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PULSE BROADENING DUE TO MULTIPLE SCATTERING IN THE INTERSTELLAR MEDIUM

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SUMMARY

It has been observed that the pulses received from pulsars are broadened at low frequencies, and this effect is thought to be due to multipath propagation in the interstellar medium. In this paper the pulse broadening is investigated in the limit of strong multiple scattering, in which ray optics is applicable. Two cases are considered: the scattering medium is assumed either to be concentrated in a small fraction of the line of sight near the pulsar, or to extend over the whole line of sight. Analytical solutions for the probability density function are found for these two cases, with the aid of the theory of Wiener integrals of Kac and Blanc-Lapierre & Fortet. Both solutions are characterized by a slow rise to a decay which rapidly becomes exponential. The relation between this exponential decay time constant and the r.m.s. broadening of the angular size of the source is found, and compared with that expected on the basis of the 'thin-slab' model in which the scattering medium is effectively confined to a plane approximately midway between source and observer.

1. INTRODUCTION

It has been suspected that irregularities in the interstellar medium cause the radiation emitted by a pulsar to travel along a variety of paths, of differing length, between source and observer (Scheuer 1968; Salpeter 1969). Even intrinsically sharp pulses will then be observed stretched out in time, and this asymmetrical broadening has indeed been observed in pulses from the Crab Nebula pulsar NP 0532 (Staelin & Sutton 1970; Rankin *et al.* 1970), from the Vela pulsar PSR 0833–45 (Ables, Komesaroff & Hamilton 1970), and from five more pulsars (Lang 1971a, b). This broadening becomes much more pronounced at lower frequencies, and follows generally the λ^4 law predicted by theory (Lang 1971a, b).

The observations of pulsar amplitude variation show clearly that the phase changes in the scattering medium are very large at these low frequencies, and in these conditions the angular spectrum of the scattered radiation is gaussian (Fejer 1953; Uscinski 1968). For a medium in which the electron density fluctuates with r.m.s. amplitude ΔN with characteristic scale size a, the mean square angular scatter per unit length D, suffered by the radiation is given by

$$D \approx \frac{r_0^2\lambda^4(\Delta N)^2}{a} \tag{1}$$

where r_0 is the classical electron radius $2{\cdot}82 \times 10^{-15}$ m and λ the wavelength of the radiation.

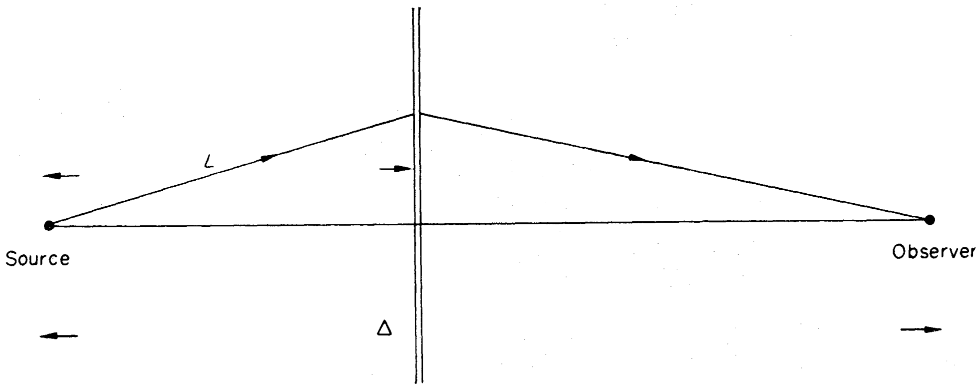

FIG. 1. *Scattering by a thin slab of fluctuating electron density.*

If the scattering occurs in a physically thin screen approximately midway between source and observer (Fig. 1), it is well known that an intrinsically sharp pulse is broadened into a pulse having a sharp rise followed by exponential decay with time constant (Cronyn 1970).

$$\frac{\Delta(\Delta-L)}{2L}\frac{\theta_0^2}{c} \tag{2}$$

θ_0 being the r.m.s. angle over which scattered radiation from the pulsar is received by the observer.

Ables *et al.* (1970) and Lang (1971a, b) find that the high frequency pulse-shape convolved with an exponential distribution gives a reasonably good fit to their observations. The distribution function for the Crab pulse NP 0532 seems to be more complex. Rankin *et al.* (1970) found initially that a function of the form xe^{-x} gave a better fit to their observations than a straightforward exponential, and this was confirmed by Counselman & Rankin (1971), who subsequently discovered however (note added in proof to latter paper) that the scattering probability distribution function varies with time.

It is of interest to discover what form the scattering p.d.f. would take in a more realistic model of the interstellar medium, when the fluctuating electron density is not confined to a small region between source and observer; and in particular what effect this would have on the rise time of the pulse. An absolutely sharp rise is not to be expected in any model in which the scattering medium is distributed throughout space, because even radiation reaching the observer exactly from the source direction has suffered some deviation on the way, and the proportion of photons which suffer no deviation is small.

The purpose of the work here described is to discover, on a purely theoretical basis, what pulse shapes are to be expected when the scattering medium is distributed throughout space. In addition to the model, already described, in which the scattering medium is confined to a thin slab (model (i)), we discuss two main models.

(ii) The medium is confined to a region surrounding the source, the observer being a great deal further away.

(iii) The medium occupies the whole of space between source and observer.

It should be emphasized that the pulse shapes derived are mean pulse shapes of an ensemble of systems which differ only statistically and in which the mean electron density is constant in space and time in those regions in which it is taken as non-zero.

The following approximations are made throughout.

(1) *Ray optics.* Although the individual diffracting irregularities are phase thin, a small fraction of the line of sight already constitutes a phase thick screen and behaves as if geometrical optics were valid.

(2) *Uniform speed of radiation.* Although the basic cause of the phenomenon is the irregular distribution of refractive index, the delays due directly to changes in group velocity are of order

$$\frac{\lambda^2}{2\pi c}(La)^{1/2}\Delta N r_0$$

and are therefore small compared to the delays due to increased geometrical path length, of order

$$\frac{DL^2}{c} \sim \frac{r_0^2\lambda^4(\Delta N)^2L^2}{ac}$$

provided that

$$\frac{DL^3}{a^2} > 1$$

where L is the extent of the medium.

This condition is similar to condition (ii) derived by Scheuer (1968) for rays from the source to reach the observer by more than one path. If the electron density fluctuations responsible for multipath broadening are also those responsible for scintillations this condition is valid and will be assumed here.

(3) *Small angle approximation.* It will be assumed that all the angular deviations are small compared to one radian—this is certainly an excellent approximation.

2. FORMULATION OF THE MATHEMATICAL PROBLEM

Two cases will be considered.

(1) The scattering medium extends uniformly from the source to some distance L, small compared with the distance to the observer; we require the distribution of the path lengths from the source of rays which emerge as one parallel beam (Fig. 2).

By reversing the paths and shifting them appropriately in the x and y directions, it is clear that the required distribution is the same as that of rays which start in the z direction and reach the plane $z = L$ at any angle (Fig. 3).

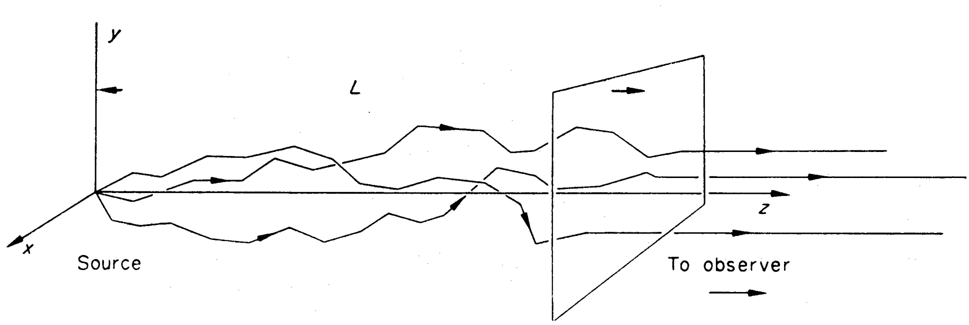

FIG. 2. *Scattering by a thick slab of medium near the source but far from the observer.*

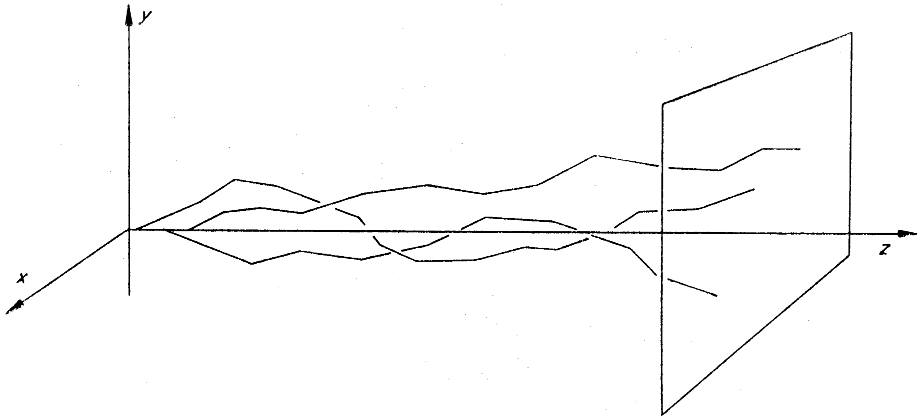

FIG. 3. *Paths starting in z-direction and reaching plane at* z = L *at any angle.*

It is also clear that the same distribution describes the situation when the scattering medium, of dimension L, is about the observer rather than the source (Fig. 2 but with directions of ray reversed).

Consider a particular ray path element at distance z from the source. Let the angle between the z-axis and the projection of the element on the xz plane be $\theta(z)$; define $\phi(z)$ similarly for the projection on to the yz plane (Fig. 4). The length of the element is $dr = dz(1+\tan^2\theta+\tan^2\phi)^{1/2}$

$$= dz\left(1+\frac{\theta^2+\phi^2}{2}\right)$$

since θ, ϕ are assumed small.

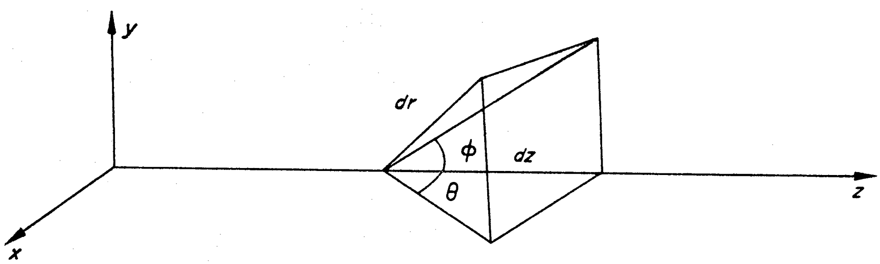

FIG. 4. *Definition of angles* θ, ϕ.

The total length of the ray-path is

$$\int_0^L \left(1+\frac{\theta^2+\phi^2}{2}\right) dz$$

and the extra time, T, that the ray takes compared with the 'straight-through' ray is

$$T = \int_0^L \frac{\theta^2+\phi^2}{2c}\, dz \tag{3}$$

taking the speed of the radiation to be c. $\theta(z)$, $\phi(z)$ perform independent one-dimensional Brownian motions as z increases, i.e. the distribution of θ, ϕ at some z is the gaussian distribution

$$\frac{1}{\sqrt{\pi D z}}\exp\left(-\frac{\theta^2}{Dz}\right) \quad \text{and similarly for } \phi \tag{4}$$

where D is the mean square angular scatter per unit length, already defined.

It is easy to calculate the mean value of T. It is just

$$\int_0^L \frac{Dz}{2c}\,dz = \frac{DL^2}{4c} \tag{5}$$

but the distribution of T cannot be so simply found, since the successive values of θ, for example, are correlated with each other.

A method of solving this problem has, however, been described by Kac (1949, 1951) and will be summarized in the next section.

(2) The scattering medium is distributed uniformly between the source and the observer.

In this case we require the probability distribution of path lengths between two fixed points distance L apart, or equivalently, the distribution of path lengths of rays starting from the source in the $+z$ direction and finishing anywhere on the surface of a sphere of radius L with the source as centre.

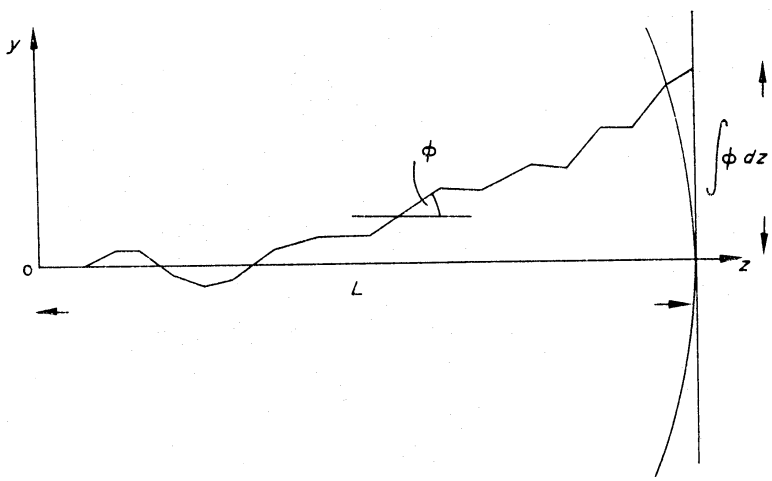

FIG. 5. *Paths starting in* z*-direction from* o *and reaching sphere radius* L.

From Fig. 5, it is clear that we require the probability distribution of

$$T = \frac{1}{2c}\int_0^L (\theta^2+\phi^2)\,dz - \frac{1}{2Lc}\left[\left(\int_0^L \theta\,dz\right)^2 + \left(\int_0^L \phi\,dz\right)^2\right] \tag{6}$$

since θ, ϕ are small, and we may neglect powers of θ, ϕ greater than the second. This problem is solved in Section 4.

3. PATHS BETWEEN A POINT AND A PLANE

As we have seen (equation (3)), we must find the probability function $P(L, T)$ of

$$T = \int_0^L \frac{\theta^2+\phi^2}{2c}\,dz$$

θ, ϕ performing independent Brownian motions as z increases, as implied by (4). As θ, ϕ are statistically independent, the distribution of the sum of

$$\int_0^L \frac{\theta^2}{2c}\,dz \quad \text{and} \quad \int_0^L \frac{\phi^2}{2c}\,dz$$

is the convolution of the distributions of each integral, which are, of course, the same. Hence, using the convolution theorem, it will be enough to find the Laplace transform of the distribution of

$$\int_0^L \frac{\theta^2}{2c}\,dz;$$

this will be the square root of the Laplace transform of the distribution of T.

We thus require the L.T. of dF/dA where

$$F(A) = Pr\left[\int_0^L \frac{\theta^2}{2c}\,dz < A\right]$$

θ having the distribution at given z of

$$\frac{1}{\sqrt{\pi D z}}\exp\left(-\frac{\theta^2}{Dz}\right).$$

We restate the problem in terms of the dimensionless variables, to conform with the notation of Kac (1951)

$$t = \frac{DL}{2} \quad \tau = \frac{Dz}{2} \quad x = \theta \quad \alpha = cDA \tag{7}$$

and write

$$f(\alpha) = F\left(\frac{\alpha}{cD}\right)$$

so that we now require the L.T. of $df/d\alpha$ where

$$f(\alpha) = Pr\left[\int_0^t x^2\,d\tau < \alpha\right]$$

$x(\tau)$ being the Wiener process, with distribution function for given τ of

$$\frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{x^2}{2\tau}\right).$$

This is one of the class of problems solved by Kac (1949, 1951). We find (Kac 1949)

$$\int_0^\infty e^{-u\alpha}\,df = (\operatorname{sech}\sqrt{2ut})^{1/2}.$$

As noted above we must invert the square of this, for our problem. Now

$$\int_0^\infty e^{-u\alpha}\frac{1}{2t^2}\left[\frac{\partial}{\partial x}\vartheta_1\left(\frac{\pi x}{2}, \frac{\pi\alpha}{2t^2}\right)\right]_{x=0} d\alpha = \operatorname{sech}\sqrt{2ut}.$$

Where the theta function

$$\vartheta_1(z,\sigma) = 2\sum_{n=0}^{\infty}(-1)^n \exp\left(-\pi(n+\tfrac{1}{2})^2\sigma\right)\sin(2n+1)z$$

and also (Jacobi's imaginary transformation)

$$\vartheta_1(z,\sigma) = -i\sigma^{-1/2}\exp(-z^2/\pi\sigma)\,\vartheta_1\left(\frac{iz}{\sigma}, \frac{1}{\sigma}\right)$$

(For definition and properties of theta-functions, see Jeffreys & Jeffreys (1962).)

So we may write the inverse Laplace transform of sech $\sqrt{2ut}$ in either of the equivalent forms

$$\sum_{n\ \mathrm{odd}}^{\infty} (-1)^{\frac{1}{2}(n-1)} \left(\frac{2}{\pi\alpha^3}\right)^{1/2} nt \exp\left(-\frac{n^2t^2}{2\alpha}\right)$$

$$\equiv \sum_{n\ \mathrm{odd}}^{\infty} (-1)^{\frac{1}{2}(n-1)} \frac{n\pi}{2t^2} \exp\left(-\frac{n^2\pi^2\alpha}{8t^2}\right).$$

Returning to our original variables (7), we find

$$P(L, T) = \left(\frac{DL^2}{2\pi cT^3}\right)^{1/2} \sum_{n\ \mathrm{odd}} (-1)^{\frac{1}{2}(n-1)}\, n \exp\left(-\frac{n^2DL^2}{8cT}\right)$$

$$\equiv \frac{2\pi c}{DL^2} \sum_{n\ \mathrm{odd}} (-1)^{\frac{1}{2}(n-1)}\, n \exp\left(-\frac{n^2\pi^2cT}{2DL^2}\right). \tag{8}$$

The effect of the scattering medium is also to increase the apparent angular diameter of the source. If we assume, as we have done so far, that we are dealing with effectively point sources, then all the observed angular diameter is due to scattering. The width of the angular distribution is usually expressed in terms of θ_d, the total half-intensity width, but it is more convenient for our purpose to consider θ_0, the r.m.s. angle. For a two-dimensional gaussian distribution the two parameters are related by

$$\theta_d^2 = 4 \log_e 2\, \theta_0^2. \tag{9}$$

We must now distinguish between the two cases, which, as we noted above, both give (8) as the probability distribution function of arrival time of rays. The first case is that of the medium extending from the source to a distance L, the second is similar but the medium extends from the observer out to a distance L.

The second case is the easier; here we have, by definition of D

$$\theta_0^2 = DL. \tag{10}$$

The case when the scattering medium is about the source is not so easily solved. It can be shown (Appendix A) that

$$\theta_0^2 = \beta DL \tag{11}$$

where $\beta = \frac{1}{3}(L/\Delta)^2$. Δ is the distance from source to observer.

$$L \ll \Delta \quad \text{so} \quad \beta \ll 1.$$

We may substitute (10) or (11) in (8) to obtain

$$P(T) = \left(\frac{\pi\tau}{4T^3}\right)^{1/2} \sum_{n\ \mathrm{odd}} (-1)^{\frac{1}{2}(n-1)}\, n \exp\left(-\frac{n^2\pi^2\tau}{16T}\right) \tag{12}$$

$$\equiv \frac{4}{\pi\tau} \sum_{n\ \mathrm{odd}} (-1)^{\frac{1}{2}(n-1)}\, n \exp\left(-\frac{n^2T}{\tau}\right)$$

where

$$\tau = \frac{2}{\pi^2} \frac{L\theta_0^2}{\beta c}. \tag{13}$$

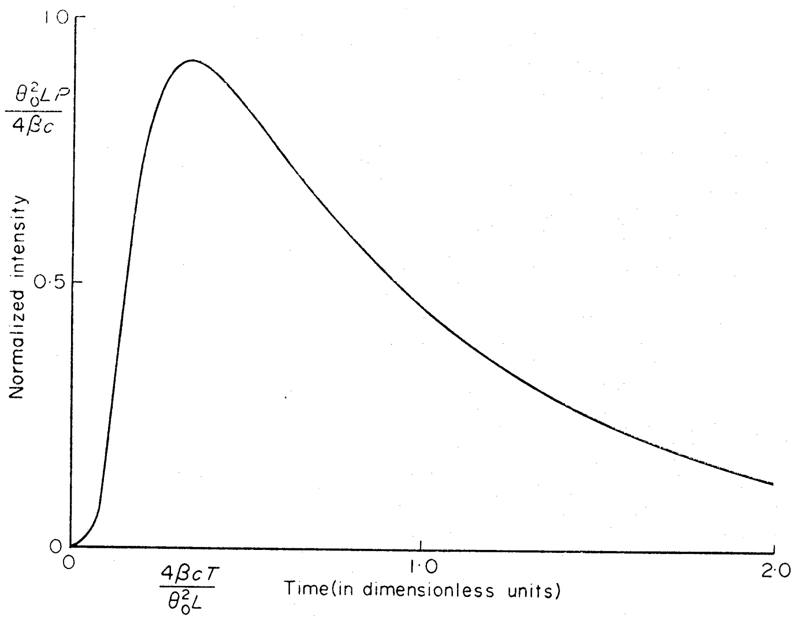

FIG. 6. *The probability distribution function for delay in time* t *for rays reaching observer through thick slab of scattering medium.* $\beta = \frac{1}{3}(L/\Delta)^2$ *if medium near source,* $= 1$ *for medium near observer.*

This function is plotted in Fig. 6. For large T, P approaches an exponential form with time constant τ whereas for T small

$$P(L, T) \simeq \left(\frac{\pi\tau}{4T^3}\right)^{1/2} \exp\left(-\frac{\pi^2\tau}{16T}\right).$$

4. PATHS BETWEEN TWO FIXED POINTS

As we have seen (equation (6)), the problem is that of finding the distribution of

$$T = \frac{1}{2c}\int_0^L (\theta^2+\phi^2)\,dz - \frac{1}{2Lc}\left[\left(\int_0^L \theta\,dz\right)^2 + \left(\int_0^L \phi\,dz\right)^2\right]$$

for the processes with distribution functions

$$\frac{1}{\sqrt{\pi Dz}}\exp\left(-\frac{\theta^2}{Dz}\right), \quad \frac{1}{\sqrt{\pi Dz}}\exp\left(-\frac{\phi^2}{Dz}\right).$$

Once again T breaks into two parts, one dependent on θ only, one on ϕ alone, so again we require only the Laplace Transform of dF/dA where

$$F(A) = Pr\left[\left\{\int_0^L \frac{\theta^2}{2c}\,dz - \frac{1}{2Lc}\left(\int_0^L \theta\,dz\right)^2\right\} < A\right].$$

Introduce, as before, the dimensionless variables t, τ, x, α (equation (7)) so that we require

$$\int_0^\infty e^{-u\alpha}\,df$$

where

$$f(\alpha) = Pr\left[\left\{\int_0^t x^2(\tau)\,d\tau - \frac{1}{t}\left(\int_0^t x(\tau)\,d\tau\right)^2\right\} < \alpha\right]$$

for the Wiener process $x(\tau)$ with distribution function

$$\frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{x^2}{2\tau}\right).$$

To solve this problem we employ a variant of a method described by Blanc-Lapierre & Fortet (1953). This method could also have been used for the problem of the previous section. It consists of discretizing the integral, and then proceeding to the limit.

Let

$$I = \int_0^t x^2(\tau)\,d\tau - \frac{1}{t}\left(\int_0^t x(\tau)\,d\tau\right)^2$$

$$= \frac{1}{t}\int_0^t \int_\tau^t [x(\tau) - x(\tau')]^2\,d\tau'\,d\tau.$$

Consider

$$I^{(m)} = \frac{t}{m^2}\sum_{i=1}^{m}\sum_{i=j}^{m}(x_i - x_j)^2 \tag{14}$$

where

$$x_{i+1} = x_i + \xi_i \qquad x_0 = 0$$

and the ξ_i are independent normal variables with zero mean and variance t/m.

In the limit $m \to \infty$

$$I^{(m)} \to I$$

$$\therefore f^{(m)}(\alpha) \to f(\alpha)$$

where

$$f^{(m)}(\alpha) = Pr[I^{(m)} < \alpha]$$

(14) can be rewritten in terms of the ξ_i

$$I^{(m)} = tA_{ij}\xi_i\xi_j \tag{15}$$

where A is chosen to be the symmetric matrix of order $m-1$ with elements

$$A_{ij} = \frac{\min(i,j)}{m} - \frac{ij}{m^2} \tag{16}$$

where i, j run from 1 to $m-1$.

Now

$$\Psi^{(m)}(u) = \int_0^\infty e^{-u\alpha}\,df^{(m)}$$

$$= E[\exp(-uI^{(m)})] \tag{17}$$

that is, the expectation value of $\exp(-uI^{(m)})$ is also the Laplace transform of $df^{(m)}/d\alpha$

$$= \underbrace{\int_{-\infty}^{\infty} \dots \int}_{m-1 \text{ times}} \exp(-utA_{ij}\xi_i\xi_j) \prod_{i=1}^{m-1} \left(\frac{m}{2\pi t}\right)^{1/2} \exp\left(-\frac{m\xi_i^2}{2t}\right) d\xi_1 \dots d\xi_{m-1}.$$

We now transform to the coordinate system whose base vectors are the orthonormal eigenvectors of A: let the components of the vector $\boldsymbol{\xi}$ in the new coordinates be $y_1 \dots y_{m-1}$. The eigenvalues of A will be denoted by $\lambda_1 \dots \lambda_{m-1}$; they are all positive, since A is positive definite. The Jacobian of the transformation is 1, since the new base vectors are chosen orthonormal.

So

$$\Psi^{(m)}(u) = \int_{-\infty}^{\infty} \dots \int \left(\frac{m}{2\pi t}\right)^{\frac{1}{2}(m-1)} \exp\left[-\sum_{i=1}^{m-1} \left(ut\lambda_i + \frac{m}{2t}\right) y_i^2\right] dy_1 \dots dy_{m-1}$$

$$= \prod_{i=1}^{m-1} \left(1 + \frac{2ut^2\lambda_i}{m}\right)^{-1/2}.$$

By considering the inverse matrix A^{-1} which is, in fact, the matrix with elements

$$A_{ij}^{-1} = 2m\delta_{ij} - m\delta_{i,j+1} - m\delta_{i,j-1}$$

it is easy to see that A has eigenvectors, with components in the original coordinate system,

$$e_j^{(n)} = \sqrt{\frac{2}{m}} \sin\left(\frac{jn\pi}{m}\right)$$

with corresponding eigenvalues

$$\lambda_n = \left\{2m\left(1 - \cos\frac{n\pi}{m}\right)\right\}^{-1}$$

so the product (18) becomes

$$\prod_{n=1}^{m-1} \left(1 + \frac{ut^2}{m^2(1 - \cos n\pi/m)}\right)^{-1/2}.$$

In the limit as $m \to \infty$, this becomes

$$\prod_{n=1}^{\infty} \left(1 + \frac{2ut^2}{n^2\pi^2}\right)^{-1/2}$$

which is the well-known infinite product expansion of

$$\left(\frac{\sinh\sqrt{2ut}}{\sqrt{2ut}}\right)^{-1/2}.$$

Therefore by (17)

$$\int_0^{\infty} e^{-u\alpha}\, df = \left(\frac{\sinh\sqrt{2ut}}{\sqrt{2ut}}\right)^{-1/2}.$$

We require the convolution of $df/d\alpha$ with itself, so we invert $\sqrt{2ut}$ cosech $\sqrt{2ut}$ rather than its square root.

Now

$$\int_0^\infty e^{-u\alpha} \frac{1}{2t^2} \left[\frac{\partial^2}{\partial x^2} \vartheta_3 \left(\frac{\pi x}{2}, \frac{\pi\alpha}{2t^2} \right) \right]_{x=1} d\alpha$$
$$= \sqrt{2ut}\ \mathrm{cosech}\ \sqrt{2ut}$$

where

$$\vartheta_3(z, \sigma) = 1 + 2 \sum_{n=1}^{\infty} \cos 2nz \exp(-\pi n^2 \sigma)$$

and also

$$\vartheta_3(z, \sigma) = \sigma^{-1/2} \exp(-z^2/\pi\sigma)\, \vartheta_3 \left(\frac{iz}{\sigma}, \frac{1}{\sigma} \right)$$

(Jeffreys & Jeffreys (1962)) so that we may write the inverse Laplace transform of $\sqrt{2ut}$ cosech $\sqrt{2ut}$ in either of the two equivalent forms

$$\frac{1}{t^2} \sum_{k=1}^{\infty} (-1)^{k+1} \pi^2 k^2 \exp\left(-\frac{\pi^2 k^2 \alpha}{2t^2} \right)$$
$$\equiv \left(\frac{2t^2}{\pi\alpha^3} \right)^{1/2} \sum_{n\ \mathrm{odd}} \left(\frac{n^2 t^2}{\alpha} - 1 \right) \exp(-n^2 t^2/2\alpha).$$

Returning to the variables L, D, T etc., we find from equations (7)

$$P(L, T) = \frac{4\pi^2 c}{DL^2} \sum_{n=1}^{\infty} (-1)^{n+1} n^2 \exp\left(-\frac{n^2 2\pi^2 cT}{DL^2} \right)$$
$$\equiv \frac{DL^2}{2\pi c T^3} \sum_{n\ \mathrm{odd}} \left(\frac{n^2 DL^2}{4cT} - 1 \right) \exp\left(-\frac{n^2 DL^2}{8cT} \right). \tag{19}$$

In this case, the observed r.m.s. angular deviation is (Appendix A)

$$\theta_0^2 = \frac{DL}{3}. \tag{20}$$

Substituting this in (19), we obtain

$$P(T) = \frac{2}{\tau} \sum_{n=1}^{\infty} (-1)^{n+1} n^2 \exp\left(-\frac{n^2 T}{\tau} \right)$$
$$\equiv \left(\frac{\pi\tau}{2T^3} \right)^{1/2} \sum_{n\ \mathrm{odd}} \left(\frac{n^2 \pi^2 \tau}{2T} - 1 \right) \exp\left(-\frac{n^2 \pi^2 \tau}{4T} \right) \tag{21}$$

where

$$\tau = \frac{3L\theta_0^2}{2\pi^2 c}.$$

This function is plotted in Fig 7.

For large T P is an exponential function with decay time τ and for small T

$$P \simeq \left(\frac{\pi^5 \tau^3}{8T^5} \right)^{1/2} \exp\left(-\frac{\pi^2 \tau}{4T} \right).$$

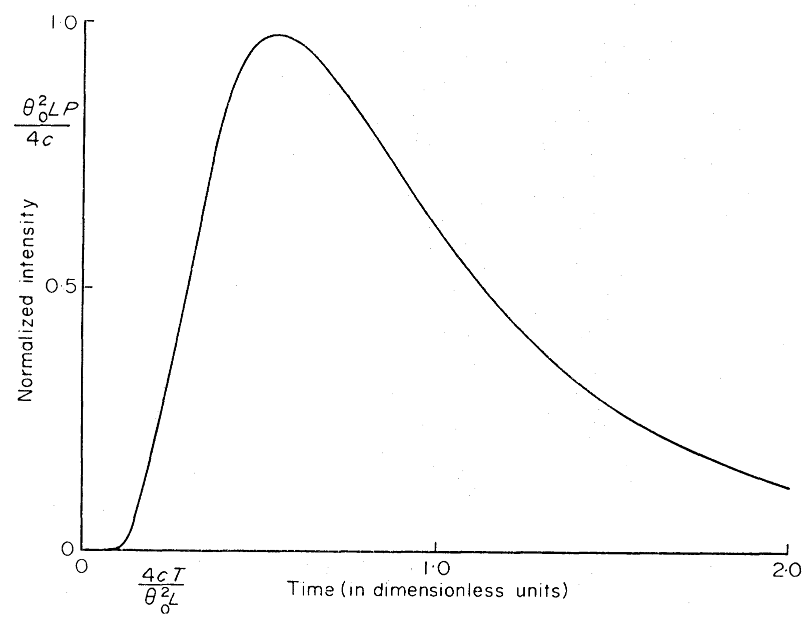

FIG. 7. *The probability distribution function for delays in time* t *for rays reaching observer from source through a scattering medium filling the whole of space.*

To provide a check on the mathematics and to see how effective the functions (12), (21) are at describing the distribution of the quantities (3), (6) when a finite number of rays reach the observer from the source, a Monte Carlo analysis using the TITAN computer of the Cambridge University Computer Laboratory was carried out. (3) and (6) were computed along 1000 ray-paths, each of 100 links, i.e. θ, ϕ were held constant over short lengths of ray-paths and given random gaussian

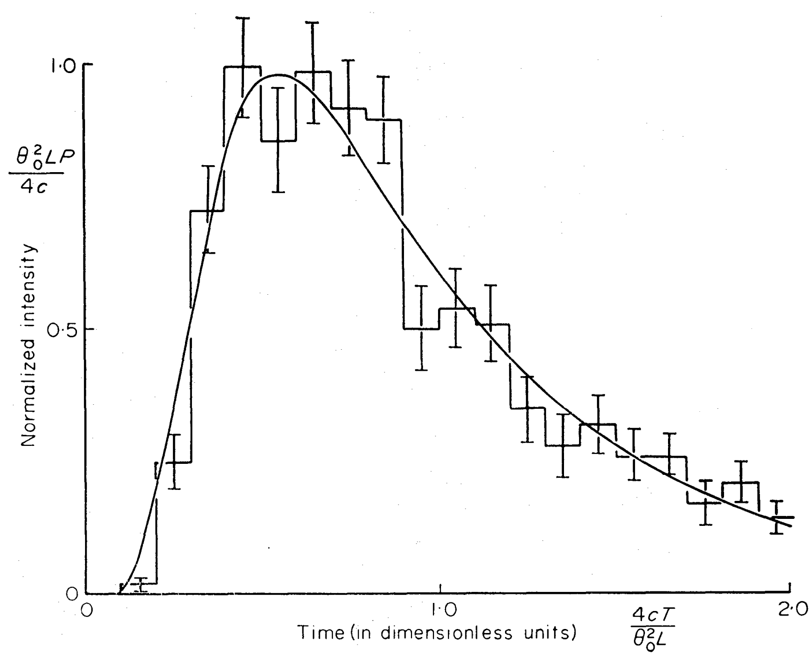

FIG. 8. *Histogram of results from a Monte Carlo simulation superimposed on the curve of Fig.* 7. 1000 *ray-paths were simulated, each ray-path consisting of* 100 *short straight lengths. The error bars extend approximately one standard deviation (calculated for the binomial distribution from the curve of Fig.* 7) *in either direction.*

increments at the end of each link. This process then corresponds to the mathematical analysis earlier in this section, except that m remains finite.

As an example of the results of this Monte Carlo analysis, the histogram of values of (6), for paths between two points is shown in Fig. 8, superimposed on the curve of Fig. 7: it is seen at once that there is good agreement.

Accordingly, (12) and (21) can be taken as good approximations to physical reality provided that assumption (2) of Section 1 is satisfied, and that a large number of rays reach the observer.

5. CONCLUSIONS

A summary of the results of the previous sections is presented in Table I. Because the functions defined in (12), (21) and plotted in Figs 6 and 7 are very shallow near the origin and for some distance along the time axis it is useful to define a 'rise point' at which the function under consideration becomes appreciable. This will be defined as the point of intersection, on the time axis, of the tangent to the function at its first point of inflexion. The 'rise time' of the pulse is defined here as the time from this 'rise point' to the maximum. Approximate values, obtained graphically, are given in Table I.

TABLE I

A comparison of the properties of the pulse shapes predicted on the basis of the four models of the interstellar medium discussed

Model	(i) Thin slab approx. mid-way between source and observer $\Delta \simeq 2L$	(iia) Thick slab near source $\Delta \gg L$	(iib) Thick slab near observer $\Delta \gg L$	(iii) Medium unconfined $\Delta = L$
Mean square size of image θ_0^2	—	$\dfrac{DL^3}{3\Delta^2}$	DL	$\dfrac{DL}{3}$
Mean delay	$\dfrac{\Delta(\Delta-L)}{2L}\dfrac{\theta_0^2}{c}$	$\dfrac{3\Delta^2}{4L}\dfrac{\theta_0^2}{c}$	$\dfrac{L}{4}\dfrac{\theta_0^2}{c}$	$\dfrac{L}{4}\dfrac{\theta_0^2}{c}$
Time constant of exponential decay τ	$\dfrac{\Delta(\Delta-L)}{2L}\dfrac{\theta_0^2}{c}$	$\dfrac{6}{\pi^2}\dfrac{\Delta^2}{L}\dfrac{\theta_0^2}{c}$	$\dfrac{2L}{\pi^2}\dfrac{\theta_0^2}{c}$	$\dfrac{3L}{2\pi^2}\dfrac{\theta_0^2}{c}$
Rise point (see Section 5)	0	0.09τ		0.29τ
Time of maximum	0	0.41τ		0.90τ
Rise time	0	0.32τ		0.61τ

Δ is the distance from source to observer.
L is the extent of the medium, except in (i) where it is the source-medium distance.

Because the apparent angular diameter is known only for the Crab Nebula pulsar NP 0532, and then not at all accurately (Bell & Hewish 1967), and it is the exponential decay time constant which is more precisely known, Fig. 9 compares the pulse shapes, predicted for the three models (i) (ii) (iii), which have the same

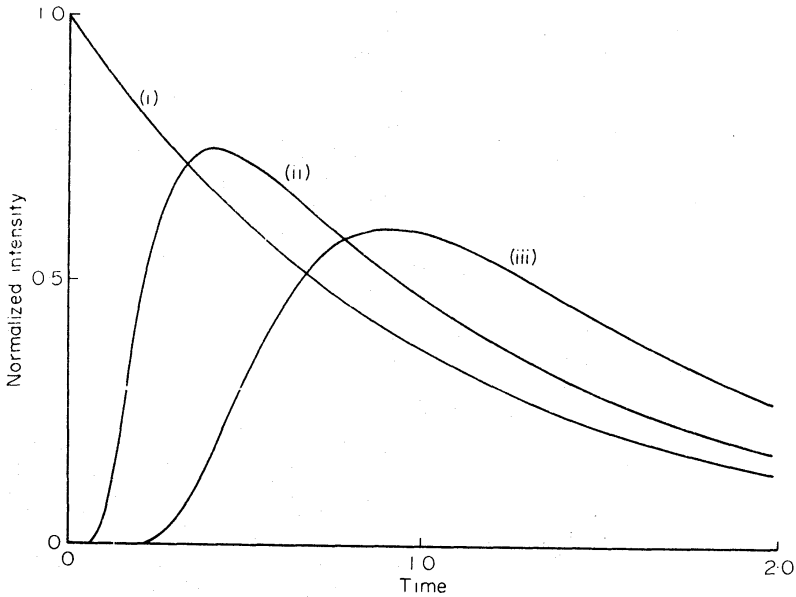

FIG. 9. *A comparison of the probability distribution functions predicted for the three cases—(i) thin slab, (ii) thick slab, (iii) medium unconfined, which have the same exponential decay at large times* (= 1 *on this diagram*).

time constant at large times. The maximum of intensity and the ensuing exponential decay are displaced to later and later times as the region to which the scattering medium is confined becomes more extensive. If the decay time constant is τ, the exponential decay is delayed, relative to model (i), by $\tau \ln(4/\pi) = 0{\cdot}24\tau$ for model (ii), and by $\tau \ln 2 = 0{\cdot}69\tau$ for model (iii).

Counselman & Rankin (1971) express the probability density function observed for the Crab pulsar NP 0532 as an expansion in terms of the Laguerre functions

$$l_n(t) = \frac{(-1)^n}{n!}(2p)^{1/2}\left[e^{x/2}\,\frac{d^n}{dx^n}(x^n\,e^{-x})\right]_{x=2pt}$$

and so it is of interest to do the same for our functions $P(T) = (1/\tau)f(T/\tau)$, say. If C_n is the coefficient of l_n in the expansion of P, then since f and all its derivatives vanish at 0, ∞

$$C_n = \int_0^\infty P(t) l_n(t)\,dt$$

$$= (2p)^{1/2}\frac{(-1)^n}{n!}\left[\frac{d^n}{ds^n}(s+p\tau)^n \bar{f}(s)\right]_{s=p\tau}$$

where

$$\bar{f}(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

which, as we have already seen,

$$= \operatorname{sech}(\pi/2\sqrt{s}) \qquad \text{for case (ii)}$$

$$= \pi\sqrt{s}\operatorname{cosech}(\pi\sqrt{s}) \quad \text{for case (iii).}$$

The magnitudes of the first few coefficients C_n, relative to C_0, for each function are given in Table II.

TABLE II

The Laguerre coefficients $C_n(C_0 = 1)$ of the functions (12), (21) *assuming* $p\tau = 1$

Model Function	(ii) Equation (12)	(iii) Equation (21)
C_1/C_0	0·441	1·153
C_2/C_0	−0·319	−0·375
C_3/C_0	0·225	0·007

A comparison of the predictions of this theory with the results of experiments will be made in a separate paper.

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APPENDIX

Apparent angular size of source due to scattering.

Case I

The scattering medium extends uniformly from the source to some distance L, the distance between source and observer being much greater than L.

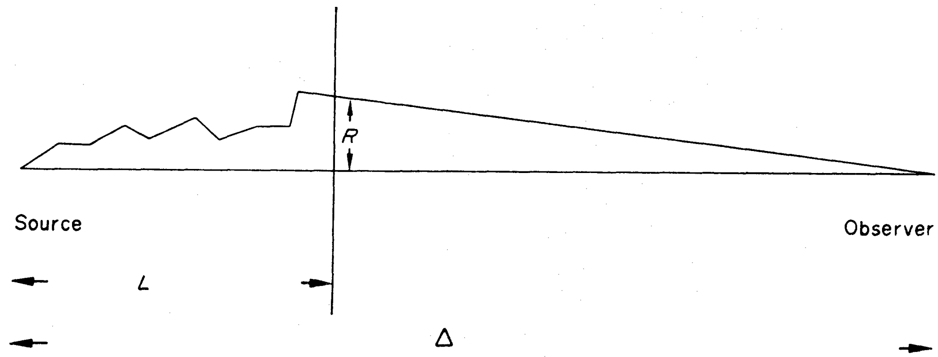

FIG. A1. *Scattering by a thick layer of uniformly fluctuating electrons about the source, observed a great distance away.*

From Fig. (A1)

$$\theta_0^2 = \frac{R^2}{\Delta^2} \tag{A1}$$

where θ_0 is the r.m.s. measure of the angular size. If Δ is large, so that rays reaching the observer are nearly parallel

$$R^2 = \left(\int_0^L \theta\, dz\right)^2 + \left(\int_0^L \phi\, dz\right)^2$$

with θ, ϕ defined as in Fig. 4, and with distribution at given z as implied by (4).

By treating the integral as the limit of a sum, and θ, ϕ as sums of independent random increments (as in (23)) it is easy to show that the distributions of $\int \theta\, dz$ and $\int \phi\, dz$ are gaussian with mean zero and variance $DL^3/6$.

Hence

$$\langle R^2 \rangle = \frac{DL^3}{3}$$

and from (A1)

$$\theta_0^2 = \frac{DL^3}{3\Delta^2} = \beta DL$$

as quoted in (2).

Case 2

The scattering medium fills uniformly all of the space between source and observer.

Here we have to find the mean square of the angle at which a ray, travelling initially in the $+z$ direction cuts the sphere radius L about the source.

This angle is easily shown to be*

$$\left\{\left(\theta(L) - \frac{1}{L}\int \theta\, dz\right)^2 + \left(\phi(L) - \frac{1}{L}\int \phi\, dz\right)^2\right\}^{1/2}$$

* We take $\theta(L)$, $\phi(L)$ at the plane rather than at the sphere itself, because the angular dispersion occurring in the short distance $\sim DL^3/L$ between the sphere and plane is only of order $(D.DL^3/L)^{1/2} = DL$ and therefore negligible compared with θ, ϕ which are of order $(DL)^{1/2}$.

So

$$\theta_0^2 = 2\left\langle\left(\theta - \frac{1}{L}\int_0^L \theta\, dz\right)^2\right\rangle$$

which by taking $\theta(z)$ as a sum of independent increments and proceeding to the limit can easily be shown to be

$$\theta_0^2 = \frac{DL}{3}$$

as quoted in (29).