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PULSATIONAL INSTABILITY OF ISOTHERMAL GAS SPHERES WITHIN THE FRAMEWORK OF GENERAL RELATIVITY

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SUMMARY

Investigation has been made of the general relativistic pulsational instability of gaseous spheres such that the equation of state is of the form $p = q\rho$, where p and ρ are the pressure and energy-density respectively, and q is a constant. Assuming that during pulsation the motion takes place adiabataically with index γ, the equation for spherical pulsation is derived by using general relativity. It is shown that the spheres, if sufficiently compressed, are unstable. The degree of compression needed to make the sphere unstable depends on the two parameters, γ and q. For given value of q, the degree of compression needed increases with γ, while for given γ, the compression needed decreases with increasing q. When q approaches zero, the equation of pulsation reduces to the corresponding one in Newtonian theory for isothermal gas spheres. Post-Newtonian effects are also briefly discussed.

1. INTRODUCTION

Recently Chandrasekhar (1972) has investigated within the framework of general relativity the structure of gas spheres such that the gas obeys the equation of state of the form $p = q\rho$ where p and ρ are the pressure and energy-density respectively, and q is a constant. The gas sphere is analogous to isothermal gas spheres in Newtonian theory. In particular the gas spheres extend to infinity unless compressed by some external media. The relation between the volume of the gas sphere and the pressure which it exerts at the boundary can be calculated both in Newtonian theory and in general relativity. In the classical case the isothermal gas sphere is a model of an H I region surrounded by an H II region (Ebert 1955); the equation of state $p = q\rho$ with $q = 1/3$ is a limiting form of highly energetic gases (Landau & Lifshitz, 1959) and even the value $q = 1$ has also been suggested (Zeldovich & Novikov 1971). Thus the general relativistic gas sphere for which $p = q\rho$ is valid can be regarded as the core of a highly energetic neutron star.

The object of the present paper is to investigate, within the framework of general relativity, the stability of the gas spheres against pulsation. As has been shown previously (Ebert 1957; Yabushita 1968) provided that the ratio of specific heats is not too large, the isothermal gas spheres in Newtonian theory are pulsationally unstable if they are sufficiently compressed. Thus it is of considerable interest to know how the result for the classical gas spheres is modified by general relativity. Chandrasekhar (1964) has derived an equation of pulsation; however, in order to make easy the comparison with the classical gas sphere, the equation of pulsation will be derived by a slightly different method. As will be seen the spheres are pulsationally unstable if they are sufficiently compressed. However, the classical isothermal gas spheres can be pulsationally unstable if they are sufficiently compressed. Hence the two instabilities, one in Newtonian theory and the other in general relativity are manifestations of essentially the same phenomenon.

In the following we shall regard q as a free parameter such that $0<q<1$. When q tends to zero, we shall recover the classical case while if q is close to $1/3$ or 1, we shall be dealing with highly relativistic case.

2. BASIC EQUATIONS

We consider a spherically symmetrical system, and following Tolman (1934), write the metric in the form

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d^2\phi + e^{\nu} d\tau^2, \; \tau = ct, \tag{2.1}$$

where c denotes the velocity of light. Assuming that λ and ν are functions of r and τ only, the energy momentum tensors

$$T_{\mu}^{\ \nu} = (p+\rho) g_{\alpha\mu} \frac{dx^{\alpha}}{ds}\frac{dx^{\nu}}{ds} - g_{\mu}^{\ \nu} p \tag{2.2}$$

have the following components;

$$\frac{8\pi G}{c^4} T_1^{\ 1} = -e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}, \tag{2.3}$$

$$\frac{8\pi G}{c^4} T_2^{\ 2} = \frac{8\pi G}{c^4} T_3^{\ 3} = -e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right) + e^{-\nu}\left(\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\lambda}\dot{\nu}}{4}\right), \tag{2.4}$$

$$\frac{8\pi G}{c^4} T_4^{\ 4} = e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}, \tag{2.5}$$

$$\frac{8\pi G}{c^4} T_4^{\ 1} = -e^{-\lambda}\frac{\dot{\lambda}}{r}, \quad \frac{8\pi G}{c^4} T_1^{\ 4} = e^{-\nu}\frac{\dot{\lambda}}{r}, \tag{2.6}$$

where the accents denote differentiation with respect to r and the dots with respect to τ and where G is the constant of gravity. All the other components vanish. The identity

$$T_4^{\ 1} = -e^{(\nu-\lambda)} T_1^{\ 4},$$

which follows from equation (2.6) should also be noted. The contravariant four velocity u has the components $(u^1, 0, 0, u^4)$. When u^1 is zero, we have that

$$T_1^{\ 1} = T_2^{\ 2} = T_3^{\ 3} = -p, \; T_4^{\ 4} = \rho. \tag{2.7}$$

For equilibrium, λ and ν are independent of t. Denoting the equilibrium quantities by suffix 0, equations (2.3) through (2.5) yield the well-known relations

$$\frac{dp_0}{dr} + (p_0 + \rho_0)\,\tfrac{1}{2}\,\frac{d\nu_0}{dr} = 0,$$

$$\frac{d}{dr}\,(re^{-\lambda_0}) = 1 - \frac{8\pi G}{c^4}\,\rho_0 r^2, \tag{2.8}$$

$$\frac{e^{-\lambda_0}}{r}\,\frac{d\nu_0}{dr} = \frac{1}{r^2}\,(1 - e^{-\lambda_0}) + \frac{8\pi G}{c^4}\,p_0.$$

When an equation of state is given, these equations are sufficient to determine equilibrium configurations of relativistic fluid spheres.

We now consider small motions about an equilibrium configuration. We write

$$\lambda = \lambda_0 + \delta\lambda(r,\tau), \qquad \nu = \nu_0 + \delta\nu(r,\tau),$$

$$p = p_0 + \delta p(r,\tau), \qquad \rho = \rho_0 + \delta\rho(r,\tau), \tag{2.9}$$

$$u^1 = \delta u^1\,(r,\tau), \qquad u^4 = u_0{}^4 + \delta u^4\,(r,\tau).$$

Clearly $\delta\lambda$, $\delta\nu$, δp, $\delta\rho$, δu^1, δu^4 are to be regarded as small quantities of first order. Then the relations (2.7) remain valid to this order of approximation. An equation which corresponds to the equation of conservation of mass in classical mechanics is

$$(T_4{}^\nu)_{;\nu} = \frac{\partial T_4{}^\nu}{\partial x^\nu} + \{\alpha\nu, \nu\}\,T_4{}^\alpha - \{4\nu, \alpha\}\,T_\alpha{}^4 = 0.$$

When appropriate expressions for the Christoffel symbols { } are used (Tolman 1934, p. 250), this equation takes the following form

$$\dot{\rho} + T_4{}^{1\prime} + \tfrac{1}{2}\dot{\lambda}(p+\rho) + T_4{}^1\left(\tfrac{1}{2}\lambda' + \tfrac{1}{2}\nu' + \frac{2}{r}\right) = 0.$$

By differentiating with respect to τ and neglecting small quantities higher than the first, the above equation yields

$$\ddot{\rho} - \frac{c^4}{8\pi G}\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\frac{e^{-\lambda_0}}{r}\,\delta\ddot{\lambda} + \tfrac{1}{2}\left\{p_0 + \rho_0 - \frac{c^4}{8\pi G}\,(\lambda_0' + \nu_0')\,\frac{e^{-\lambda_0}}{r}\right\}\delta\ddot{\lambda} = 0. \tag{2.10}$$

But owing to equations (2.8), the last term vanishes.

For small motions about an equilibrium, equations (2.3) through (2.5) yields

$$\frac{1}{r}\,e^{-\nu_0}\,\delta\ddot{\lambda} = \frac{8\pi G}{c^4}\left\{\delta p' + \frac{\nu'}{2}\,(\delta p + \delta\rho) + \tfrac{1}{2}\,(p_0 + \rho_0)\,\delta\nu'\right\}.$$

When this relation is inserted into the above equations one finds that

$$\frac{\partial\rho^2}{\partial\tau^2} - \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\left\{e^{\nu_0 - \lambda_0}\left[\delta p' + \frac{\nu_0'}{2}\,(\delta p + \delta\rho) + \frac{\delta\nu'}{2}\,(p_0 + \rho_0)\right]\right. = 0. \tag{2.11}$$

Introducing the function $f(r, t)$ through the relation

$$\delta\rho = \frac{1}{r^2}\,\frac{\partial f}{\partial r} \tag{2.12}$$

and assuming that all the perturbations depend on τ through the factor exp ($\sigma\tau$), the above equation can be integrated to give

$$\sigma^2 f - r^2\, e^{\nu_0-\lambda_0}\left[\delta p' + \frac{\nu_0'}{2}(\delta p+\delta\rho) + \frac{\delta\nu'}{2}(p_0+\rho_0)\right] = C, \tag{2.13}$$

where C is a constant of integration. Since the perturbations δp, $\delta\rho$ and $\delta\nu$ are to remain finite at $r = 0$, the constant C must clearly be equal to zero.

The equation (2.13) has to be supplemented by a relation which connects $\delta\rho$ to δp. Following Chandrasekhar & Friedman (1972), define the adiabatic index γ by the relation

$$\frac{\Delta p}{p} = \gamma\,\frac{\Delta\rho}{p+\rho}, \tag{2.14}$$

where Δp and $\Delta\rho$ are the Lagrangian changes of p and ρ respectively. To the approximation adopted in the present paper we have that

$$u_0{}^4\,\delta\dot{p} + \delta u^1\, p_0' = \gamma\,\frac{p_0}{p_0+\rho_0}\,[u_0{}^4\,\delta\dot{\rho} + \delta u^1\,\rho_0']. \tag{2.15}$$

On the other hand, $u_0{}^4 = e^{-\nu_0/2}$, while δu^1 is given from equation (2.6) as

$$\frac{8\pi G}{c^4}(p_0+\rho_0)\,u_0{}^4\,\delta u^1 = -\frac{\delta\dot{\lambda}}{r}\,e^{-\nu_0-\lambda_0}$$

and the perturbation $\delta\rho$ is given from equation (2.5) by the relation

$$\frac{8\pi G}{c^4}\,\delta\rho = \frac{1}{r^2}\frac{d}{dr}(re^{-\lambda_0}\,\delta\lambda).$$

Hence equation (2.15) can be integrated to give the relation

$$\delta p = \frac{p_0'}{p_0+\rho_0}\frac{f}{r^2} + \gamma\left[\frac{1}{r^2}\frac{df}{dr} - \frac{\rho_0'}{p_0+\rho_0}\frac{f}{r^2}\right]. \tag{2.16}$$

The perturbation $\delta\nu$ is obtainable from equation (2.3) which yields that

$$\frac{8\pi G}{c^4}\,\delta p = e^{-\lambda_0}\frac{\delta\nu'}{r} - e^{-\lambda_0}\delta\lambda\left(\frac{\nu_0'}{r}+\frac{1}{r^2}\right), \tag{2.17}$$

and since δp has been expressed in terms of f, $\delta\nu'$ can also be regarded as given in terms of f. In this way one finally arrives at the following equation for spherical pulsation;

$$\frac{\sigma^2}{q} f e^{\lambda_0-\nu_0} = \frac{\gamma}{1+q}\frac{d^2 f}{dr^2} + g(r)\frac{df}{dr} + h(r) f, \tag{2.18}$$

where

$$g(r) = -\frac{2}{r}\frac{\gamma}{(1+q)} - \frac{\rho_0'}{\rho_0}\frac{\gamma}{1+q} + \frac{4\pi G}{c^4}\,e^{\lambda_0}\,r\,(p_0+\rho_0)\,\frac{\gamma}{1+q},$$

$$h(r) = \frac{1+q-\gamma}{(1+q)^2}\left(\frac{1}{r^2}\frac{\rho_0'}{\rho_0}\right)' - \frac{q(1+q-\gamma)}{(1+q)^3}\left(\frac{\rho_0'}{\rho_0}\right)^2 + \frac{4\pi G}{c^4}(p_0+\rho_0)$$
$$\times e^{\lambda_0}\left\{1 - \frac{2q}{1+q}\frac{r\rho_0'}{\rho_0} + rq\left[\frac{\rho_0}{p_0+\rho_0}\times\frac{2+q-\gamma}{1+q} - \frac{\rho_0\rho_0'}{(p_0+\rho_0)^2}\right]\right\}.$$

Following Chandrasekhar (1972), write

$$r = \alpha\xi, \qquad \alpha^2 = \frac{c^4 q}{4\pi G \rho_c (1+q)} \tag{2.19}$$

$$\rho_0 = \rho_c \, e^{-\psi} \quad p_0 = q\rho_c \, e^{-\psi}$$

where ρ_c is the value of energy-density at $r = 0$. When the change of variables (2.19) has been made, equation (2.18) takes the following form;

$$\begin{aligned} &\frac{\gamma}{1+q}\frac{d^2 f}{d\xi^2} + \frac{\gamma}{1+q}\left(-\frac{2}{\xi} - \frac{\rho_0'}{\rho_0} + q\xi \, e^{-\psi} \, e^{\lambda_0}\right)\frac{df}{d\xi} \\ &+ \left\{\frac{1+q-\gamma}{1+q}\left[\frac{\xi^2}{1+q}\left(\frac{1}{\xi^2}\frac{\rho_0'}{\rho_0}\right)' - \frac{q}{(1+q)^2}\left(\frac{\rho_0'}{\rho_0}\right)^2\right]\right. \\ &\left. + e^{-\psi} \, e^{\lambda_0}\left[1 - \frac{2q\xi}{1+q}\frac{\rho_0'}{\rho_0} + q\xi\frac{\rho_0'}{\rho_0}\frac{(1+q-\gamma)}{(1+q)^2}\right]\right\} f = \frac{\sigma^2 c^4}{4\pi G\rho_c(1+q)} \, e^{\lambda_0 - \nu_0} f, \end{aligned} \tag{2.20}$$

where the accents denote differentiation with respect to ξ.

3. BOUNDARY CONDITIONS

The eigenvalue equation (2.20) has to be solved under appropriate boundary conditions. Since $\delta\rho$ is given from f by the relation

$$\delta\rho = \frac{1}{\alpha^3 \, \xi^2}\frac{df}{d\xi} \tag{3.1}$$

one must have that

$$\frac{df}{d\xi} = 0 \text{ at } \xi = 0. \tag{3.2}$$

Another condition is given by considering the perturbation at the boundary; Let us contend that at the boundary, $\delta p = 0$. From the relation (2.16), this condition is found to be equivalent to the relation

$$\frac{\gamma}{1+q}\frac{df}{d\xi} + \frac{1+q-\gamma}{(1+q)^2}\frac{\rho_0'}{\rho_0} f = 0, \text{ at } \xi = \xi_b, \text{ say.} \tag{3.3}$$

It should be noted that in the limit $q \to 0$, $e^{\lambda_0} \to 1$ and $e^{\nu_0} \to 1$, while the function ψ reduces to the Emden function ψ for isothermal gas spheres. Hence, in this limit equation (2.20) reduces to the form

$$\frac{d^2 f}{d\xi^2} + \left(-\frac{2}{\xi} + \frac{d\psi}{d\xi}\right)\frac{df}{d\xi} + \gamma^{-1}\left[e^{-\psi} + (1-\gamma)\left(\frac{2}{\xi}\frac{d\psi}{d\xi} - \frac{d^2\psi}{d\xi^2}\right)\right] f = \frac{\sigma^2 c^4}{4\pi G\rho_c} f. \tag{3.4}$$

Equation (3.4) is exactly the same as the equation for pulsation of isothermal gas spheres in Newtonian theory (see equation (2.19) of Yabushita (1968) where ψ is defined by $\rho = \rho_c \, e^{\psi}$). The structure of gas spheres for which the equation of state $p = q\rho$ is valid has been given by Chandrasekhar; and in the preceding paper (Yabushita 1973) the relation has been obtained between the volume of the gas sphere and the pressure which it exerts at the boundary; also the expressions for λ_0 and ν_0 have been given. Hence, all the unperturbed quantities that appear in equation (2.20) can be regarded as given.

We now proceed to considering the eigenvalue problem (2.20). Instead of calculating the eigenvalues σ^2, it is easier to adopt the following procedure. Equation (3.1) is of Sturm–Liouville type; hence $\sigma^2 = 0$ corresponds to marginal stability. Hence, it is only sufficient to see if the equation admits a solution which corresponds to $\sigma^2 = 0$ and which satisfies the boundary conditions (3.2) and (3.3).

To do this numerically, let us first note that if one puts

$$f = \xi^n (1 + a\xi + b\xi^2 + \dots) \quad \xi \ll 1,$$

then equation (2.21) gives that $n(n-3) = 0$. Since $\delta\rho$ is to remain finite at $\xi = 0$, one must have that $n = 3$. Therefore one should integrate the differential equation (2.21) with $\sigma^2 = 0$ from the initial conditions

$$f = \xi^3, \quad \xi \ll 1.$$

The numberical integration is continued until the boundary condition (3.3) is satisfied at a certain value of ξ. If a value ξ_b of ξ is found such that the condition (3.3) holds there, $\sigma^2 = 0$ is one of the eigenvalues to equation (2.20). A similar precedure was used by Ebert (1957) in his stability analysis of infinitely large isothermal gas spheres. On the other hand if no value of ξ is found for which the condition (3.3) is satisfied, the gas sphere is thoroughly stable.

4. NUMERICAL RESULT

The eigenvalue problem posed by equation (2.20) with σ^2 put equal to zero contain two independent parameters, namely q and γ. The value of ξ_b where the condition (3.3) first is satisfied will depend upon the values of the two parameters. When q is small, the pulsational stability will be almost identical to the stability within the framework of Newtonian theory. Therefore we shall carry out numerical computations for the cases $q = 0.1$, $q = 1/3$ and $q = 1$.

TABLE I

The value of ξ_b such that a gas sphere with the boundary at ξ_b is marginally stable. The spheres with boundaries with values of ξ greater than ξ_b are pulsationally unstable

	$q = 0\cdot1$	$q = 1/3$	$q = 1\cdot0$
$\gamma = 1\cdot0$	2·82	1·90	1·16
1·1	3·05	2·01	1·22
1·2	3·30	2·12	1·28
1·3	3·58	2·23	1·31
1·4	3·88	2·34	1·39
1·5	4·23	2·44	1·45
1·6	4·64	2·54	1·50
1·7	5·11	2·65	1·56
1·8	5·73	2·76	1·62

In Table I we give the values of ξ_b at which the boundary condition (3.3) is satisfied. For given values of q and γ, gas spheres with the boundary value of ξ greater than ξ_b are pulsationally unstable. In Fig. 1, we plot the perturbation of energy-density, $\delta\rho$ which corresponds to the eigenvalue $\sigma^2 = 0$.

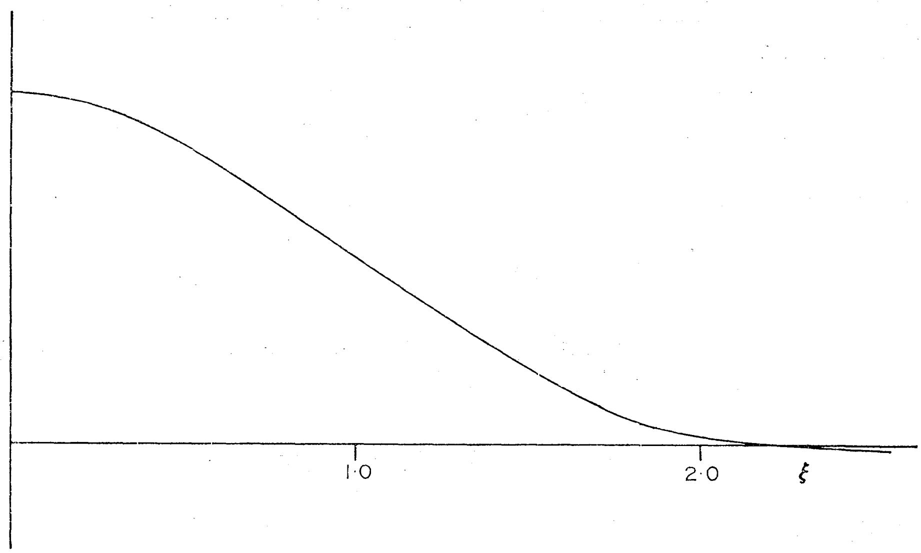

FIG. 1. *Perturbation* $\delta\rho$ *of energy-density.* $q = 1/3$ *and* $\gamma = 1{\cdot}6$. $\delta\rho$ *for other values of* q *and* γ *are similar to the case shown here.*

5. DISCUSSIONS

We now come to discuss the implications of the result given in Table I. As has been explained in the preceding paper (Yabushita 1973), the greater is the value of ξ_b at the boundary of the gas sphere, the greater is the compression toward the centre.

It may be seen from Table I that for given values of q, the degree of compression toward the centre required to make the sphere pulsationally unstable increases with the value of γ. This feature is common to the isothermal gas spheres in Newtonian theory. It should also be noted that for given value of γ, the degree of compression needed to make the sphere dynamically unstable decreases with increasing q. This feature is understandable because in general relativity pressure also contributes to gravity and owing to the equation of state $p = q\rho$, the role of pressure increases with increasing q.

In order to discuss some qualitative feature of the eigenvalue equation (2.21), let us first note that the equations (2.8) admit a particular solution

$$e^{-\psi} = Q/\xi^2,\ Q = 2(1+q)/[(1+q)^2+4q] \tag{5.1}$$

that has been found by Chandrasekhar (1972). This particular solution has a singularity at the centre $\xi = 0$. However, in order to make comparison with the classical case, let us adopt it and insert it to the eigenvalue equation (2.20). One then readily obtains the following form of the eigenvalue problem;

$$\frac{\gamma}{1+q}\frac{d^2f}{d\xi^2}+\frac{\gamma}{1+q}\frac{2q}{1+q}\frac{1}{\xi}\frac{df}{d\xi}+\left[\frac{1+q-\gamma}{1+q}\cdot\frac{6-2q}{(1+q)^2}+\frac{2(1+5q)}{(1+q)^2}\right]$$
$$\times\frac{f}{\xi^2} = \frac{\sigma^2c^4\,e^{\lambda_0-\nu_0}}{4\pi G\rho_c\,(1+q)}f. \tag{5.2}$$

If q is sufficiently small so that only the first power in q need be retained, the above equation reduces to

$$\gamma \frac{d^2 f}{d\xi^2} + \gamma 2q \frac{1}{\xi}\frac{df}{d\xi} + [8 - 6\gamma + 14q\gamma]\frac{f}{\xi^2} = \frac{\sigma^2 c^4}{4\pi G} e^{\lambda_0 - \nu_0} f.$$

By writing $f = g\xi^{-q}$, the above equation reduces to

$$\frac{d^2 g}{d\xi^2} + (8 - 6\gamma + 14\gamma q + q)\frac{g}{\xi^2} = \frac{\sigma^2 c^4}{4\pi G} e^{\lambda_0 - \nu_0} g.$$

Since $e^{\lambda_0 - \nu_0}$ is every where positive, one easily finds that when $\xi_b = \infty$ positive eigenvalues σ^2 cannot exist if the inequality

$$8 - 6\gamma + 14\gamma q + q < 0, \text{ or } \gamma > \frac{4}{3}\left(1 + \frac{118}{48} q\right), \tag{5.3}$$

is satisfied. This inequality is not the necessary condition for the stability of the spheres, but a sufficient condition for the stability. It clearly shows an effect of general relativity upon the stability of isothermal gas spheres. As $q \to 0$, the inequality (5.3) reduces to the corresponding one in classical theory, namely $\gamma > 4/3$.

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