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Pulsating States for Quantal Harmonic Oscillator

Izuru FUJIWARA and Kunio MIYOSHI
Department of Mathematical Sciences
University of Osaka Prefecture, Sakai 591

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In contrast to its coherent state, the quantal harmonic oscillator allows the existence of another state of completely nonclassical character. It is represented by a Gaussian wave packet, of which the center remains at rest and the width varies sinusoidally with time. Thus we may term it pulsating state.

It has been pointed out recently by one of the present authors[1] that, for a one-dimensional system specified by a time-independent Hamiltonian H, the position probability density at an instant of time t is represented as

$$|\langle q'|U(t)|\psi\rangle|^2=\langle\psi|\delta[q(t)-q']|\psi\rangle\,, \tag{1}$$

where the initial statevector, the evolution operator and the Heisenberg operator for position are denoted by $|\psi\rangle$, $U(t)=\exp(-itH/\hbar)$ and $q(t)=U^\dagger(t)qU(t)$ with $q(0)=q$ respectively. The remarkable advantage of Eq. (1) is that we need not care about the phase factor appearing in the time-dependent wave function $\psi(q',t)\equiv\langle q'|U(t)|\psi\rangle$.

In the case of a one-dimensional harmonic oscillator specified by the Hamiltonian $H=p^2/2m+m\omega^2q^2/2$, the time-dependence of the position operator is given by

$$\begin{aligned} q(t)&=q\cos(\omega t)+p\sin(\omega t)/m\omega \\ &=[a^\dagger\exp(i\omega t)+a\exp(-i\omega t)]/k \end{aligned} \tag{2}$$

with a constant $k=\sqrt{2m\omega/\hbar}$ and the construction operators $a=k(q+ip/m\omega)/2$ and $a^\dagger=k(q-ip/m\omega)/2$ such that $[a,a^\dagger]=1$. As is well known, the normalized ground state $|0\rangle$ of the system is such that $a|0\rangle=0$, $\langle 0|a^\dagger=0$ and $\langle 0|0\rangle=1$. If the initial state in Eq. (1) is defined by $|\psi\rangle=X|0\rangle$ in terms of a unitary operator X such that

$$X^\dagger q(t)X=[a^\dagger u(t)+au^*(t)+w(t)]/k\,, \tag{3}$$

then Eq. (1) is evaluated as

$$\begin{aligned} &2\pi\langle 0|\delta[X^\dagger q(t)X-q']|0\rangle \\ &=\int dy\langle 0|\exp[iy(a^\dagger u+au^*+w-kq')/k]|0\rangle \\ &=\int dy\exp[iy(w/k-q')-y^2|u|^2/2k^2] \\ &\quad\times\langle 0|\exp(iyua^\dagger/k)\cdot\exp(iyu^*a/k)|0\rangle \\ &=\sqrt{2\pi}\,(k/|u|)\exp[-(w-kq')^2/2|u|^2]. \end{aligned} \tag{4}$$

Here use is made of the operator identity: $\exp(\alpha a^\dagger+\beta a)=\exp(\alpha a^\dagger)\exp(\beta a)\exp(\alpha\beta/2)$.

First we shall consider the case when the unitary operator X is identified with the displacement operator $D(z)=\exp(za^\dagger-z^*a)$ such that $D^\dagger(z)aD(z)=a+z$ and $D^\dagger(z)a^\dagger D(z)=a^\dagger+z^*$ for an arbitrary complex number $z=|z|\exp(i\phi)$. Then from Eqs. (2) and (3) we see that $u(t)=\exp(i\omega t)$ and $w(t)=uz^*+u^*z=2|z|\cos(\omega t-\phi)$. If we write $|z\rangle=D(z)|0\rangle$, then Eq. (3) gives $w(t)=k\langle z|q(t)|z\rangle$, so that Eq. (4) is simplified to

$$\begin{aligned} &|\langle q'|U(t)|z\rangle|^2=(m\omega/\pi\hbar)^{1/2} \\ &\quad\times\exp[-m\omega\{q'-\langle z|q(t)|z\rangle\}^2/\hbar]. \end{aligned} \tag{5}$$

The statevector $|z\rangle$ represents the coherent state[2] due to Glauber, and we may characterize it as being "as classical as possible" according to Carruthers and Nieto.[3]

In contrast to the above we can think of the case where the displacement $w(t)$ vanishes in Eq. (3). Then for a unitary operator V replacing X we must have

$$V^\dagger[a^\dagger \exp(i\omega t)+a\exp(-i\omega t)]V$$
$$=a^\dagger u(t)+au^*(t). \quad (6)$$

Then Eq. (4) represents a Gaussian wave packet, of which the center remains at rest and the width varies with time as

$$|\langle q'|U(t)V|0\rangle|^2=[m\omega/\pi\hbar|u(t)|^2]^{1/2}$$
$$\times\exp[-m\omega q'^2/\hbar|u(t)|^2]. \quad (7)$$

The unitary operator V in Eq. (6) may be assumed to be

$$V(x)=\exp[x(r^*a^{\dagger 2}-ra^2+2isa^\dagger a)/2], \quad (8)$$

wherein a nonnegative parameter x is variable while $r=\exp(2i\rho)$ and a real parameter s are kept constant. On account of the quadratic nature of the exponent in Eq. (8) we shall have

$$V^\dagger(x)a^\dagger V(x)=a^\dagger f(x)+ag(x) \quad (9)$$

with complex functions $f(x)$ and $g(x)$ such that $f(0)=1$ and $g(0)=0$. By virtue of $[a,a^\dagger]=1$ it follows from Eq. (9) and its Hermitean conjugate that

$$|f(x)|^2-|g(x)|^2=1. \quad (10)$$

Now Eq. (9) is differentiated with respect to the continuous parameter x to yield

$$a^\dagger f'(x)+ag'(x)$$
$$=[a^\dagger f(x)+ag(x),\, r^*a^{\dagger 2}-ra^2+2isa^\dagger a]/2 \quad (11)$$

and accordingly

$$\left.\begin{aligned} f'(x)&=r^*g(x)-isf(x),\\ g'(x)&=rf(x)+isg(x). \end{aligned}\right] \quad (12)$$

Since $|r|=1$, these are combined to give

$$f''(x)=c^2f(x) \text{ and } g''(x)=c^2g(x) \quad (13)$$

with $c^2=1-s^2$. Hence for $|s|<1$ the constant c is equal to $\sqrt{1-s^2}$ and one obtains

$$\left.\begin{aligned} f(x)&=\cosh(cx)-(is/c)\sinh(cx),\\ g(x)&=(r/c)\sinh(cx), \end{aligned}\right] \quad (14)$$

for which the condition (10) holds evidently. Moreover it will be easily seen that

$$\left.\begin{aligned} f(x)&=|f(x)|\exp(-2i\theta),\\ g(x)&=|g(x)|\exp(2i\rho) \end{aligned}\right] \quad (15)$$

with $\tan 2\theta=s\sinh(cx)/c\cosh(cx)$. For $|s|=1$ we are only to take the limit $c\to 0$ in the above, and for $|s|>1$ we have $c=i\sqrt{s^2-1}$.

The function $u(t)$ in Eq. (6) is evaluated on the basis of Eq. (15) as

$$u(t)=f(x)\exp(i\omega t)+g^*(x)\exp(-i\omega t)$$
$$=|f(x)|\exp[i(\omega t-2\theta)]$$
$$+|g(x)|\exp[-i(\omega t+2\rho)], \quad (16)$$

and accordingly one obtains

$$|u(t)|^2=|f(x)|^2+|g(x)|^2$$
$$+2|f(x)g(x)|\cos 2(\omega t+\rho-\theta), \quad (17)$$

of which the extremal values are $[|f(x)|\pm|g(x)|]^2$. Thus for a quantal harmonic oscillator a new time-dependent position probability density function is afforded by Eq. (7) and shows a striking contrast to Eq. (5) for the coherent states. The normalized state

$$|x;r,s\rangle=V(x)|0\rangle \quad (18)$$

defined with the unitary operator (8) may be termed the pulsating state, since the width of the wave packet (7) varies periodically with time according to Eq. (17).

All the foregoing considerations can be transferred straightforwardly to the momentum space. Since

$$p(t)=im\omega[a^\dagger\exp(i\omega t)-a\exp(-i\omega t)]/k, \quad (19)$$

we have, corresponding to Eq. (3),

$$X^\dagger p(t)X=im\omega[a^\dagger v(t)$$

$$-av^*(t)-iw'(t)]/k. \quad (20)$$

The momentum probability density function at an instant of time t is given in just the same way as Eq. (4) by

$$|\langle p'|U(t)X|0\rangle|^2=\langle 0|\delta[X^\dagger p(t)X-p']|0\rangle$$
$$=(2\pi)^{-1}\int dy\langle 0|\exp[iy\{im\omega(a^\dagger v-av^*-iw')/k-p'\}]|0\rangle$$
$$=(2\pi)^{-1}\int dy\exp[iy(m\omega w'/k-p')-(m\omega y|v|/k)^2/2]$$
$$\times\langle 0|\exp(-m\omega yva^\dagger/k)\times\exp(m\omega yv^*a/k)|0\rangle$$
$$=[\pi\hbar m\omega|v(t)|^2]^{-1/2}\times\exp[-(m\omega w'/k-p')^2/\hbar m\omega|v(t)|^2]. \quad (21)$$

For a coherent state $|z\rangle=D(z)|0\rangle$ with $X=D(z)$ one obtains $v(t)=\exp(i\omega t)$ and $w'(t)=k\langle z|p(t)|z\rangle/m\omega$, so that Eq. (21) is transcribed as

$$|\langle p'|U(t)|z\rangle|^2=(\pi\hbar m\omega)^{-1/2}\times\exp[-(p'-\langle z|p(t)|z\rangle)^2/\hbar m\omega]. \quad (22)$$

Then for a pulsating state (18) we see in view of Eq. (6) that we have to find out a function $v(t)$ such that

$$V^\dagger[a^\dagger\exp(i\omega t)-a\exp(-i\omega t)]V=a^\dagger v(t)-av^*(t), \quad (23)$$

from which it follows with the aid of Eq. (9) that

$$v(t)=f(x)\exp(i\omega t)-g^*(x)\exp(-i\omega t). \quad (24)$$

Then Eqs. (15) and (16) lead at once to

$$|v(t)|^2=|f(x)|^2+|g(x)|^2-2|f(x)g(x)|\cos 2(\omega t+\rho-\theta). \quad (25)$$

Therefore the momuntum probability density function (21) is simplified to

$$|\langle p'|U(t)V(x)|0\rangle|^2=[\pi\hbar m\omega|v(t)|^2]^{-1/2}\times\exp[-p'^2/\hbar m\omega|v(t)|^2]. \quad (26)$$

As for the uncertainties in position and momentum for a pulsating state, they can be easily evaluated on the basis of Eqs. (7) and (26) as

$$\left.\begin{aligned}(\Delta q)_t^2&=|u(t)|^2\hbar/2m\omega,\\ (\Delta p)_t^2&=|v(t)|^2\hbar m\omega/2,\end{aligned}\right] \quad (27)$$

so that the uncertainty product varies with time as

$$(\Delta p\cdot\Delta q)_t=|u(t)v(t)|\hbar/2\geq\hbar/2, \quad (28)$$

wherein we see from Eqs. (10), (17) and (25) that

$$|u(t)v(t)|^2=1+4|f(x)g(x)|^2\times\sin^2 2(\omega t+\rho-\theta). \quad (29)$$

Subsuming all the above discussions we can easily construct a pulsating coherent state. On factorizing the unitary operator X in Eq. (3) as $D(z)V(x)$ one obtains

$$|z,x;r,s\rangle=D(z)V(x)|0\rangle, \quad (30)$$

of which the position probability density function is evaluated with the aid of Eqs. (4), (5), and (7) as

$$|\langle q'|U(t)|z,x;r,s\rangle|^2$$
$$=\langle 0|\delta[V^\dagger(x)D^\dagger(z)q(t)D(z)V(x)-q']|0\rangle$$
$$=[m\omega/\pi\hbar|u(t)|^2]^{1/2}\times\exp[-m\omega(q'-\langle z|q(t)|z\rangle)^2/\hbar|u(t)|^2]. \quad (31)$$

In addition the expectation value and the uncertainty of the Hamiltonian $H=\hbar\omega(a^\dagger a+1/2)$ in the state (30) are evaluated respectively as

$$\langle H\rangle_{z,x}=\hbar\omega[2|z|^2+|f(x)|^2+|g(x)|^2]/2, \quad (32)$$

$$(\Delta H)_{z,x}=\hbar\omega[2|fg|^2+|zf+z^*g^*|^2]^{1/2}. \quad (33)$$

If we want to treat a coherent state and a pulsating state separately, then we have to set $x=0$, $f(0)=1$, $g(0)=0$ for the former and $z=0$ for the latter respectively.

Finally we have to remark that the existence of a harmonic oscillator wave packet corresponding to our pulsating coherent state has recently been noticed by Marhic.[4),*)] Moreover the time-dependent uncertainty product (28) was investigated by Stoler[5)] by making use of the unitary operator (8) for $s=0$. Detailed discussions concerning the unitary transformation (9) were given by Yuen[6)] in connection to the two-photon coherent states.

The physical implication of the mathematical structures discussed above is now under investigation.

1) I. Fujiwara, Prog. Theor. Phys. **62** (1979), 1438.
2) R. J. Glauber, Phys. Rev. **131** (1963), 2766.
3) R. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40** (1968), 411.
4) M. E. Marhic, Lett. Nuovo Cim. **22** (1978), 376.
5) D. Stoler, Phys. Rev. **D1** (1970), 3217.
6) H. P. Yuen, Phys. Rev. **A13** (1976), 2226.

*) A similar wave packet was found also by Prof. H. Wergeland in Trondheim as we heard from him late in 1979.