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Pulsational instability of accretion disks to axially symmetric oscillations

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Summary. The pulsational instability of accretion disks to axially symmetric oscillations is examined in the approximation that conditions are quasi-adiabatic and quasi-inviscid. The essential difference from the usual stellar pulsation is that the shear motion in the unperturbed state has two (thermal and dynamical) effects on the stability of oscillations. Both effects act so as to excite oscillations if the coefficient of viscosity increases by a certain degree in the compressed phase of oscillations, in comparison with the expanded phase. The numerical condition of the growth is examined, in particular for the nearly radial oscillations whose radial wavelength is shorter than the radius of the disk (i.e. local oscillations) but longer than the thickness of the disk. The examination is made mainly for optically thin disks, and secondarily for an optically thick disk. In the particular disk examined in the optically thick case, the radiative diffusion of thermal energy in the vertical direction contributes positively to the excitation of oscillations by the very fact that the flows of oscillations are nearly radial. The instability of accretion disks to local oscillations will be important as a possible cause of turbulence leading to viscosity.

1 Introduction

Accretion disks have been extensively examined in relation to binary X-ray sources, quasars and galactic nuclei. Recent theoretical investigations of stationary disk models have revealed that some models are secularly unstable (Lightman & Eardley 1974) or thermally unstable (Pringle, Rees & Pacholczyk 1973; Shibazaki & Hoshi 1975; Shakura & Sunyaev 1976; Pringle 1976). The presence of instabilities is of interest because periodic, quasi-periodic or chaotic variability have been observed in some binary X-ray sources and quasars.

Among possible instabilities of accretion disks, however, pulsational instability has not been examined, and the purpose of this paper is to remedy this omission. Pulsational instability is particularly interesting in relation to periodic components of variability. For example, Ozernoi & Usov (1977) thought that disk models had difficulty in explaining any quasi-periodic light variations of quasars, and regarded this as one of the defects of such

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models. If pulsational instability is present, the situations are different from what they assumed. The global oscillations of disks are of interest in this respect but, in this paper, the instability to local oscillations is studied as the first step. Such a study also has its own importance. If a disk is sufficiently unstable to local oscillations, acoustic and eddy turbulence will be generated and these turbulent motions may lead to a large viscosity.

The mechanism of pulsational instability of accretion disks is also interesting as an extension of the theory of stellar pulsation. The reason is that two processses due to the shear motion of the disk become important for pulsational instability, although they are not so in normal stellar pulsation. One of these processes is thermal and the other is dynamical. First we remember that, in the usual accretion disks, thermal energy is supplied by viscous dissipation of shear motions. If the coefficient of viscosity increases in the compressed phase of oscillations in comparison with the expanded phase, the thermal energy generation by shear motions increases correspondingly. This leads to amplification of the oscillations by the same mechanism as applies to nuclear energy generation in stellar pulsation. This effect of viscous energy generation on the criterion of pulsational instability, although unimportant in normal stellar pulsation because the contribution from viscous energy is a minor part of the thermal energy, is important in accretion disks. The dynamical effect of shear motions is the following. If the coefficient of viscosity varies during an oscillation, the viscous force in the longitudinal direction varies in such a way that it has a component in phase with the longitudinal motion of the oscillation. If it is in the positive sense, the viscous force gives energy to the oscillation and amplifies it.

In addition to the above, in optically thick disks the radiative diffusion in the vertical direction acts, at least in a particular case, so as to amplify oscillations by the very fact that they are nearly radial, as will be shown. The fact that the unperturbed state has a slow accretion flow also affects the stability condition, but it is not important unless oscillations of global scale are considered.

In Section 2, the basic equations to be used are summarized. A general expression for the criterion of pulsational instability to axially symmetric oscillations is given in Section 3. Section 4 is devoted to a study of local oscillations with nearly radial motions. Sections 5 and 6 show that local oscillations are overstable if the coefficient of viscosity varies, during an oscillation, with temperature by a certain power larger than a critical one. The final section discusses the results.

2 Basic equations for perturbations

The unperturbed disk is assumed to be in a stationary accreting state. In the lowest order of approximations, a thermal balance and a hydrostatic equilibrium in the vertical direction hold. The latter implies that the pressure gradient in the vertical direction of the disk is balanced by the component of the gravitational attraction of the central body normal to the disk. (The self-gravitation of the disk has been neglected.) Such a stationary state will not, in general, be realized with a radial flow v_{0r} alone; a much slower vertical flow v_{0z} will inevitably accompany it. The effects of this slow stationary flow cannot be neglected in deriving a general criterion of pulsational instability of the disk.

Axially symmetric, small-amplitude oscillations are considered over the above stationary state. The perturbed quantities associated with the oscillations are referred to by attaching the subscript 1, and those in the unperturbed state by the subscript 0. In the cylindrical polar coordinate frame (r, ϕ, z), where the z axis is the axis of the disk rotation, the equations describing the small perturbations are

$$\frac{\partial \rho_1}{\partial t}+\frac{\partial}{r\partial r}(r\rho_0 v_r)+\frac{\partial}{\partial z}(\rho_0 v_z)=0, \tag{2.1}$$

$$\frac{\partial v_r}{\partial t}-2\Omega v_\phi+\frac{\partial p_1}{\rho_0\partial r}-\frac{\rho_1}{\rho_0^2}\frac{\partial p_0}{\partial r}=\frac{1}{\rho_0}N_{1r}+W_{1r}, \tag{2.2}$$

$$\frac{\partial v_\phi}{\partial t}+\frac{\kappa^2}{2\Omega}v_r=\frac{1}{\rho_0}N_{1\phi}-\frac{\rho_1}{\rho_0^2}N_{0\phi}+W_{1\phi}, \tag{2.3}$$

$$\frac{\partial v_z}{\partial t}+\frac{1}{\rho_0}\frac{\partial p_1}{\partial z}-\frac{\rho_1}{\rho_0^2}\frac{\partial p_0}{\partial z}=\frac{1}{\rho_0}N_{1z}+W_{1z} \tag{2.4}$$

and

$$\frac{\partial p_1}{\partial t}+v_r\frac{\partial p_0}{\partial r}+v_z\frac{\partial p_0}{\partial z}+\rho_0 c^2\left[\frac{\partial}{r\partial r}(rv_r)+\frac{\partial}{\partial z}v_z\right]=(\gamma-1)(-\operatorname{div}\mathbf{F}_1+\Phi_1), \tag{2.5}$$

where v_r, v_ϕ and v_z are the velocity components over the rotational motion $(0, \Omega r, 0)$, c is the velocity of sound, $(\gamma p_0/\rho_0)^{1/2}$ and γ is the ratio of the specific heats. The epicyclic frequency $\kappa(r)$ will be equated to the angular velocity $\Omega(r)$ of the disk rotation in Section 4, because the rotation is Keplerian in a system with no self-gravitation.

The right-hand sides of equations (2.2) to (2.5) represent the effects on oscillations both of non-conservative processes and of the stationary accretion flow in the unperturbed state. The symbols N_{1r}, $N_{1\phi}$ and N_{1z} represent the perturbed components of the viscous force per unit volume:

$$N_{1r}=-\frac{\partial}{r\partial r}(rT_{rr})+\frac{1}{r}T_{\phi\phi}-\frac{\partial}{\partial z}T_{rz}, \tag{2.6}$$

$$N_{1\phi}=-\frac{\partial}{r^2\partial r}(r^2T_{\phi r})-\frac{\partial}{\partial z}T_{\phi z}, \tag{2.7}$$

and

$$N_{1z}=-\frac{\partial}{r\partial r}(rT_{zr})-\frac{\partial}{\partial z}T_{zz}, \tag{2.8}$$

where the T's are the components of the viscous stress tensor in the perturbed state. Their expressions in terms of the coefficient of viscosity are given in the Appendix. The r, ϕ and z components of the viscous force per unit mass are $\rho_0^{-1}[N_{1r}-(\rho_1/\rho_0)N_{0r}]$, $\rho_0^{-1}[N_{1\phi}-(\rho_1/\rho_0)N_{0\phi}]$ and $\rho_0^{-1}[N_{1z}-(\rho_1/\rho_0)N_{0z}]$, respectively but, except for the longitudinal component, the second terms in the brackets have been neglected in equations (2.2) and (2.4), because they are negligibly small. The expression for $N_{0\phi}$, the viscous force (per unit volume) in the longitudinal direction in the unperturbed state, is

$$N_{0\phi}=\frac{\partial}{r^2\partial r}\left(\eta_0 r^3\frac{d\Omega}{dr}\right), \tag{2.9}$$

where η_0 is the coefficient of viscosity in the unperturbed state.

The terms, W_{1r}, $W_{1\phi}$ and W_{1z} in equations (2.2)–(2.4) represent the effects of stationary flow in the unperturbed state. In the case of $v_{0r}=v_{0r}(r, z)$ and $v_{0z}=v_{0z}(r, z)$, their expressions are

$$W_{1r} = -\frac{\partial}{\partial r}(v_{0r}v_r) - v_{0z}\frac{\partial v_r}{\partial z} - v_z\frac{\partial v_{0r}}{\partial z}, \tag{2.10}$$

$$W_{1\phi} = -v_{0r}\frac{\partial}{r\partial r}(rv_\phi) - v_{0z}\frac{\partial v_\phi}{\partial z}, \tag{2.11}$$

and

$$W_{1z} = -v_{0r}\frac{\partial v_z}{\partial r} - v_r\frac{\partial v_{0z}}{\partial r} - \frac{\partial}{\partial z}(v_{0z}v_z). \tag{2.12}$$

The above expressions can be simplified by neglecting some terms if the accretion flow in the unperturbed state and the characteristics of the oscillations are specified. We will not discuss these problems here, however.

In equation (2.5), Φ_1 represents the perturbation of the rate of thermal energy generation (per unit volume) by the viscous dissipation:

$$\Phi_1 = 2\eta_0 r^2\frac{d\Omega}{dr}\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \eta_1\left(r\frac{d\Omega}{dr}\right)^2, \tag{2.13}$$

while div $\mathbf{F}_1$ represents the perturbation of the rate of loss of thermal energy by radiation. For an optically thin disk we take

$$\text{div}\,\mathbf{F}_1 = \rho_0\mathscr{L}_0\left(\frac{\rho_1}{\rho_0} + \frac{\mathscr{L}_1}{\mathscr{L}_0}\right), \tag{2.14}$$

where $\mathscr{L}(\rho, T)$ is the so-called cooling function per unit mass. For an optically thick disk we adopt

$$\text{div}\,\mathbf{F}_1 = -\,\text{div}\left[\frac{16\sigma T^3}{3\kappa_a\rho}\,\text{grad}\,T\right], \tag{2.15}$$

where κ_a is the opacity and σ is the Stefan–Boltzmann constant.

On the right-hand side of equation (2.5), there are some additional terms which result from the presence of stationary flow in the unperturbed state. The terms, however, have been neglected from the beginning in equation (2.5), because they are negligible compared with the term $(\gamma - 1)(-\text{div}\,\mathbf{F}_1 + \Phi_1)$ in such low entropy systems as disks (the internal energy is much less than the rotational energy). In the equation of continuity (2.1), the right-hand side also has some additional terms representing the effects of the flow in the unperturbed state. The effects of these terms again are negligible in disks (this can be shown easily by retaining the terms until further transformation of the equations). In order to avoid unnecessary complications, we have neglected the terms from the beginning.

Elimination of v_ϕ, ρ_1 and p_1 from the set of equations (2.1)–(2.5) gives two relations between v_r and v_z. One is derived by taking the time derivative of equation (2.2) and by substituting equations (2.1), (2.3) and (2.5) in it. The result is

$$\left[\frac{\partial^2}{\partial t^2} + \kappa^2 - r\frac{\partial}{\partial r}\left(\frac{1}{r\rho_0}\frac{\partial p_0}{\partial r}\right)\right](\rho_0 v_r) - \frac{\partial}{\partial r}\left(c^2\rho_0\frac{\partial}{\partial r}(rv_r)\right)$$
$$-\frac{\partial}{\partial r}\left(v_z\frac{\partial p_0}{\partial z}\right) + \frac{1}{\rho_0}\frac{\partial p_0}{\partial r}\frac{\partial}{\partial z}(\rho_0 v_z) - \frac{\partial}{\partial r}\left(c^2\rho_0\frac{\partial v_z}{\partial r}\right) = -\frac{\partial}{\partial r}[(\gamma-1)(-\text{div}\,\mathbf{F}_1 + \Phi_1)]$$
$$+\frac{\partial}{\partial t}N_{1r} + 2\Omega\left(N_{1\phi} - \frac{\rho_1}{\rho_0}N_{0\phi}\right) + \frac{\partial}{\partial t}\rho_0 W_{1r} + 2\Omega W_{1\phi}. \tag{2.16}$$

The other relation between v_r and v_z is obtained from equation (2.4) by eliminating ρ_1 and p_1 by use of equations (2.1) and (2.5):

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}\right)\right](\rho_0 v_z)-\frac{\partial}{\partial z}\left(c^2\rho_0\frac{\partial v_z}{\partial z}\right)-\frac{\partial}{\partial z}\left(v_r\frac{\partial p_0}{\partial r}\right)+\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}\frac{\partial}{r\partial r}(r\rho_0 v_r)$$
$$-\frac{\partial}{\partial z}\left[c^2\rho_0\frac{\partial}{r\partial r}(rv_r)\right]=-\frac{\partial}{\partial z}[(\gamma-1)(-\operatorname{div}\mathbf{F}_1+\Phi_1)]+\frac{\partial}{\partial t}(N_{1z}+\rho_0 W_{1z}). \tag{2.17}$$

3 General criterion of pulsational instability of accretion disk

When the right-hand sides are neglected, equations (2.16) and (2.17) give, under suitable boundary conditions, the set of purely periodic oscillations of frequencies $\omega_1, \omega_2, \ldots$. We shall now write the eigenfunctions, v_r and v_z, corresponding to the mth eigenvalue ω_m as

$$\left.\begin{aligned}&v_r(r,z,t;\omega_m)=\operatorname{Re}\exp(i\omega_m t)v_{rm}\\&\text{and}\\&v_z(r,z,t;\omega_m)=\operatorname{Re}\exp(i\omega_m t)v_{zm}.\end{aligned}\right\} \tag{3.1}$$

Our present purpose is to examine how the above oscillations are amplified or damped when the effects of the right-hand sides of equations (2.16) and (2.17) are taken into account as small perturbations upon the oscillations.

To do so, we shall first show that the eigenfunctions are orthogonal when the right-hand sides of equations (2.16) and (2.17) are neglected. The outline of the proof is noted only briefly, since the procedures are well known when equations are self-ajoint. Equations (2.16) and (2.17) for v_{rm} and v_{zm} (the right-hand sides of the equations are now taken to be zero) are multiplied by v_{rn}^* and v_{zn}^* (the asterisk denotes the complex conjugate), respectively, and integrated over the whole volume of the disk. The resulting two equations are summarized after performing integrations by parts. The surface integrals vanish if the density vanishes at the surface. The equation thus obtained is symmetric with respect to the simultaneous changes of v_{rm} to v_{rn}^* and v_{zm} to v_{zn}^*, except for the term of ω_m^2. In the proof of this symmetry property, the relation $(\partial p_0/\partial r)\cdot(\partial\rho_0/\partial z)=(\partial p_0/\partial z)(\partial\rho_0/\partial r)$ has been used. This relation means $p_0=p_0(\rho_0)$ and actually holds since Ω is a function of r alone. (Strictly speaking, the relation $p_0=p_0(\rho_0)$ and thus the orthogonal relation, do not hold in general because there is a slow accretion flow in the unperturbed state. The effect of violation of the orthogonal relation on the instability criterion is, however, negligible.) Similarly, starting from equations (2.16) and (2.17) for v_{rn}^* and v_{zn}^*, we integrate them over the whole volume after multiplying them by v_{rm} and v_{zm}, respectively. The summation of these two equations gives an equation similar to that above. The difference of these two equations gives the orthogonality relation:

$$\iiint\rho_0[v_{rm}v_{rn}^*+v_{zm}v_{zn}^*]d\mathbf{x}=0,\qquad m\neq n, \tag{3.2}$$

where the integrations are performed over the whole volume V of the disk.

The use of the above orthogonality relation allows us to estimate how ω_m is changed by the non-conservative effects and by the effects of stationary accretion flow in the unperturbed state. By regarding the right-hand sides of equations (2.16) and (2.17) as small perturbations, we see that ω_m, v_{rm} and v_{zm} are slightly changed as

$$\omega_m \to \omega_m + \omega_m', \qquad v_{rm} \to v_{rm} + v_{rm}' \qquad \text{and} \qquad v_{zm} \to v_{zm} + v_{zm}', \tag{3.3}$$

where

$$v_{rm}' = \sum_{n \neq m} a_n v_{rn} \qquad \text{and} \qquad v_{zm}' = \sum_{n \neq m} b_n v_{zn}, \tag{3.4}$$

and a_n and b_n are expansion coefficients. Equations (3.1), (3.3) and (3.4) are substituted into equations (2.16) and (2.17), and the resulting two equations are integrated over the whole volume V after multiplying by v_{rm}^* and v_{zm}^*, respectively. The sum of these two equations gives, with the orthogonality relation (3.2),

$$\begin{aligned} &-2\omega_m \omega_m' \iiint \rho_0 [v_{rm} v_{rm}^* + v_{zm} v_{zm}^*]\, d\mathbf{x} \\ &= -\iiint \left(v_{rm}^* \frac{\partial}{\partial r} + v_{zm}^* \frac{\partial}{\partial z} \right) [(\gamma - 1)(-\operatorname{div} \mathbf{F}_1 + \Phi_1)]\, d\mathbf{x} \\ &+ \iiint \left[v_{rm}^* \left\{ \frac{\partial}{\partial t} N_{1rm} + 2\Omega \left(N_{1\phi m} - \frac{\rho_{1m}}{\rho_0} N_{0\phi} \right) \right\} + v_{zm}^* \frac{\partial}{\partial t} N_{1zm} \right] d\mathbf{x} \\ &+ \iiint \rho_0 \left[v_{rm}^* \left(\frac{\partial}{\partial t} W_{1rm} + 2\Omega W_{1\phi m} \right) + v_{zm}^* \frac{\partial}{\partial t} W_{1zm} \right] d\mathbf{x}, \end{aligned} \tag{3.5}$$

where the subscript m to N_1's, W_1's and ρ_1 means the coefficients of exp $(i\omega_m t)$ when they are written in the forms like equations (3.1). The use of integrations by parts, in addition to $\partial/\partial t = i\omega_m$ and $i\omega_m v_{\phi m} = -(\kappa^2/2\Omega) v_{rm}$, reduces equation (3.5) to

$$\begin{aligned} 2i\omega' \iiint \rho_0 (v_r v_r^* + v_z v_z^*)\, d\mathbf{x} &= \iiint \frac{\delta T^*}{T_0} (-\operatorname{div} \mathbf{F}_1 + \Phi_1)\, d\mathbf{x} \\ &+ \iiint \left[v_r^* N_{1r} + \frac{4\Omega^2}{\kappa^2} v_\phi^* \left(N_{1\phi} - \frac{\rho_1}{\rho_0} N_{0\phi} \right) + v_z^* N_{1z} \right] d\mathbf{x} \\ &+ \iiint \rho_0 \left[v_r^* W_{1r} + \frac{4\Omega^2}{\kappa^2} v_\phi^* W_{1\phi} + v_z^* W_{1z} \right] d\mathbf{x}, \end{aligned} \tag{3.6}$$

where and hereinafter the subscript m is dropped in order to avoid complications, and δ represents the Lagrangian variation. If the real part of the right-hand side of equation (3.6) is positive, the oscillations are overstable, since Re $(i\omega') > 0$.

The physical meaning of equation (3.6) is clear. The first integral of the right-hand side represents the well-known criterion of pulsational instability (e.g. Ledoux 1958). In our present case, however, the viscous generation of thermal energy Φ_1 is included in addition to $-\operatorname{div} \mathbf{F}_1$. The second integral represents the mechanical work done by viscous forces on the oscillations. The presence of the factor $4\Omega^2/\kappa^2$ before v_ϕ^* results from the following fact. The real velocity component in the longitudinal direction associated with a displacement is not v_ϕ, but $v_\phi + (r d\Omega/dr)\xi_r = (4\Omega^2/\kappa^2) v_\phi$, where ξ_r is the radial displacement. The third integral represents the effects of the flow in the unperturbed state.

If there are no shear motions in the unperturbed state, the effects of viscous processes are only to damp oscillations, but the situation is different if shear motions are present.

This can be seen clearly by expressing the right-hand side of equation (3.6) in terms of the cylindrical coordinates. Performing the second integration in equation (3.6) by parts, under the condition that at the surface there is no tangential shear stress or normal pressure, we have

$$
\begin{aligned}
2i\omega' \iiint & \rho_0(|v_r|^2 + |v_z|^2)\,d\mathbf{x} \\
= \iiint & \eta_0 \left(r\frac{d\Omega}{dr}\right)^2 \frac{\delta T^*}{T_0}\left[-\frac{\operatorname{div}\mathbf{F}_1}{\operatorname{div}\mathbf{F}_0} + 2\left(\frac{d\Omega}{dr}\right)^{-1}\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{\eta_1}{\eta_0}\right]d\mathbf{x} \\
+ \iiint & \eta_0\left[-2\left|\frac{\partial v_r}{\partial r}\right|^2 - \frac{2}{r^2}|v_r|^2 - \left|\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right|^2 - 2\left|\frac{\partial v_z}{\partial z}\right|^2 - \frac{4\Omega^2}{\kappa^2}\left|\frac{\partial v_\phi}{\partial z}\right|^2\right. \\
& - r^2\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\frac{\partial}{\partial r}\left(\frac{4\Omega^2}{\kappa^2}\frac{v_\phi^*}{r}\right) - \frac{1}{3}\left|\frac{\partial}{r\partial r}(rv_r) + \frac{\partial v_z}{\partial z}\right|^2 \\
& \left. - \frac{\rho_1}{\rho_0}v_\phi^*\frac{4\Omega^2}{\eta_0\kappa^2 r^2}\frac{\partial}{\partial r}\left(\eta_0 r^3\frac{d\Omega}{dr}\right) - \frac{\eta_1}{\eta_0}\frac{\partial}{\partial r}\left(\frac{4\Omega^2}{\kappa^2}\frac{v_\phi^*}{r}\right)r^2\frac{d\Omega}{dr}\right]d\mathbf{x} \\
+ \iiint & \rho_0\left[v_r^* W_{1r} + \frac{4\Omega^2}{\kappa^2}v_\phi^* W_{1\phi} + v_z^* W_{1z}\right]d\mathbf{x},
\end{aligned}
\tag{3.7}
$$

where $|\ |$ represents the absolute values. In rewriting the first integration of equation (3.6) we have used $\operatorname{div}\mathbf{F}_0 = \Phi_0 = \eta_0(rd\Omega/dr)^2$.

On the right-hand side of equation (3.7) the last terms in the first two integrals represent the main effects of the shear motions. The former one with η_1/η_0 simply represents the thermal effect that, if the coefficient of viscosity increases in the compressed phase ($\delta T > 0$) of oscillations in comparison with the expanded phase ($\delta T < 0$), it acts so as to excite oscillations. The latter term proportional to $-\eta_1 d\Omega/dr \cdot \partial(v_\phi^*/r)/\partial r$ represents the dynamical effect of the shear. If the variation of the shear force in the longitudinal direction, i.e. $\partial(\eta_1 r^3 d\Omega/dr)/r^2\partial r$, is in phase in the positive sense with $(4\Omega^2/\kappa^2)v_\phi$, i.e. $(4\Omega^2 v_\phi^*/\kappa^2)\partial(\eta_1 r^3 d\Omega/dr)/r^2\partial r > 0$, positive work is done on the oscillations, and the oscillations are amplified. The above expression of the work is reduced to $-\eta_1\partial(4\Omega^2 v_\phi^*/r\kappa^2)/\partial r \cdot r^2 d\Omega/dr > 0$ if it is integrated by parts. Most of the other terms in the second integral are negative, and represent the usual viscous damping of oscillations.

The third integral on the right-hand side of equation (3.7), representing the effects of the stationary accretion flow in the unperturbed state of oscillations, is comparable with other integrals only for global oscillations whose radial wavelength λ is of the order of the radius R of the disk. The momentum flow $\rho_0 v_{0r}$ in the radial direction in the unperturbed state is $(2\Omega/\kappa^2)\partial[r^3\eta_0 d\Omega/dr]/r^2\partial r$. Thus, the integrand of the third integral is at most of the order of $\eta_0|v_r|^2/R\lambda$. On the other hand, integrands of the first and the second integrals on the right-hand side of equation (3.7) are of the order $\eta_0|v_r|^2/\lambda^2$. This means that the third integral is smaller than the others by the factor λ/R unless global oscillations are considered. In the following sections we shall examine only the local oscillations, so the third integral will be neglected hereinafter.

4 Nearly radial local oscillations

The purely periodic zeroth-order oscillations are obtained by solving equations (2.16) and (2.17), after regarding the right-hand sides of the equations as zero. To solve the equations in general, however, is beyond the scope of the paper. The following simplest case is considered here: the radial wavelength, λ, of oscillations is taken as sufficiently shorter than the characteristic radial dimension of the disk, R, but sufficiently longer than the thickness of the disk, D, i.e. $D \ll \lambda \ll R$. Furthermore, the oscillations are taken to be nearly radial in the sense that $\partial v_r/\partial z \ll v_r/D$. Although the flow is nearly radial, it cannot be completely so in general. As is seen later, or seen from the combination of the equation of motion in the vertical direction and the equation of continuity, the vertical velocity v_z is of the order of $v_z \sim (D/\lambda)v_r$. In addition, we use $-(1/\rho_0)\partial p_0/\partial z = GMz/r^3 = \Omega^2 z$, $\kappa = \Omega$ and $c \ll \Omega\lambda$.

In the lowest order of approximations, neglecting small quantities of the orders of D/λ, λ/R and $c/\Omega\lambda$, we can reduce equations (2.16) and (2.17) to, respectively,

$$\left(\frac{\partial^2}{\partial t^2} + \Omega^2\right) v_r = 0, \tag{4.1}$$

and

$$\left(\frac{\partial^2}{\partial t^2} + \Omega^2\right) v_z - \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(c^2\rho_0\frac{\partial v_z}{\partial z}\right) + (\gamma - 1)\Omega^2 z\frac{\partial v_r}{\partial r} = 0. \tag{4.2}$$

Equation (4.1) shows that the frequency of oscillations is $\Omega(=\kappa)$, independent of wavelength. This is obvious because the pressure force has been neglected in the radial oscillations. In equation (4.2) the first term $(\partial^2/\partial t^2 + \Omega^2)v_z$ vanishes because the frequency of oscillations is found to be Ω by equation (4.1). The derivative $\partial/\partial z$ to $c^2\rho_0\partial v_z/\partial z$ in the second term of equation (4.2) is operated only to $c^2\rho_0$ in the lowest order of approximations, because the variation of $\partial v_z/\partial z$ in the vertical direction is weak (the validity of this approximation is shown by the result, equation (4.3)). Thus from equation (4.2) we have

$$\frac{\partial v_z}{\partial z} = -\frac{\gamma-1}{\gamma}\frac{\partial v_r}{\partial r} \quad \text{and} \quad v_z = -\frac{\gamma-1}{\gamma} z\frac{\partial v_r}{\partial r}. \tag{4.3}$$

Substitution of equation (4.3) into the equation of continuity gives

$$\frac{\delta\rho}{\rho_0} = i\frac{1}{\gamma\Omega}\frac{\partial v_r}{\partial r}. \tag{4.4}$$

The other relations which we need later are

$$v_\phi = \frac{1}{2}iv_r, \quad \frac{\delta T}{T_0} = (\gamma-1)\frac{\delta\rho}{\rho_0}, \tag{4.5}$$

and

$$\left.\begin{aligned} \frac{\rho_1}{\rho_0} &= \frac{\delta\rho}{\rho_0} + \frac{i}{\Omega}v_z\frac{\partial \ln\rho_0}{\partial z} = \left[1 - (\gamma-1)\frac{\partial\ln\rho_0}{\partial\ln z}\right]\frac{\delta\rho}{\rho_0}, \\ \frac{T_1}{T_0} &= \frac{\delta T}{T_0} + \frac{i}{\Omega}v_z\frac{\partial\ln T_0}{\partial z} = \left[1 - \frac{\partial\ln T_0}{\partial\ln z}\right]\frac{\delta T}{T_0}. \end{aligned}\right\} \tag{4.6}$$

Important characteristics of nearly radial oscillations, which are shown in equations (4.3)–(4.5), are that in the lowest order of approximations, v_r, v_ϕ, $\delta\rho/\rho_0$ and $\delta T/T_0$ are constants with respect to z, but v_z varies linearly with respect to z. We have derived the above results from equations (2.16) and (2.17), but they can be checked directly from the basic equations (2.1)–(2.5).

For nearly radial and local oscillations considered above, the condition of pulsational instability, equation (3.7), is reduced to

$$2i\omega' \int \rho_0 dz = \frac{9(\gamma-1)}{4\gamma^2} k^2 \int \eta_0 \left[-\frac{\operatorname{div} \mathbf{F}_1}{\operatorname{div} \mathbf{F}_0} \left(\frac{\delta\rho}{\rho_0}\right)^{-1} - \frac{2}{3}\gamma + \frac{\eta_1}{\eta_0}\left(\frac{\delta\rho}{\rho_0}\right)^{-1} \right] dz$$
$$+ \frac{k^2}{\gamma^2} \int \eta_0 \left[-3\gamma^2 - 2(\gamma-1)^2 - \frac{1}{3} + 3\gamma \frac{\eta_1}{\eta_0}\left(\frac{\delta\rho}{\rho_0}\right)^{-1} \right] dz, \tag{4.7}$$

where k represents the radial wavenumber of perturbations, defined by $(\partial v_r/\partial r)(\partial v_r/\partial r)^* = k^2 v_r v_r^*$.

5 Pulsational instability of optically thin disk

The condition of pulsational instability depends on how the coefficient of viscosity η changes during the pulsation. We assume that, to density and temperature variations, η responds as $\eta \propto \rho^\alpha T^\beta$ i.e. $\eta_1/\eta_0 = \alpha\rho_1/\rho_0 + \beta T_1/T_0$, where α and β are parameters. For a gas of neutral particles we have, approximately, $\eta \propto T^{1/2}$ (e.g. Jeans 1904), and $\eta \propto T^{5/2}$ for a fully ionized gas (e.g. Cowling 1953). In both cases η is independent of density, because $\eta \sim \rho c l$ and the mean free path l is proportional to the inverse of the density. The molecular viscosity, however, will be negligible in comparison with the turbulent (or magneto-turbulent) one in actual accretion disks. In the case of turbulence, the mean free path (or the eddy size) will not depend directly on the gas density (it may be of the order of the vertical thickness of the disk or of the vertical scale height of density stratification), and it may be allowed to take $\alpha = 1$. In the following we shall consider the two cases of $\alpha = 0$ and 1, focusing our attention mainly on the case of $\alpha = 1$. A further brief discussion on η will be given in the last section.

First we shall consider an optically thin disk, radiating energy by bremsstrahlung, and take $-\operatorname{div} \mathbf{F} = -\rho\mathscr{L}$ with $\mathscr{L} \propto \rho T^{1/2}$. In the unperturbed state, there is a thermal balance $\rho_0 \mathscr{L}_0 = \eta_0(r d\Omega/dr)^2$, i.e. $\rho_0^2 T_0^{1/2} \propto \rho_0^\alpha T_0^\beta$, and the hydrostatic balance $(1/\rho_0)\partial p_0/\partial z = -\Omega^2 z$ in the vertical direction. Then the vertical distributions of temperature $T_0(z)$ and density $\rho_0(z)$ become

$$\left.\begin{aligned} &T_0(z) = T_{00}(1 - z^2/z_b^2) \\ &\text{and} \\ &\rho_0(z) = \rho_{00}(1 - z^2/z_b^2)^{(\beta - 1/2)/(2-\alpha)} \end{aligned}\right\} \tag{5.1}$$

with

$$z_b^2 = [2 + (2\beta - 1)/(2-\alpha)]\mathscr{R} T_{00}/\Omega^2, \tag{5.2}$$

where T_{00} and ρ_{00} are the temperature and the density at $z = 0$, respectively, and z_b is the half-thickness of the disk, $\mathscr{R}$ being the gas constant.

For the perturbed state, we have $\operatorname{div} \mathbf{F}_1/\operatorname{div} \mathbf{F}_0 = 2\rho_1/\rho_0 + (1/2)T_1/T_0$ as well as $\eta_1/\eta_0 = \alpha\rho_1/\rho_0 + \beta(T_1/T_0)$. In addition, the differences between the Eulerian perturbations and the Lagrangian perturbations (see equation (4.6)) are taken into account. Equation (4.7) is then reduced to

$$2i\omega' = \frac{k^2 \eta_{00}}{\gamma^2 \rho_{00}} B\left(\frac{4 + 4\beta - 3\alpha}{2(2-\alpha)}, \frac{1}{2}\right) \Big/ B\left(\frac{3 + 2\beta - 2\alpha}{2(2-\alpha)}, \frac{1}{2}\right)$$
$$\times \left[\frac{3}{4}(7\gamma - 3)\{(\gamma - 1)\beta + (\alpha - 1)\} - \frac{1}{8}(\gamma - 1)(37\gamma - 7) - \frac{1}{3}\right], \tag{5.3}$$

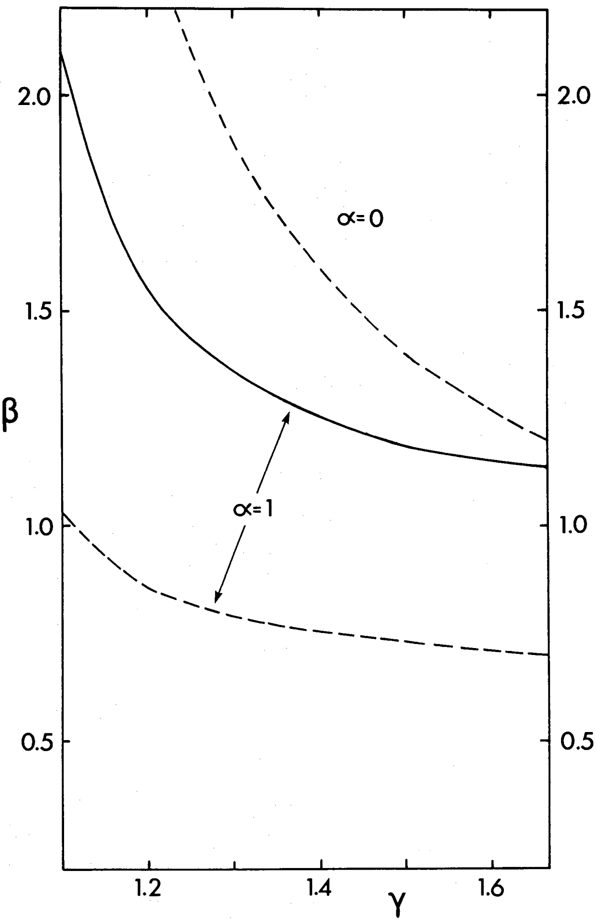

Figure 1. The critical value of β as a function of γ (the ratio of specific heats) for optically thin disks. The cooling function $\mathscr{L}$ per unit mass of the disk is taken to be $\mathscr{L} \propto \rho T^{1/2}$. The coefficient of viscosity, η, varies during the oscillation as $\eta_1/\eta_0 = \alpha\rho_1/\rho_0 + \beta T_1/T_0$. The solid curve is for the case where the coefficient of viscosity in the unperturbed state, η_0, changes with z as $\rho_0 T_0^{\beta}$, while the broken curves are for the case where η_0 goes as $\rho_0 T_0^{1/2}$.

where $B(a, b)$ is the beta function of arguments a and b. The condition of overstability is that the term in the brackets is positive. This is realized if β is larger than a critical value. This critical value of β is shown in Fig. 1 as a function of γ for $\alpha = 1$.

In the above, the temperature dependence of η in the unperturbed state is taken to be same as that in the perturbed state, i.e. $\eta_0 \propto \rho_0 T_0^{\beta}$ and $\eta_1/\eta_0 = \rho_1/\rho_0 + \beta T_1/T_0$. This is, however, not the case in general. For example, if we consider the case where $\eta_0 \sim \rho_0 c_0 l_0$ and l_0 is independent of z with $l_0 \sim D$, the vertical variation of η_0 in the unperturbed state is specified by $\eta_0 \propto \rho_0 T_0^{1/2}$. In this case, the temperature in the unperturbed state is distributed in the vertical direction in the same form as equation (5.1), but the unperturbed density is homogeneous in the vertical direction. Then, instead of equation (5.3), we have

$$2i\omega' = \frac{k^2 \eta_{00}}{2\gamma^2 \rho_{00}} B\left(\frac{3}{2}, \frac{1}{2}\right) \times \left[\frac{3}{4}(7\gamma - 3)\{3(\gamma - 1)\beta + (\alpha - 1)\} - \frac{1}{8}(\gamma - 1)(79\gamma - 25) - \frac{1}{3}\right]. \tag{5.4}$$

The critical β for overstability determined by this equation is also shown in Fig. 1 as a function of γ, for $\alpha = 1$ and 0.

6 Pulsational instability of optically thick disk

What we need now to know is $-\mathrm{div}\,\mathbf{F}_1$ expressed in terms of $\delta\rho/\rho_0$. In general, $-\partial F_{1z}/\partial z$ is proportional to $\delta\rho/\rho_0$ with a constant proportionality coefficient A, i.e. $-\partial F_{1z}/\partial z = A\delta\rho/\rho_0$. The coefficient A is determined by quantities of the unperturbed state. To know the value of A, however, detailed forms of $T_0(z)$ and $\rho_0(z)$ are required. These forms are obtained by solving the equation of hydrostatic balance $(1/\rho_0)\partial p_0/\partial z = -\Omega^2 z$ and the equation of energy balance

$$\frac{\partial}{\partial z}\left(\frac{16\sigma T_0^3}{3\kappa_{a0}\rho_0}\frac{\partial T_0}{\partial z}\right) + \eta_0\left(r\frac{d\Omega}{dr}\right)^2 = 0 \tag{6.1}$$

with $\eta_0 \propto \rho_0^\alpha T_0^\beta$. To make such a disk model is beyond the scope of this paper, and we shall be satisfied here by demonstrating for a particular case that the radiative diffusion in the vertical direction can contribute positively to the excitation of oscillations.

The case which we shall consider is that in which the radiation pressure p_r dominates over the gas pressure p_g, and the source of opacity is electron scattering alone, i.e. κ_a = constant. Since $p_r \gg p_g$, the temperature gradient is directly related to the pressure gradient. The use of hydrostatic balance in the vertical direction thus reduces equation (6.1) to

$$-\frac{4\pi}{\kappa_a}\Omega^2 + \eta_0\left(r\frac{d\Omega}{dr}\right)^2 = 0. \tag{6.2}$$

Now we assume that the unperturbed disk is homogeneous in the vertical direction, and the temperature $T_0(z)$ varies so that the hydrostatic balance holds. Since equation (6.2) shows that η_0 must be independent of z, we take $\eta_0 \propto \rho_0^\alpha T_0^\beta$ with an arbitrary α and $\beta = 0$.

The quantity $-\mathrm{div}\,\mathbf{F}_1$ can be calculated as

$$-\mathrm{div}\,\mathbf{F}_1 = \frac{\partial}{\partial z}\left(\frac{16\sigma T^3}{3\kappa_a\rho}\frac{\partial T}{\partial z}\right)_1 = \frac{\partial}{\partial z}\left[\frac{4\pi}{\kappa_a}\left(\frac{1}{\rho}\frac{\partial p_r}{\partial z}\right)_1\right]$$
$$= \frac{\partial}{\partial z}\left[\frac{4\pi}{\kappa_a}\left(-\frac{\partial v_z}{\partial t}\right)\right] = \eta_0\left(r\frac{d\Omega}{dr}\right)^2(\gamma-1)\frac{\delta\rho}{\rho_0}. \tag{6.3}$$

In deriving the last equality, $\partial v_z/\partial t \cong i\Omega v_z$, equations (4.3) and (4.4) have been used. An important point which is shown in equation (6.3) is that the radiative diffusion in the vertical direction by electron scattering in a radiation-dominated gas acts so as to amplify oscillations, since $-\mathrm{div}\,\mathbf{F}_1$ is positive in the compressed phase $(\delta\rho/\rho_0 > 0)$. This is independent of the forms of η_0, as far as oscillations are nearly horizontal.

The result of equation (6.3) is related to the following fact. In the compressed phase $(\delta\rho/\rho_0 > 0)$, the gas is compressed in the radial direction, but is expanded in the vertical direction with a larger displacement for larger z. This relation between $\delta\rho/\rho_0$ and the vertical motion is in anti-phase with that in a purely vertical oscillation. In the latter case, the radiative diffusion never acts so as to excite oscillation.

Substitution of equation (6.3) into equation (4.7) gives

$$2i\omega' = \frac{k^2\eta_0}{4\gamma^2\rho_0}\left[3(7\gamma-3)(\alpha-1) - (\gamma-1)(17\gamma-8) - 4/3\right]. \tag{6.4}$$

This shows that oscillations become overstable for $\alpha > 1.33$ when $\gamma = 4/3$. This relatively large value of α results from the adoption of $\beta = 0$. In general, β in the unperturbed state, $\eta_0 \propto \rho_0^\alpha T_0^\beta$, and β in the perturbed state, $\eta_1/\eta_0 = \alpha\rho_1/\rho_0 + \beta T_1/T_0$, need not be equal, as mentioned before. If we adopt $\eta_1/\eta_0 = \alpha\rho_1/\rho_0 + \beta T_1/T_0$ with an arbitrary β, a term $3(\gamma-1)(7\gamma-3)\beta$ is added in the brackets of equation (6.4). In this case, the condition of overstability is $\beta > 56/57$, when $\alpha = 1$ and $\gamma = 4/3$.

7 Discussion

Pulsational instability of accretion disks to axially symmetric oscillations has been examined for local ($\lambda \ll R$) oscillations with nearly radial motions ($v_r \gg v_z$). The radial wavelength of oscillations was taken to be larger than the thickness of the disk, i.e. $D \ll \lambda$. The case of optically thin disks has been examined, with supplementary discussion for a thick disk. The stability criterion depends strongly on the density and temperature dependencies of the coefficient of viscosity during the oscillations. We have considered, in particular, the two cases where η varies as $\eta \propto \rho T^\beta$ or as $\eta \propto T^\beta$, where β is a parameter. The results show that larger values of β are more conductive to instability. The critical value of β over which oscillations become overstable has been shown in Fig. 1 for optically thin disks. We cannot conclude whether pulsational instability occurs in real disks until we know the actual temperature and density dependencies of η.

To know how the coefficient of viscosity changes during the oscillations is a difficult problem for the case when η comes from turbulent processes. If the time-scale τ_e of turbulent eddy motions is much longer than the oscillation period τ_p, eddies cannot respond to the rapid change of the mean state during the oscillation; η will then be effectively constant during the oscillation. On the other hand, if $\tau_p \gg \tau_e$, the viscosity changes, responding instantly to the variation of the mean state. The turbulent viscosity is of the order of ρcl, where l is the mean eddy size. Thus, in the case of $\tau_p \gg \tau_e$, we might have $\eta_1/\eta_0 = \rho_1/\rho_0 + (3/2)T_1/T_0$, i.e. $\alpha = 1$ and $\beta = 3/2$, if we assume that the mean free path l is proportional to the disk thickness during the oscillation. The reason is that the above assumption means that $l_1/l_0 \sim D_1/D_0$ (where l_1 and D_1 are perturbations of the mean free path and the disk thickness, respectively) and this is reduced to $D_1/D_0 \sim \xi_z/D_0 \sim i(\gamma-1)(\partial v_r/\partial r)/\Omega \sim \delta T/T_0$, where ξ_z is the vertical displacement.

The characteristic time τ_e of eddy motions, whose sizes are comparable with D, is $\tau_e \sim D/c \sim \Omega^{-1}$, if the velocity of eddy motions is taken to be c. This time-scale is same as the period of oscillations, τ_p. Thus, in the practically important case, we have $\tau_p \sim \tau_e$. It is difficult in this case to know the real temperature and density dependencies of η, because the oscillations and the eddy motions are strongly coupled to each other. By this very fact of strong coupling, the case $\tau_p \sim \tau_e$ has another importance, related to a possible cause of large viscosity leading to rapid accretion. In general, large-amplitude local oscillations with random phases may become a possible source of turbulence, but a situation particularly favourable to strong turbulence will be expected in the case of $\tau_p \sim \tau_e$. In this case, oscillations excited by our pulsational instability may feed back strongly to turbulence so as to enhance it. By this feedback process, a strong turbulent state may be maintained in a disk. In stellar pulsation there are similar cases where pulsation and turbulence (convection) are coupled. A red giant star has a deep outer convection zone, and the characteristic time of eddy motions, which transport energy outward in the zone, is comparable with the pulsation period of the star as a whole. Thus the pulsational instability criterion depends on how the coupling between eddy motions and the pulsation occurs. This coupling problem is not solved strictly, although some theories (Unno 1967; Gough 1977) have been proposed.

Here we shall discuss briefly the growth rate of local oscillations. As is seen from equation (4.7), the growth rate, if any, is of the order of $\eta_0/\rho_0\lambda^2$. This increases with decreasing wavelength, but for a too-short wavelength the result cannot be applied because it is based on $\lambda \gg D$. We can, however, consider the wavelength dependence of the growth rate until $\lambda \gtrsim D$. If the wavelength decreases, the following three effects, which were not taken into account in our treatment, appear. The first is that the kinetic energy of the vertical oscillation becomes non-negligible in comparison with that in the disk plane. In other words, the term of $|v_z|^2$ in the integrand of the left-hand side of equation (3.7) becomes comparable with the term of $|v_r|^2$. Since $v_z v_z^* = [(\gamma - 1)/\gamma]^2 k^2 z^2 \sim (kD)^2 \propto k^2$, this first effect is to saturate the growth rate when λ approaches D. The second effect is that, if the wavelength becomes comparable with the disk thickness, the frequency of oscillations increases by the effect of the pressure force. Furthermore, such characteristics as v_r, v_ϕ, $\delta\rho/\rho_0$ and $\delta T/T_0$ are no longer constant with respect to z. Investigation of the equations shows that they act so as to decrease the growth rate. The third effect, which is most important, is the radiative energy diffusion in the radial direction during the oscillations. This always acts so as to damp the oscillations. Considering the above three effects, we suppose that the growth rate reaches a maximum at a certain λ not far from D.

It is implicit in the above discussion that the oscillations are standing. In the lowest order of approximation the oscillations are actually standing (see equation (4.1)), but they propagate as pressure waves if we proceed to the next order of approximation. We should estimate to what extent the propagation works so as to damp the oscillations. The time-scale in which a wave propagates a characteristic length L of an unstable region in a disk is L/c, which is $(\eta_0/\rho_0\lambda^2)^{-1}$ times LD/λ^2 if we take $\eta_0 \sim \rho_0 c D$. This means that, if the unstable region extends so widely that $LD/\lambda^2 \gg 1$, the wave actually grows to a finite amplitude. If not, careful considerations will be required such as what fraction of oscillations can be trapped in the unstable region without penetrating to a surrounding stable region.

Finally, the growth rate is compared with that of the thermal instability. The typical growth rate of thermal imbalance, if any, will be of the order of $\Phi_0/\rho_0 c_v$ (c_v is the specific heat $\sim \eta_0\Omega^2/\rho_0 c^2$, which is $\eta_0/\rho_0\lambda^2$ times $(\lambda/D)^2$). In other words, the growth rate of local oscillations is lower than that of the thermal instability by the factor $(D/\lambda)^2$.

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Appendix: expressions for viscous stress tensors

The viscous stress tensors in the perturbed state are expressed in the cylindrical polar coordinate frame as

$$T_{rr} = -\eta_0 \left(2\frac{\partial v_r}{\partial r} + \frac{1}{3}\operatorname{div}\mathbf{v}\right), \tag{A1}$$

$$T_{\phi\phi} = -\eta_0 \left(\frac{2}{r} v_r + \frac{1}{3}\operatorname{div}\mathbf{v}\right), \tag{A2}$$

$$T_{zz} = -\eta_0 \left(2\frac{\partial v_z}{\partial z} + \frac{1}{3}\operatorname{div}\mathbf{v}\right), \tag{A3}$$

$$T_{r\phi} = T_{\phi r} = -\eta_0 r \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) - \eta_1 r \frac{d\Omega}{dr}, \tag{A4}$$

$$T_{rz} = T_{zr} = -\eta_0 \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right), \tag{A5}$$

$$T_{\phi z} = T_{z\phi} = -\eta_0 \frac{\partial v_\phi}{\partial z} \tag{A6}$$

where

$$\operatorname{div}\mathbf{v} = \frac{\partial}{r\partial r}(r v_r) + \frac{\partial v_z}{\partial z}. \tag{A7}$$