm A}}}{1 - \dfrac{\varpi_{\rm A}^2}{\varpi^2}}\right].\end{aligned} \tag{A25}$$

Substituting equation (A25) into equation (A22) by using equations (A14) and (A15), we obtain the expressions for $\mathcal{D}$ and $\mathcal{N}$ given in equations (112) and (113).

Appendix 3. Expression for the Derivative of $d\ln\zeta/d\ln\sigma$ at S_{F}

From (125), (127), and (133) one obtains a form of $\mathcal{D}_{\mathrm{a}}$ expressed in terms of w and σ, i.e.,

$$\mathcal{D}_{\mathrm{a}}=\left(1-\frac{\zeta_{\mathrm{A}}}{\zeta_{\mathrm{B}}}\right)\left[\sigma^{2}\frac{1+w}{w}-\mu_{\varepsilon}^{2}\frac{w}{(1+w)^{2}}\right]. \tag{A26}$$

The derivative of $\mathcal{D}_{\mathrm{a}}$ at S_{F} becomes

$$\begin{aligned}\left(\frac{\partial\mathcal{D}_{\mathrm{a}}}{\partial\varpi}\right)_{\mathrm{F}}&=\left(1-\frac{\zeta_{\mathrm{A}}}{\zeta_{\mathrm{B}}}\right)\left(\sigma^{2}\frac{1+w}{w}\right)_{\mathrm{F}}\\&\quad\times\left[\frac{\partial}{\partial\varpi}\ln\left(\sigma^{2}\frac{1+w}{w}\right)-\frac{\partial}{\partial\varpi}\ln\frac{w}{(1+w)^{2}}\right]_{\mathrm{F}}\\&=\left(1-\frac{\zeta_{\mathrm{A}}}{\zeta_{\mathrm{B}}}\right)\gamma_{\mathrm{F}}^{2}w_{\mathrm{F}}\left(\frac{\partial\ln\sigma}{\partial\varpi}\right)_{\mathrm{F}}\\&\quad\times\left[2+3\frac{\partial}{\partial\ln w}\ln\left(\frac{1+w}{w}\frac{\partial\ln\zeta}{\partial\ln\sigma}\right)\right]_{\mathrm{F}}.\end{aligned} \tag{A27}$$

On the other hand, the derivative of $\mathcal{N}_{\mathrm{a}}$ at S_{F} becomes from (134)

$$\begin{aligned}\left(\frac{\partial\mathcal{N}_{\mathrm{a}}}{\partial\varpi}\right)_{\mathrm{F}}&=\left(1-\frac{\zeta_{\mathrm{A}}}{\zeta_{\mathrm{B}}}\right)\gamma_{\mathrm{F}}^{2}\frac{\sigma_{\mathrm{F}}^{2}}{w_{\mathrm{F}}^{2}}\left(\frac{\partial\ln\sigma}{\partial\varpi}\right)_{\mathrm{F}}^{2}\\&\quad\times\left[\frac{\partial\ln\varpi}{\partial\ln\sigma}\frac{\partial}{\partial\varpi}\ln\left(\sigma^{2}\varpi^{2}\frac{\partial\ln\sigma}{\partial\varpi}\right)\right.\\&\qquad\left.-\frac{\partial\ln G_{\mathrm{a}}w^{2}}{\partial\ln w}\frac{\partial\ln\zeta}{\partial\ln\sigma}\right]_{\mathrm{F}}.\end{aligned} \tag{A28}$$

Substitution of (A27) and (A28) into (160) yields

$$\left(\frac{\partial\ln\zeta}{\partial\ln\sigma}\right)_{\mathrm{F}}^{2}-2K_{1}\left(\frac{\partial\ln\zeta}{\partial\ln\sigma}\right)_{\mathrm{F}}+K_{2}=0, \tag{A29}$$

$$K_{1}=\frac{\gamma_{\mathrm{F}}^{2}}{6}\left(\frac{\partial\ln G_{\mathrm{a}}w^{4}}{\partial\ln w}\right)_{\mathrm{F}}, \tag{A30}$$

$$K_{2}=\frac{\gamma_{\mathrm{F}}^{2}}{3}\left(\frac{\partial\varpi}{\partial\ln\sigma}\right)_{\mathrm{F}}\left(\frac{\partial}{\partial\varpi}\ln\sigma^{2}\varpi^{2}\frac{\partial\ln\sigma}{\partial\varpi}\right)_{\mathrm{F}}. \tag{A31}$$

Substitution of G_{a} in equation (135) into the above equations yields equations (162) and (163).

In the nonrelativistic limit, K_1 and K_2 reduce to

$$K_{1}=\frac{2}{3}\frac{\varepsilon^{2}-4\varepsilon\delta-3\delta^{2}}{(\varepsilon+\delta)(\varepsilon-3\delta)}, \tag{A32}$$

$$K_{2}=\frac{2}{3}+\frac{1}{\varpi_{\mathrm{F}}}\left(\frac{\partial\ln\varpi}{\partial\ln\sigma}\right)_{\mathrm{F}}+\frac{1}{3}\left(\frac{\partial\varpi}{\partial\ln\sigma}\right)_{\mathrm{F}}^{2}\left(\frac{\partial^{2}\ln\sigma}{\partial\varpi^{2}}\right)_{\mathrm{F}}, \tag{A33}$$

which coincide with those given in (A4) in Paper III.

Appendix 4. Nonrelativistic Limits of Various Quantities and Expressions

Nonrelativistic limits of various quantities and expressions for $c\to\infty$ are given here, partly because of checking the results by comparing with the nonrelativistic results already given in Papers I and III and partly because of showing no discontinuities between the relativistic and nonrelativistic theories.

(4-I) The nonrelativistic limits of μ_δ and μ_ε are

$$\mu_{\delta}\approx1+\frac{\delta}{c^{2}},\quad\mu_{\varepsilon}\approx1+\frac{\varepsilon}{c^{2}} \tag{A34}$$

[see equation (2.6) in Paper I for δ and ε]. Then, equations (45) and (51) reduce to

$$\mu_{\varepsilon}\approx1+\frac{1}{c^{2}}\left(\delta-\frac{\alpha\beta}{4\pi\eta}\right),\quad\varepsilon=\delta-\frac{\alpha\beta}{4\pi\eta}=\delta+\alpha^{2}\varpi_{\mathrm{A}}^{2}. \tag{A35}$$

To obtain the nonrelativistic limit of ζ_{A} and ζ_{B} in equations (58) and (68), using equations (A34) gives

$$\zeta_{\mathrm{A}}=4\pi\eta\alpha\varpi_{\mathrm{A}}^{2},\quad\zeta_{\mathrm{B}}=\frac{4\pi\eta\varepsilon}{\alpha}, \tag{A36}$$

where one needs to subtract a relativistic factor, $4\pi\eta c^2/\alpha$, in ζ_{B} [see equation (2.12) in Paper II].

(4-II) The nonrelativistic definition of σ is $\sigma=\alpha^2\Phi/[4\pi\eta\,(2\varepsilon)^{3/2}]=(\zeta/\zeta_{\mathrm{B}})\,[v_{\mathrm{p}}/(2\varepsilon)^{1/2}]$ in equation (3.13) of Paper III, where $\zeta_{\mathrm{B}}=4\pi\eta\varepsilon/\alpha$ in equation (3.11a). If one defines the nonrelativistic w as

$$\begin{aligned}w&=\left(-\frac{\alpha\varpi B_{\mathrm{t}}}{4\pi}B_{\mathrm{p}}\right)\Big/\left(\frac{1}{2}\rho v_{\mathrm{p}}^{2}v_{\mathrm{p}}\right)\\&=\left(\frac{\varpi B_{\mathrm{t}}}{\zeta_{\mathrm{B}}}\right)\left(\frac{v_{\mathrm{p}}^{2}}{2\varepsilon}\right)^{-1},\end{aligned} \tag{A37}$$

then in the asymptotic domain

$$w=\frac{\zeta}{\zeta_{\mathrm{B}}}\Big/\left(1-\frac{\zeta}{\zeta_{\mathrm{B}}}\right),\quad\text{or}\quad\frac{\zeta}{\zeta_{\mathrm{B}}}=\frac{w}{1+w}. \tag{A38}$$

Thus,

$$\sigma=\frac{w}{2(1+w)^{3/2}},\quad\frac{dw}{d\sigma}=\frac{4(1+w)^{5/2}}{2-w}. \tag{A39}$$

Thus, one obtains $w_{\mathrm{F}}=2$ at $\sigma_{\mathrm{F}}=1/3\sqrt{3}$ and $w_{\mathrm{EQ}}=1$ at $\sigma_{\mathrm{EQ}}=1/4\sqrt{2}$.

(4-III) In the nonrelativistic limit, equations (A34), (136), and (137) give

$$v_{\mathrm{pF}}^{2}=\frac{2\varepsilon}{3},\quad\frac{\zeta_{\mathrm{F}}}{\zeta_{\mathrm{B}}}=\frac{2}{3}, \tag{A40}$$

which are coincident with equation (4.4) in Paper III, since $y_{\mathrm{F}}\equiv v_{\mathrm{pF}}/v_{\mathrm{p}\infty}=(1/3)^{1/2}$ and $\zeta_{\mathrm{B}}=4\pi\eta\varepsilon/\alpha$. Also, using equation (69), $\alpha^2\varpi_{\mathrm{A}}^2=\varepsilon-\delta$, and

$$G_{\mathrm{a}}(w_{\mathrm{F}})\to\frac{1}{3c^{4}}(\varepsilon-3\delta)(\varepsilon+\delta)\quad\text{for }c\to\infty, \tag{A41}$$

from equation (156), equation (157) reduces to

$$\frac{\varpi_{\mathrm{F}}^{2}}{\varpi_{\mathrm{A}}^{2}}=-\frac{(\varepsilon-3\delta)(\varepsilon+\delta)}{2\varepsilon(\varepsilon-\delta)}\left(\frac{\partial\ln\varpi}{\partial\ln\sigma}\right)_{\mathrm{F}}, \tag{A42}$$

which coincides with equation (5.2b) in Paper III.

(4-IV) In the limit of $c\to\infty$, one obtains $\varepsilon=\delta+\alpha^2\varpi_{\mathrm{A}}^2$, and then (139)–(141) become, respectively,

$$\zeta_{\rm F} = \frac{2}{3}\frac{4\pi\eta\varepsilon}{\alpha} = \frac{2}{3}\zeta_{\rm B}, \quad (\varpi\, v_{\rm t})_{\rm F} = \frac{\varepsilon - 3\delta}{3\alpha}, \qquad \text{(A43)}$$
$$(\varpi\, B_{\rm t})_{\rm F} = -\frac{2\zeta_{\rm B}}{3},$$

which coincides with equations (4.6b), (4.7a, b) in Paper III. Also, equation (148) reduces to

$$\frac{\alpha^2\,\Phi_{\rm F}}{4\pi\,\eta\,(2\varepsilon)^{3/2}} = \left(\frac{1}{3}\right)^{3/2}. \qquad \text{(A44)}$$

We defined $\sigma \equiv (\alpha^2\,\Phi)/[4\pi\,\eta\,(2\varepsilon)^{3/2}]$ for a nonrelativistic wind, and hence one can reconfirm the eigenvalue of the criticality problem at $S_{\rm F}$, i.e., $\sigma_{\rm F} = (1/3)^{3/2}$ [see equations (3.2) and (4.4) in Paper III].

(4-V) By taking $D_{\rm a}^2 \to 1$, $\mu_\delta \to 1$, and $\zeta_{\rm B}/c^2 \to 4\pi\,\eta/\alpha$ for $c \to \infty$, it can be seen that equation (178) reduces to

$$\frac{\varpi}{R} = \frac{\zeta^2}{4\pi\,\eta\alpha\Phi}\frac{\partial\zeta}{\partial P} = \frac{4\pi\,\eta\alpha^2\Pi^4}{v_{\rm p}\,\zeta}\frac{\partial\ln\zeta}{\partial P}, \qquad \text{(A45)}$$

which coincides with equation (6.10) in Paper I, where $v_{\rm p} = \alpha\Phi/\eta$ and $\alpha\Pi^2 = \zeta/(4\pi\,\eta)$ are used. It is also easy to confirm that $|\varrho_{\rm e}\,E_{\rm p}|/|j_{\|}\,B_{\rm t}/c| \sim O(c^{-2})$ in equation (182).

(4-VI) The nonrelativistic limit of equation (194) is given by

$$\frac{\varpi}{R} = \frac{\sqrt{\varepsilon}}{\sqrt{2}\alpha}\frac{\left(1 - \dfrac{v_{\rm p}^2}{2\varepsilon}\right)}{\left(\dfrac{v_{\rm p}}{\sqrt{2\varepsilon}}\right)}\frac{\partial}{\partial P}\left[\zeta_{\rm B}\left(1 - \frac{v_{\rm p}^2}{2\varepsilon}\right)\right]. \qquad \text{(A46)}$$

If one substitutes $v_{\rm p}^2/2\varepsilon = 1 - \zeta/\zeta_{\rm B}$ into equation (A46), one obtains equation (6.1b) in Paper III. Thus, $\varpi/R \to 0$ for $v_{\rm p}^2/2\varepsilon \to 1$ or *vice versa*.

References

Begelman, M. C. 1998, ApJ, 493, 291
Begelman, M. C., & Li, Z.-Y. 1994, ApJ, 426, 269
Blandford, R. 1993, in Astrophysical Jets, ed. D. Burgarella, M. Livio, & C. O'Dea (Cambridge: Cambridge University Press), 15
Blandford, R. D. 2002, in Lighthouse of the Universe, ed. M. Gilfanov, R. Sunyaev, & E. Churazov (Berlin: Springer-Verlag), 381
Blandford, R. D., & Payne, D. G. 1982, MNRAS, 199, 883
Bogovalov, S., & Tsinganos, K. 1999, MNRAS, 305, 211
Chiueh, T., Li, Z.-Y., & Begelman, M. C. 1991, ApJ, 377, 462
Chiueh, T., Li, Z.-Y., & Begelman, M. C. 1998, ApJ, 505, 835
Eichler, D. 1993, ApJ, 419, 111
Goldreich, P., & Julian, W. H. 1969, ApJ, 157, 869
Goldreich, P., & Julian, W. H. 1970, ApJ, 160, 971
Heyvaerts, J., & Norman, C. 1989, ApJ, 347, 1055
Heyvaerts, J., & Norman, C. 2003a, ApJ, 596, 1240
Heyvaerts, J., & Norman, C. 2003b, ApJ, 596, 1256
Heyvaerts, J., & Norman, C. 2003c, ApJ, 596, 1270
Kennel, C. F., & Coroniti, F. V. 1984, ApJ, 283, 694
Kennel, C. F., Fujimura, F. S., & Okamoto, I. 1983, Geophys. Astrophys. Fluid Dyn., 26, 147
Lyubarsky, Y., & Kirk, J. G. 2001, ApJ, 547, 437
Macdonald, D., & Thorne, K. S. 1982, MNRAS, 198, 345
Mestel, L. 1968, MNRAS, 138, 359
Michel, F. C. 1969, ApJ, 158, 727
Okamoto, I. 1974, MNRAS, 166, 683
Okamoto, I. 1978, MNRAS, 185, 69
Okamoto, I. 1992, MNRAS, 254, 192
Okamoto, I. 1997, A&A, 326, 1277
Okamoto, I. 1999, MNRAS, 307, 253 (Paper I)
Okamoto, I. 2000, MNRAS, 318, 250 (Paper II)
Okamoto, I. 2001, MNRAS, 327, 55 (Paper III)
Okamoto, I. 2002, ApJ, 573, L31 (Paper IV)
Okamoto, I. 2003, ApJ, 589, 671
Okamoto, I. 2006, PASJ, 58, 1047
Phinney, E. S. 1983, in Astrophysical Jets, ed. A. Ferrari & A. G. Pacholczyk (Dordrecht: D. Reidel Publishing Company), 201
Rees, M. J., & Gunn, J. E. 1974, MNRAS, 167, 1
Sakurai, T. 1985, A&A, 152, 121
Sakurai, T. 1987, PASJ, 39, 821
Spruit, H. C. 1996, in Evolutionary Processes in Binary Stars, ed. R. A. M. J. Wijers, M. B. Davies, & C. A. Tout (Dordrecht: Kluwer Academic Publishers), 249
Takahashi, M., & Shibata, S. 1998, PASJ, 50, 271
Tomimatsu, A. 1994, PASJ, 46, 123
Weber, E. J., & Davis, L., Jr. 1967, ApJ, 148, 217