Physica D 165 (2002) 176–198
PHYSICA D
www.elsevier.com/locate/physd

Pulse–pulse interaction in reaction–diffusion systems

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Received 8 February 2001; accepted 6 February 2002
Communicated by Y. Kuramoto

Abstract

It had been long believed that one-dimensional travelling pulses and the corresponding two-dimensional expanding rings and spiral waves arising in excitable reaction–diffusion systems annihilate when they closely approach one another. However, recently it has been numerically confirmed that if the velocity is very slow, expanding rings and spiral do not necessarily annihilate. In particular, in some situation, two closely approaching pulses reflect, as if they were elastic like objects. By using the center manifold theory, we show that if there are travelling pulses which primarily and super-critically bifurcate from a standing pulse when some parameter is varied, they possess reflection mechanism if the velocity is very slow.

Keywords: Pulse–pulse interaction; Bifurcation of standing pulse; Reflection of travelling pulses

1. Introduction

A variety of regular and irregular complex spatio-temporal patterns have been observed in biological, physiological and chemical systems. Among these systems, it is well known that excitable reaction–diffusion systems generate travelling pulses in one dimension and expanding rings and spiral waves in two dimensions. It is a typical feature that these patterns exhibit annihilation on collision. However, even in such excitable systems, it has been recently reported that annihilation of travelling pulses does not necessarily occur but either reflection or extinction appears before collision in some parameter regime [7]. As an example, the authors considered in the previous paper [6] the following two-component excitable reaction–diffusion system:

$$\begin{aligned}\frac{\partial u}{\partial t} - \Delta u &= \frac{1}{\varepsilon}(-au + k(u)v) \equiv \frac{1}{\varepsilon} f(u, v),\\ \frac{\partial v}{\partial t} - d\Delta v &= h(v^* - v) - k(u)v \equiv g(u, v) \quad \text{in } \Omega, t > 0,\end{aligned} \tag{1.1}$$

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PII: S0167-2789(02)00379-2

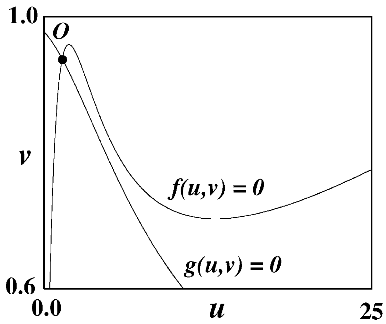

Fig. 1.1. Nulclines of f and g in (1.1) where $a = 2.0$, $c = 5.0$, $h = 45.0$ and $v^* = 1.0$.

where d is the ratio of the diffusion rates of u and v, ε the time constant between the dynamics of u and v. a, h and v^* are some positive constants. If $k(u) = u^2$, (1.1) is called the Gray–Scott model which describes a cubic autocatalytic reaction process [2]. If $k(u) = \exp(u/(1+u/c))$ with positive constant c, (1.1) describes the first step exothermic reaction, where u is the temperature and v the concentration of chemical reactant [6]. Let us consider here the latter case for (1.1). When $a = 2.0$, $c = 5.0$, $h = 45.0$ and $v^* = 1.0$ for instance, the nulclines of f and g are drawn in Fig. 1.1 where there is only one critical point, say $O = (\bar{u}, \bar{v})$.

For the diffusionless system of (1.1), one finds that O is globally stable and that if ε is suitably small, the system possesses excitability mechanism. Under this situation, we consider (1.1) for different values of d. We first take $d = 0.5$ so that there is a stable travelling pulse in one dimension, as in Fig. 1.2(a). Correspondingly, we consider the situation where three stimuli are initially given to the constant state O in a rectangular domain with zero-flux boundary condition. Then three rings are uniformly expanding and annihilate when they collide or hit the boundary, as in Fig. 1.2(b). We next take $d = 4.5$ where a travelling pulse still exists, as in Fig. 1.3(a). A different feature from the previous pulse in Fig. 1.2(a) is that the velocity is much slower. Though the initial condition is the same as the one in Fig. 1.2(b), the resulting patterns are totally changed, as in Fig. 1.3(b). Some features can be

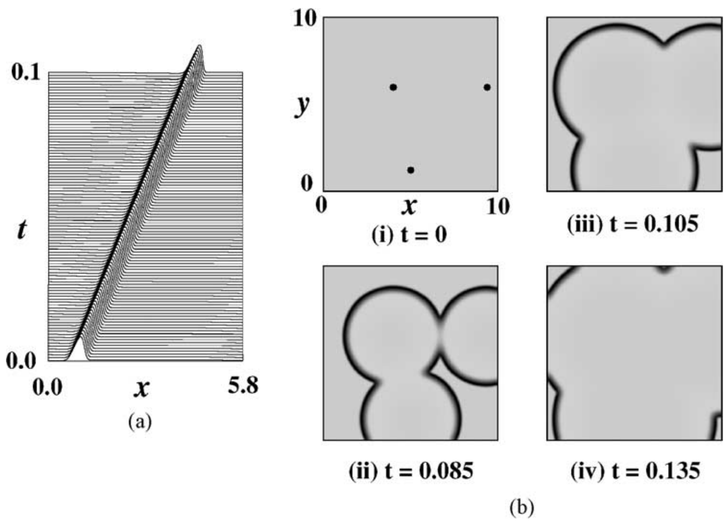

Fig. 1.2. (a) Travelling pulse; (b) annihilation of expanding rings where $a = 2.0$, $c = 5.0$, $h = 45.0$, $v^* = 1.0$, $d = 0.5$ and $\varepsilon = 0.001$.

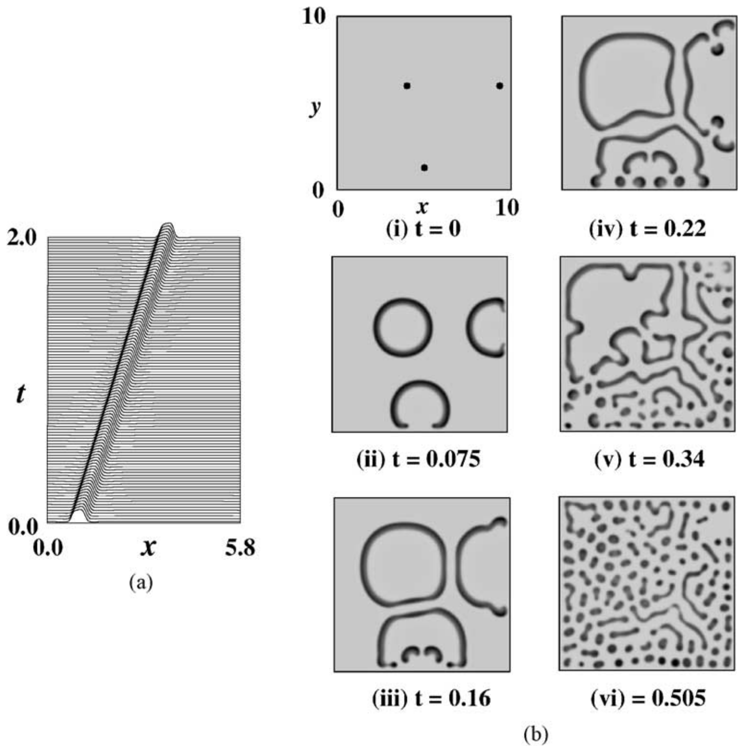

Fig. 1.3. (a) Travelling pulse; (b) breakdown of three ring patterns where the parameters except $d = 4.5$ are the same as those in Fig. 1.2.

observed: (i) Expanding rings do not necessarily annihilate when they approach closely, that is, they either repel each other or fade out before collision. (ii) Some part of the expanding rings shrinks after repulsion so that the ring splits into several pieces. (iii) After large time, there appear very dynamic spot-patterns which never decay at all. It is numerically shown that such complex spatio-temporal patterns occur under the situation where very slowly travelling pulses stably exist in one dimension, which will be discussed in Section 2. The results in this paper are as follows: (i) The bifurcation structure of travelling pulse solutions is made clear in a general framework. Very slowly travelling pulse solutions can be constructed as primarily and super-critically bifurcation from a standing pulse when some parameter is varied. The dynamics of a pulse-like solution near the bifurcation point can be described by some specific system of ODEs (Sections 3 and 4). (ii) By using the reduced ODEs, the interaction of two very slowly travelling-like pulses is discussed and then the reflection of two travelling-like pulse solutions can be shown (Sections 5–7). The tool, which we use here, is the center manifold theory and complementarily numerical methods. Finally, in Section 8 we give concluding remarks on our results.

2. Interaction of two expanding rings

We first numerically consider the interaction of two expanding completely circular rings. For the same values of parameters as in Fig. 1.3, we take specific initial conditions such that two point stimuli of u are given to the constant state O where the distance between the two stimuli, say L, is varied. For $L = 1.4$, the resulting rings annihilate when they collide (Fig. 2.1(a)). For $L = 2.8$, the parts of two rings where they closely approach fade out before collision

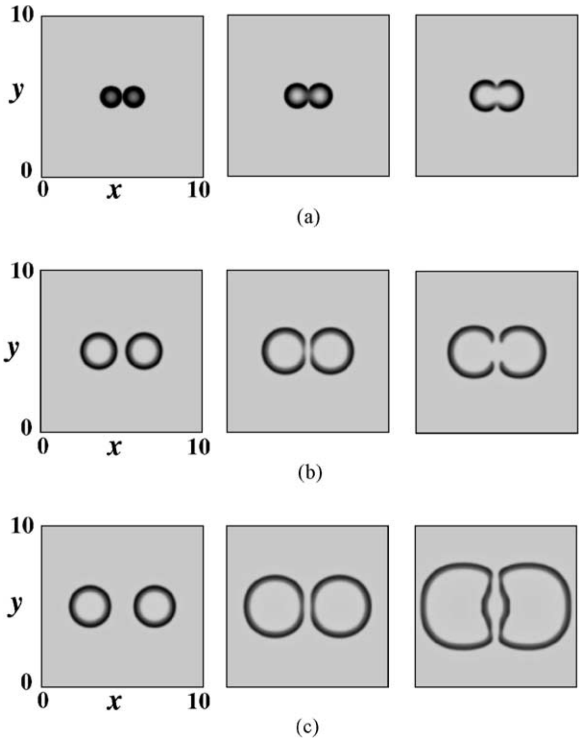

Fig. 2.1. Collision of two expanding rings where the parameters are the same as the ones in Fig. 1.3: (a) $L = 1.4$; (b) $L = 2.8$; (c) $L = 4.0$.

(Fig. 2.1(b)). For $L = 4.0$, the expanding rings never annihilate and some parts exhibit reflection (Fig. 2.1(c)). We thus find that the interaction of expanding rings sensitively depend on the magnitude (or radius) of the expanding rings when they closely approach.

We next consider the dependency of the radius of a single completely circular ring on the expanding velocity. It is demonstrated in Fig. 2.2 that the velocity of the ring monotonously decreases with its radius and that, as the radius

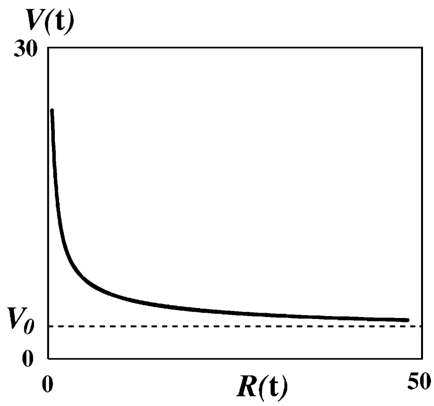

Fig. 2.2. Radius $R(t)$ dependence on velocity $V(t)$ of an expanding ring, where V_0 is the velocity of one-dimensional travelling pulse and the parameters are the same as the ones in Fig. 1.3.

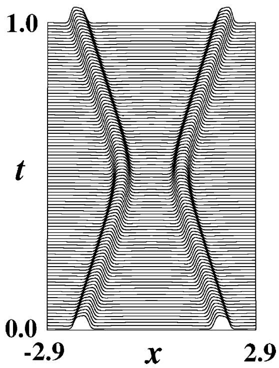

Fig. 2.3. Repulsion of two travelling pulses where the parameters are the same as the ones in Fig. 1.3.

becomes very large, the velocity converges to some constant value, which gives the velocity of the one-dimensional travelling pulse. These results clearly indicate that either annihilation or non-annihilation of rings depends on expanding velocity of rings. Roughly speaking, if the velocity is relatively fast, expanding rings annihilate, while if it is very slow, they reflect. The limiting situation where the radius tends to infinity, suggests us that one-dimensional travelling pulses reflect one another when they approach, as in Fig. 2.3.

We finally consider the dependency of the velocity of one-dimensional travelling pulses on annihilation or non-annihilation properties. In order to change the velocity of travelling pulses, we take d as a free parameter by fixing the other parameters. When d is small, the velocity is relatively fast so that two travelling pulses annihilate as one can expect (Fig. 2.4(a)). If d increases slightly, the velocity becomes slow so that two approaching pulses fade out before collision (Fig. 2.4) and if d increases slightly further where the values of d is the same as in Fig. 2.3, the velocity is very slow so that the pulses reflect one another. Thus numerical results indicate that the dependency of velocity on the interaction of two travelling pulses is qualitatively similar to that of expanding rings shown in Fig. 2.1. We thus conjecture that spatio-temporal complex patterns as in Fig. 1.3(b) appear in two dimensions, if there is the situation where one-dimensional travelling pulses possess reflection property.

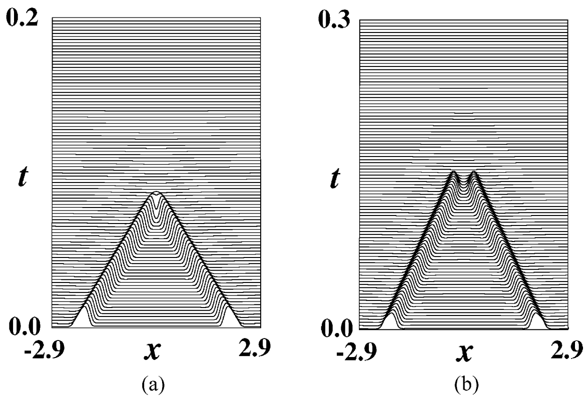

Fig. 2.4. (a) Annihilation of two travelling pulses where $d = 2.0$; (b) fading out of two travelling pulses where $d = 3.1$. The parameters except the value of d are the same as those in Fig. 1.3.

3. Bifurcation of a standing pulse

In this section, we consider the reason why such very slowly traveling pulses possibly appear in the system (1.1). We rely on numerical method to study the existence and stability of standing pulse solutions of (1.1) when d and ε are both free parameters fixing other parameters to be the same as in Fig. 1.3. Taking d as a bifurcation parameter with fixed $\varepsilon = 1.025 \times 10^{-3}$, we find that there coexist four standing (equilibrium) pulses in certain interval of d where the solutions of types (B)–(D) are always unstable, while the solution of type (A) changes its stability, depending on values of d (Fig. 3.1(a) and (b)). Fig. 3.2(a) demonstrates the existence and stability regions of the standing pulse solutions of type (A) in (d, ε)-space. We should remark that there are two bifurcation curves in this region; one is the Hopf bifurcation curve OB and other is the translational one TB where two curves intersect at one point. This structure indicates that with $\varepsilon = 1.025 \times 10^{-3}$, there is the critical value $d_{\mathrm{TB}}(\varepsilon)$ which is larger than $d_{\mathrm{OB}}(\varepsilon)$ so that there appear travelling pulse solutions which bifurcate primarily and super-critically from the standing pulse

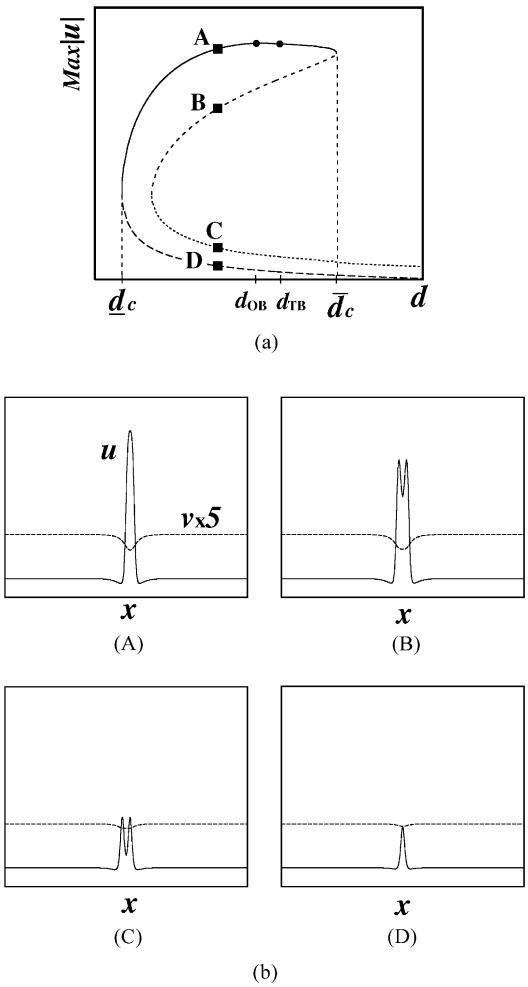

Fig. 3.1. (a) Global structure of standing pulses with the parameter d where $\varepsilon = 0.001$; (b) spatial profiles of standing pulses where $\varepsilon = 0.001$, $d = 3.0$.

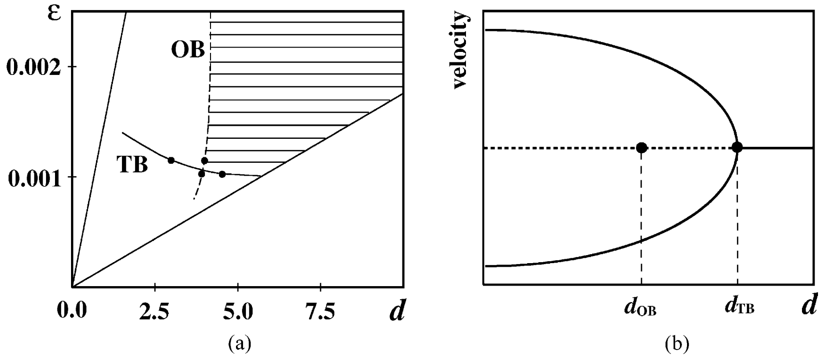

Fig. 3.2. (a) Existence region of standing pulses in the (d, ε)-plane where the solid line is the curve of TB and dotted line the curve of OB. Stable standing pulses are in the hatched region. (b) Schematic bifurcation diagram of standing and travelling pulses when the parameter d is varied, where the vertical axis indicates the velocity of travelling pulses.

solution of type (A) (Fig. 3.2(b)). This implies that very slowly stable travelling pulses can be arbitrarily obtained if d is taken to be suitably close to $d_{\mathrm{TB}}(\varepsilon)$.

4. Construction of very slowly travelling pulses

In this section, we consider the theoretical basis on the bifurcation problem where travelling pulse solutions appear as the destabilization of the standing pulse solution.

Let us consider the following general form of an N-component reaction–diffusion system with a bifurcation parameter k:

$$\boldsymbol{U}_t = D\Delta \boldsymbol{U} + F(\boldsymbol{U}; k) = \mathcal{L}(\boldsymbol{U}; k), \quad t > 0, \quad x \in \boldsymbol{R}, \tag{4.1}$$

where $\boldsymbol{U} = (u_1, u_2, \dots, u_N)$, D is a diagonal matrix with elements $\{d_j\}$ $(j = 1, 2, \dots, N)$. We suppose there exists k_{c} such that (4.1) has a standing symmetric pulse solution $P(x)$, where there exist a positive constant α and a non-zero vector $\boldsymbol{a} \in \boldsymbol{R}^N$ such that

$$P(x) \to \mathrm{e}^{-\alpha|x|}\boldsymbol{a} \quad \text{as } x \to \pm\infty.$$

Simply write the linearized operator $\mathcal{L}'(P(x); k_{\mathrm{c}})$ with respect to $P(x)$ as L. It is obvious to see that $LP_x = 0$ holds. We suppose that L has Jordan block at $k = k_{\mathrm{c}}$, namely:

- (H1) There exists a function $\Psi(x)$ such that $L\Psi = -P_x$. Let Σ_{c} be the spectrum of L.
- (H2) Σ_{c} consists of two sets $\Sigma_0 = \{0\}$ and $\Sigma_1 \subset \{z \in \boldsymbol{C}; \mathrm{Re}(z) < -\gamma_0\}$ for some positive constant γ_0. The generalized eigenspace associated to 0 is spend by P_x and Ψ.

We consider (4.1) in a neighborhood of $k = k_{\mathrm{c}}$. In order to do it, we introduce a new parameter η such that $k = k_{\mathrm{c}} + \eta$ and rewrite (4.1) simply as

$$\boldsymbol{U}_t = \mathcal{L}(\boldsymbol{U}) + \eta g(\boldsymbol{U}), \tag{4.2}$$

where $\mathcal{L}(\boldsymbol{U}) = \mathcal{L}(\boldsymbol{U}; k_{\mathrm{c}})$, $g(\boldsymbol{U}) = g(\boldsymbol{U}; \eta)$ and $\eta g(\boldsymbol{U}; \eta) = \mathcal{L}(\boldsymbol{U}; k_{\mathrm{c}} + \eta) - \mathcal{L}(\boldsymbol{U})$. Let $S(x; r) = P(x) + r\Psi(x)$. Define the translation operator τ by $(\tau(l)\boldsymbol{U})(x) = \boldsymbol{U}(x - l)$ and put $\mathcal{M}(r^*) = \{\tau(l)S(\cdot; r); l \in \boldsymbol{R}, |r| < r^*\}$.

Theorem 4.1. *Let $\boldsymbol{U}(t)$ be a solution of* (4.2). *There exist positive constants C_0, r^*, η^* and a neighborhood U_N of $\mathcal{M}(r^*)$ such that if the initial data $\boldsymbol{U}(0) \in U_N$, then there exist functions $l(t)$ and $r(t)$ such that*

$$\|\boldsymbol{U}(t) - \tau(l(t))S(\cdot; r(t))\|_\infty \le C_0(|r(t)|^2 + |\eta|) \tag{4.3}$$

holds as long as $|r(t)| < r^*$ and $|\eta| < \eta^*$, where $l(t)$ and $r(t)$ are estimated as

$$\dot{l} = r + \mathrm{O}(|r|^2 + |\eta|^2), \qquad \dot{r} = \mathrm{O}(|r|^2 + |\eta|^2). \tag{4.4}$$

Proof will be given in Section 7.

Theorem 4.1 indicates that the movement of a single pulse-like solution $U(t, x)$ of (4.2) is essentially described by $l(t)$.

Next, we shall give the explicit form of the evolutional equations of $l(t)$ and $r(t)$. Let L^* be the adjoint operator of L. Then, L^* also has the same properties as L. Specially, there exist Φ^* and Ψ^* such that L^* satisfies $L^*\Phi^* = 0$ and $L^*\Psi^* = -\Phi^*$, where Φ^* is exponentially decaying, that is, there is a non-zero vector $\boldsymbol{a}^* \in \boldsymbol{R}$ such that

$$\Phi^*(x) \to \pm \mathrm{e}^{-\alpha|x|}\,\boldsymbol{a}^* \quad \text{as } x \to \pm\infty. \tag{4.5}$$

We show the following proposition without proof.

Proposition 4.1. *Ψ, Φ^* and Ψ^* are uniquely determined by the normalization*

$$\langle \Psi, P_x \rangle_{L^2} = 0, \qquad \langle P_x, \Psi^* \rangle_{L^2} = 1, \qquad \langle \Psi, \Psi^* \rangle_{L^2} = 0.$$

We note that

$$\langle \Psi, \Phi^* \rangle_{L^2} = 1, \qquad \langle P_x, \Phi^* \rangle_{L^2} = 0$$

are automatically satisfied.

In addition to (H1) and (H2), we assume:

(H3) $\Psi(x)$, $\Phi^*(x)$, $\Psi^*(x)$ are all odd functions with respect to $x = 0$.

This hypothesis is numerically confirmed for the specific system (1.1), as in Fig. 4.1.

Proposition 4.2. *$\Psi(x)$ and $\Psi^*(x)$ satisfy*

$$\Psi(x) \to \mathrm{e}^{-\alpha x}\{(\beta_1 x + \beta_2)\boldsymbol{a} + \boldsymbol{b}\},$$

and

$$\Psi^*(x) \to \mathrm{e}^{-\alpha x}\{(-\beta_1 x + \beta_2^*)\boldsymbol{a}^* + \boldsymbol{b}^*\}$$

as $x \to \infty$, where $\beta_1 = -(\langle \boldsymbol{a}, \boldsymbol{a}^* \rangle / 2\langle D\boldsymbol{a}, \boldsymbol{a}^* \rangle)$, and β_2, β_2^* are some constants, and *$\boldsymbol{b}$, $\boldsymbol{b}^*$ are vectors satisfying $\langle \boldsymbol{b}, \boldsymbol{a}^* \rangle = \langle \boldsymbol{b}^*, \boldsymbol{a} \rangle = 0$.*

Proof. See Appendix A. □

Let $E = \mathrm{span}\{P_x, \Psi\}$ and $E^\perp = \{\boldsymbol{U}; \langle \boldsymbol{U}, \Phi^* \rangle_{L^2} = \langle \boldsymbol{U}, \Psi^* \rangle_{L^2} = 0\}$. Let functions ζ_1 and ζ_2 in $E^\perp$ be the unique solutions of the equations

$$L\zeta_1 + \tfrac{1}{2}F''(P(x))\Psi \cdot \Psi + \Psi_x = 0 \quad \text{and} \quad L\zeta_2 + g(P(x)) = 0,$$

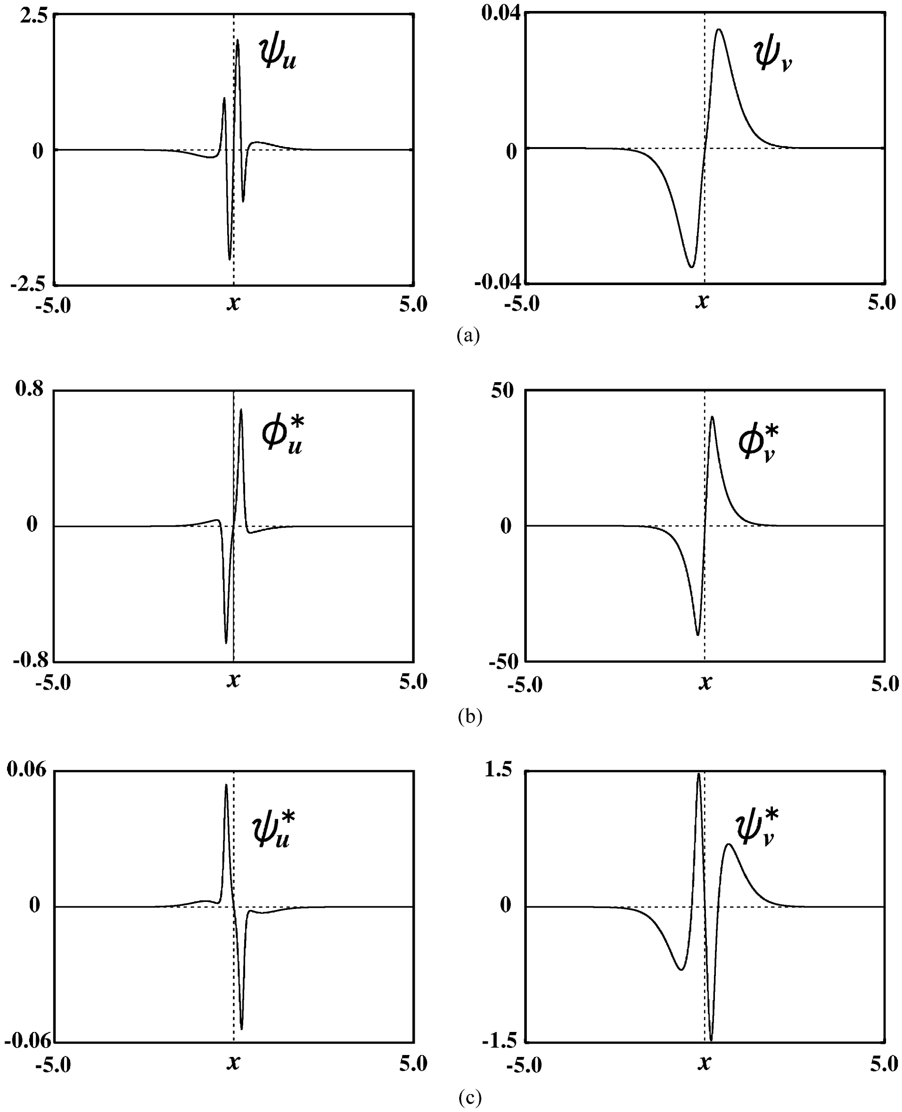

Fig. 4.1. Eigenfunctions of the linearized operator for (1.1) where $\varepsilon = 0.001075$, $a = 2.0$, $c = 5.0$, $v^* = 1.0$ and $d = 5.0$: (a) $\Psi(x) = (\psi_u(x), \psi_v(x))$; (b) $\Phi^*(x) = (\phi_u^*(x), \phi_v^*(x))$; (c) $\Psi^*(x) = (\psi_u^*(x), \psi_v^*(x))$.

respectively, where $F(U, k_c)$ is simply rewritten as $F(U)$. By using these functions, we define two constants M_1 and M_2 by

$$M_1 = -\{\langle \partial_x \zeta_1, \Phi^* \rangle_{L^2} + \langle F''(P)\psi \cdot \zeta_1, \Phi^* \rangle_{L^2} + \tfrac{1}{6}\langle F'''(P)\Psi^3, \Phi^* \rangle_{L^2}\},$$
$$M_2 = \{\langle \partial_x \zeta_2, \Phi^* \rangle_{L^2} + \langle F''(P)\Psi \cdot \zeta_2, \Phi^* \rangle_{L^2} + \langle g'(P)\Psi, \Phi^* \rangle_{L^2}\}.$$

Then we obtain the following theorem.

Theorem 4.2. *Let $l(t)$ and $r(t)$ be the functions defined in* Theorem 4.1. *Then*

$$\dot{l} = r + \mathrm{O}(|r|^3 + |\eta|^{3/2}), \tag{4.6}$$

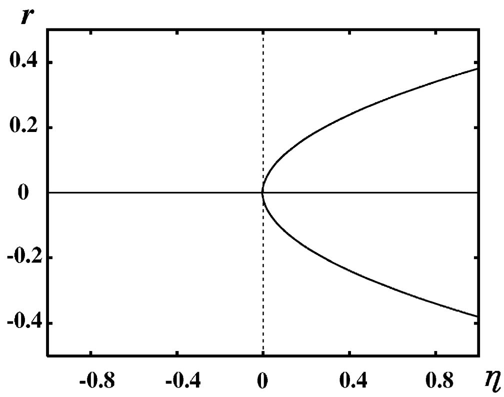

Fig. 4.2. Bifurcation diagram of (4.7) for the exothermic reaction–diffusion system (1.1) where $M_1 \approx 0.729577$, $M_2 \approx 0.110267$ where the parameters except $d = 5.0$ and $\varepsilon = 0.001075$ are the same as those in Fig. 1.2.

$$\dot{r} = K(r;\eta) + \mathrm{O}(|r|^4 + |\eta|^2) \tag{4.7}$$

hold as long as $|r(t)| < r^*$ *and* $|\eta| < \eta^*$, *where* $K(r;\eta) = -M_1 r^3 + M_2 \eta r$.

Proofs of Theorems 4.1 and 4.2 are stated in Section 7.

By (4.6), r can be regarded as the velocity of the pulse P, because l denotes the position of the pulse.

In the following, we consider the bifurcation structure of travelling pulses in the neighborhood of $k = k_c$. Suppose there exist trivial solutions $P(x;\eta)$ for sufficiently small η, which satisfy $\mathcal{L}(P(x;\eta); k_c + \eta) \equiv 0$, $P(x;0) = P(x)$ and $P(x;\eta)$ is symmetric for $x = 0$. Let $L(\eta) = \mathcal{L}'(P(x;\eta); k_c + \eta)$. Note that 0 is always an eigenvalue of $L(\eta)$ with the associated eigenfunction $P_x(x;\eta)$. Then it is expected that there is another eigenvalue, say $\lambda(\eta)$ near 0 (critical eigenvalue) approaching 0 as $\eta \to 0$. In fact, it is true and the movement of the critical eigenvalue of $L(\eta)$ with respect to sufficiently small η is clearly known by Theorem 4.2 as in the following corollary.

Corollary 4.1. *Suppose there exist trivial solutions* $P(x;\eta)$ *represented by* $P(x;\eta) = P(x) + \eta P_1(x) + \eta^2 P_2(x;\eta)$ *such that* $P_1(x)$ *and* $P_2(x;\eta)$ *are symmetric for* $x = 0$. *Then* $L(\eta)$ *has a critical eigenvalue* $\lambda(\eta) = M_2\eta + \mathrm{O}(\eta^2)$ *with the associated eigenfunction* $\Phi(\eta) = (1 + \mathrm{O}(|\eta|))P_x + \lambda(\eta)\Psi + \mathrm{O}(|\eta|^2)$.

This corollary means that the constant M_2 is positive if and only if the critical eigenvalue $\lambda(\eta)$ crosses 0 at $\eta = 0$ from negative to positive with non-zero speed for η.

On the other hand, Theorem 4.2 means that the constant M_1 determines the direction of the bifurcation of travelling pulses. Suppose M_2 is positive, then M_1 is positive if and only if the bifurcation diagram near $k = k_c$ is super-critical with no degeneracy as in Fig. 4.2. If the constants M_1 and M_2 are both positive (in fact, this situation holds for the exothermic reaction–diffusion system (1.1)), the velocities of bifurcating travelling pulses for $\eta > 0$ are approximately given by $\pm\sqrt{M_2\eta/M_1}$. We can thus choose the velocity to be small arbitrarily if η is positive but very small.

5. Interaction of very slowly travelling pulses

In this section, we consider the interaction of two very slowly travelling pulses.

Let $P(x;h) = P(x) + P(x-h)$, $\xi(x;h,\boldsymbol{r}) = r_1\Psi(x) + r_2\Psi(x-h)$ with $\boldsymbol{r} = (r_1, r_2)$ and $S(x;h,\boldsymbol{r}) = P(x;h) + \xi(x;h,\boldsymbol{r})$. Define a set

$$\mathcal{M}(h^*, r^*) = \{\tau(l)S(\cdot;h,\boldsymbol{r}); l \in \boldsymbol{R}, h > h^*, |r_j| < r^*\}$$

for positive constants h^* and r^* and define a quantity

$$\delta = \delta(h) = \sup_{x\in \boldsymbol{R}} |\mathcal{L}(P(x;h))|.$$

Here we note that $\delta(h) = \mathrm{O}(\mathrm{e}^{-\alpha h})$.

Theorem 5.1. *Let* $\Delta_1 = \Delta_1(h, \boldsymbol{r}, \eta) = \delta(h) + |\boldsymbol{r}|^2 + |\eta|$, $\Delta_2 = \Delta_2(h, \boldsymbol{r}, \eta) = \delta(h) + |\boldsymbol{r}| + |\eta|^2$ *and* $\Delta_3 = \Delta_3(h, \boldsymbol{r}, \eta) = \delta^2(h) + |\boldsymbol{r}|^2 + |\eta|^2$. *There exist positive constants* C_0, h^*, r^*, η^* *and a neighborhood* U *of* $\mathcal{M}(h^*, r^*)$ *such that if the initial data* $\boldsymbol{U}(0) \in U$, *then there exist functions* $l(t), h(t) \in \boldsymbol{R}$ *and* $\boldsymbol{r}(t) \in \boldsymbol{R}^2$ *such that*

$$\|\boldsymbol{U}(t) - \tau(l(t))S(h(t), \boldsymbol{r}(t))\|_\infty \le C_0\Delta_1(h(t), \boldsymbol{r}(t), \eta) \tag{5.1}$$

holds as long as $h(t) > h^*$, $|\boldsymbol{r}(t)| < r^*$ *and* $|\eta| < \eta^*$, *where* $\boldsymbol{U}(t)$ *is a solution of* (4.2). $l(t)$, $h(t)$ *and* $\boldsymbol{r}(t)$ *are estimated by*

$$\dot{l}, \dot{h} = \mathrm{O}(\Delta_2), \qquad \dot{\boldsymbol{r}} = \mathrm{O}(\Delta_3). \tag{5.2}$$

Theorem 5.2. *A time evolutional system of* $l(t)$, $h(t)$ *and* $\boldsymbol{r}(t)$ *in* Theorem 5.1 *is described by the following ODEs*:

$$\dot{l} = r_1 - \tilde{H}_1(h) + \mathrm{O}(\delta^2 + |\boldsymbol{r}|^3 + |\eta|^{3/2}), \tag{5.3}$$

$$\dot{r}_1 = K(r_1;\eta) + H_1(h) + \mathrm{O}(\delta^2 + |\boldsymbol{r}|^4 + |\eta|^2), \tag{5.4}$$

$$\dot{r}_2 = K(r_2;\eta) + H_2(h) + \mathrm{O}(\delta^2 + |\boldsymbol{r}|^4 + |\eta|^2), \tag{5.5}$$

$$\dot{h} = r_2 - r_1 + \tilde{H}_1(h) - \tilde{H}_2(h) + \mathrm{O}(\delta^2 + |\boldsymbol{r}|^3 + |\eta|^{3/2}) \tag{5.6}$$

as long as $h(t) > h^*$ *and* $|\boldsymbol{r}(t)| < r^*$ *hold, where* $\delta = \delta(h) = \sup_{x\in\boldsymbol{R}}|\mathcal{L}(P(x;h))|$,

$$H_j(h) = \langle \mathcal{L}(P(\cdot + h_j; h)), \Phi^*\rangle_{L^2}, \qquad \tilde{H}(h) = \langle \mathcal{L}(P(\cdot + h_j; h)), \Psi^*\rangle_{L^2},$$

and $h_1 = 0$, $h_2 = h$.

The functions $H_j(h)$ and $\tilde{H}_j(h)$ in the above theorems are explicitly represented.

Theorem 5.3. *The functions* $H(h)$ *and* $\tilde{H}(h)$ *are represented by*

$$H_1(h) = M_0\,\mathrm{e}^{-\alpha h}(1 + \mathrm{O}(\mathrm{e}^{-\gamma_1 h})), \tag{5.7}$$

$$H_2(h) = -M_0\,\mathrm{e}^{-\alpha h}(1 + \mathrm{O}(\mathrm{e}^{-\gamma_1 h})), \tag{5.8}$$

$$\tilde{H}_1(h) = \tilde{M}_0\,\mathrm{e}^{-\alpha h}(1 + \mathrm{O}(\mathrm{e}^{-\gamma_1 h})), \tag{5.9}$$

$$\tilde{H}_2(h) = -\tilde{M}_0\,\mathrm{e}^{-\alpha h}(1 + \mathrm{O}(\mathrm{e}^{-\gamma_1 h})) \tag{5.10}$$

for a constant $\gamma_1 > 0$ *and the constants* M_0, $\tilde{M}_0$ *given by*

$$M_0 = 2\alpha\langle D\boldsymbol{a}, \boldsymbol{a}^*\rangle, \qquad \tilde{M}_0 = \int_{\boldsymbol{R}} \mathrm{e}^{\alpha x}\langle\{F'(P(x)) - F'(\boldsymbol{0})\}\boldsymbol{a}, \Psi^*(x)\rangle\,\mathrm{d}x.$$

If two travelling pulses are located in symmetric position at $x = 0$, for instance, the four-dimensional system of ODEs (5.3)–(5.6) simply reduces to a two-dimensional one as follows.

Corollary 5.1. *If the initial data $\boldsymbol{U}(0)$ is given by $P(l_0) + P(-l_0)$ with any fixed $l_0 > 0$, then*

$$\|\boldsymbol{U}(t) - \tau(-l(t))S(2l(t), r(t), -r(t))\|_\infty \leq C_0 \Delta_1(2l(t), r(t), \eta),$$

and the dynamics of $l(t)$ and $r(t)$ is described by solutions of

$$\dot{l} = r + \tilde{H}_1(2l) + \mathrm{O}(\delta^2(2l) + |r|^3 + |\eta|^{3/2}), \qquad \dot{r} = K(r;\eta) + H_1(2l) + \mathrm{O}(\delta^2(2l) + |r|^4 + |\eta|^2).$$

6. The reduced system of ODEs

We consider the interaction of two travelling pulses which locate symmetric with $x = 0$, assuming all constants M_1, M_2, M_0 and $\tilde{M}_0$ are positive, which holds for (1.1). The resulting system is reduced to the following ODEs for the unknowns $l(t)$ and $r(t)$ by Corollary 5.1 and Theorem 5.3:

$$\dot{l} = r + \tilde{M}_0 \exp(-2\alpha l), \qquad \dot{r} = -M_1 r^3 + M_2 \eta r + M_0 \exp(-2\alpha l). \tag{6.1}$$

We impose the following initial conditions:

$$l(0) = l_0 \gg 1 \quad \text{and} \quad r(0) = -\sqrt{\frac{M_2 \eta}{M_1}} = -v(\eta), \tag{6.2}$$

which indicates the situation where there is initially a travelling pulse propagating to the left direction. Putting $z = \exp(-2\alpha l)$, then (6.1) can be rewritten as

$$\dot{z} = -2\alpha z(r + \tilde{M}_0 z) \equiv f(z, r), \qquad \dot{r} = -M_1 r^3 + M_2 \eta r + M_0 z \equiv g(z, r), \tag{6.3}$$

and the corresponding initial conditions as

$$z(0) = \exp(-2\alpha l_0) \ll 1 \quad \text{and} \quad r(0) = -v(\eta). \tag{6.4}$$

It is obvious to see that the critical points of (6.3) in (r, z)-plane are $(-v(\eta), 0)$, $(v(\eta), 0)$ and $(0, 0)$. The first two ones correspond to the velocities of travelling pulses of the original RD system, while the last does to the standing one. When η is positive and suitably small, the phase plane analysis on the (r, z)-plane reveals that the solution $(r(t), z(t))$ of (6.3) and (6.4) tends to $(v(\eta), 0)$ for large time, as shown in Fig. 5.1(a). The corresponding solution $l(t)$ is drawn in Fig. 5.1(b), which clearly shows that a pulse propagating with the velocity $-v(\eta)$ to the left direction reflects near $x = 0$ and then moves to the right direction, as if it rebounds near the wall $x = 0$ and the velocity becomes $v(\eta)$. This situation is similar to the reflection of two travelling pulses as was seen in Fig. 2.3.

We next consider the movement of a single travelling pulse in a finite interval $(0, L)$ where L is very large. By (6.1), the system which we are treating is

$$\dot{l} = r + \tilde{M}_0 \exp(-2\alpha l) - \exp(-2\alpha(L - l)),$$
$$\dot{r} = -M_1 r^3 + M_2 \eta r + M_0 \exp(-2\alpha l) - \exp(-2\alpha(L - l)). \tag{6.5}$$

Putting $z = \exp(-2\alpha l)$ again, we have

$$\dot{z} = -2\alpha z \left(r + \tilde{M}_0 \left(\frac{z - \exp(-2\alpha L)}{z}\right)\right) \equiv F(z, r),$$
$$\dot{r} = -M_1 r^3 + M_2 \eta r + M_0 \left(\frac{z - \exp(-2\alpha L)}{z}\right) \equiv G(z, r). \tag{6.6}$$

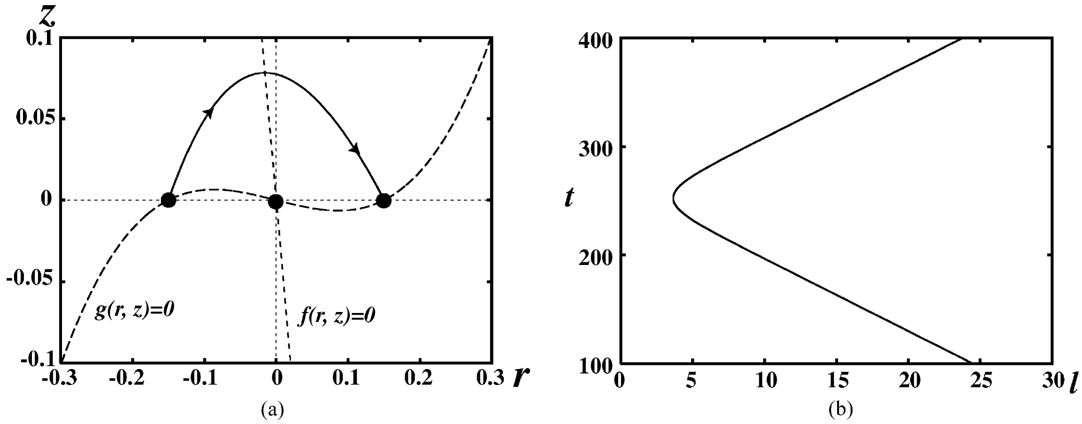

Fig. 5.1. Reflection dynamics of ODEs (6.3) where $a = 0.35$, $M_1 = 0.5$, $M_2 = 0.0225$, $K = 0.1$, $H = 0.2$, $\eta = 0.5$: (a) trajectory of (6.3) in the (r, z)-plane; (b) trajectory of (6.2) in the (l, r)-plane.

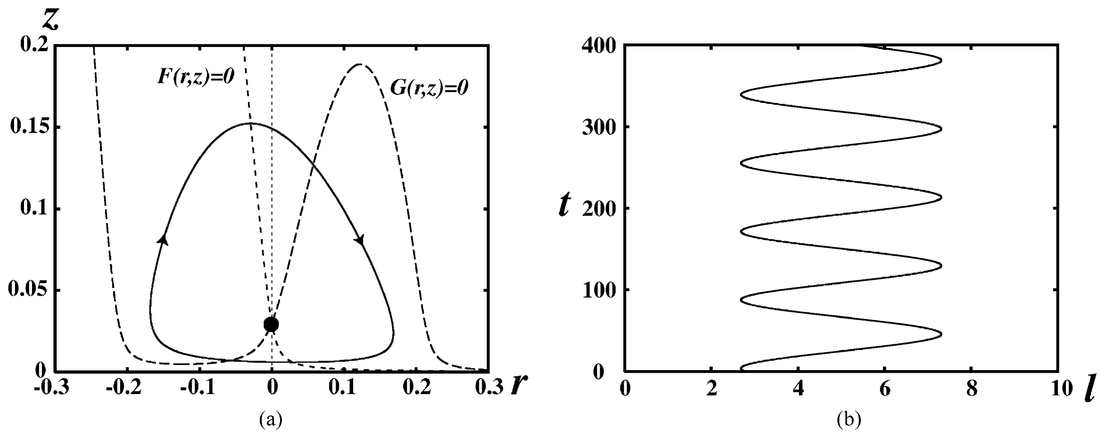

Fig. 5.2. Oscillatory solution of ODEs (6.6) where the parameters except $\eta = 1.0$, $L = 10.0$ are the same as those in Fig. 5.1: (a) trajectory of (6.6) in the (r, z)-plane; (b) time evolution of $l(t)$ of (6.5).

The nulclines of F and G are drawn in Fig. 5.2(a). We find that the critical point of (6.6) is $(r, z) = (\exp(-\alpha L), 0)$ which indicate the standing pulse locating on the center of the interval $(0, L)$. It should be noted that the nonlinearities F and G are qualitatively similar to the ones of the Van der Pol equations. By using the Poincare–Bendixon theorem, we found that there is a limit cycle in the (r, z)-plane, if L is large, as in Fig. 5.2(a). The corresponding trajectory of $l(t)$ is drawn in Fig. 5.2(b). Fig. 5.3 demonstrates the rebounding behavior of a travelling pulse of the RD system (1.1) with two walls.

7. Proof of the theorems

In this section, let C denote a positive constant independent of sufficiently large $h > h^*$ and sufficiently small η with $|\eta| < \eta^*$ where we take suitably large h^* and small η^*.

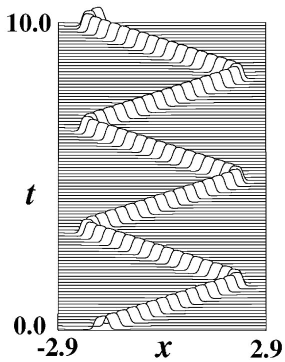

Fig. 5.3. Rebounding of travelling pulse of the RD system (1.1) where the parameters are the same as the ones in Fig. 1.3.

7.1. *Proof of Theorem 4.1*

Let $X = \{L^2(\boldsymbol{R})\}^N$. We start the following proposition without showing the proof, because it is quite similar to Ei [1].

Proposition 7.1. *There exists a neighborhood $U_N \subset X$ of $M(r^*)$ such that any $\boldsymbol{U} \in U_N$ is represented as*

$$\boldsymbol{U} = \tau(l)\{S(x;r) + \boldsymbol{W}\}$$

by uniquely determined $l \in \boldsymbol{R}$, r with $|r| < r^*$ and $\boldsymbol{W} \in E^{\perp}$.

We transform Eq. (4.2) of $\boldsymbol{U}$ to that of $(\boldsymbol{W}, l, r)$ by

$$\boldsymbol{U}(t,x) = \tau(l)\{S(x;r) + \boldsymbol{W}\}$$

for $l \in \boldsymbol{R}$, $|r| < r^*$ and $\boldsymbol{W} \in E^{\perp}$. Since $\tau'(l) = -\tau(l)(\partial/\partial x)$ holds, we have

$$\begin{aligned}\boldsymbol{U}_t &= \dot{l}\tau'(l)\{S(x;r) + \boldsymbol{W}\} + \tau(l)\{\dot{r}\Psi + \boldsymbol{W}_t\} = \tau(l)\{-\dot{l}(S_x + \boldsymbol{W}_x) + \dot{r}\Psi + \boldsymbol{W}_t\} \\ &= \tau(l)\{-\dot{l}(P_x + r\Psi_x + \boldsymbol{W}_x) + \dot{r}\Psi + \boldsymbol{W}_t\},\end{aligned}$$

and

$$\mathcal{L}(\boldsymbol{U}) + \eta g(\boldsymbol{U}) = \tau(l)\{\mathcal{L}(S(x;r) + \boldsymbol{W}) + \eta g(S(x;r) + \boldsymbol{W})\}.$$

Hence

$$-\dot{l}(P_x + r\Psi_x + \boldsymbol{W}_x) + \dot{r}\Psi + \boldsymbol{W}_t = \mathcal{L}(S(x;r) + \boldsymbol{W}) + \eta g(S(x;r) + \boldsymbol{W}) \tag{7.1}$$

holds. Let Q and R be the projections from X to E and $E^{\perp}$, respectively. By the normalization of eigenfunctions of L and L^* as in Proposition 4.1, it follows that

$$Q\boldsymbol{V} = \langle \boldsymbol{V}, \Psi^* \rangle_{L^2} P_x + \langle \boldsymbol{V}, \Phi^* \rangle_{L^2} \Psi.$$

Therefore operating Q on (7.1), we obtain

$$-\dot{l}(1 + \langle \boldsymbol{W}_x, \Psi^* \rangle_{L^2}) = \langle \mathcal{L}(S(x;r) + \boldsymbol{W}) + \eta g(S(x;r) + \boldsymbol{W}), \Psi^* \rangle_{L^2}, \tag{7.2}$$

$$-\dot{l}\langle \boldsymbol{W}_x, \Phi^* \rangle_{L^2} + \dot{r} = \langle \mathcal{L}(S(x;r) + \boldsymbol{W}) + \eta g(S(x;r) + \boldsymbol{W}), \Phi^* \rangle_{L^2}. \tag{7.3}$$

Let X^ω be the fractional powered space of X with respect to L such that X^ω is imbedded into $BU^1(\boldsymbol{R})$, where $BU^1(\boldsymbol{R})$ is the functional space of uniformly continuous and bounded functions on $\boldsymbol{R}$ up to their first derivative. Then, it follows:

$$\begin{aligned}&\mathcal{L}(S(x;r)+\boldsymbol{W})+\eta g(S(x;r)+\boldsymbol{W})\\&\quad= L\boldsymbol{W}-rP_x+\tfrac{1}{2}r^2F''(P)\Psi^2+\tfrac{1}{6}r^3F'''(P)\Psi^3\\&\qquad+rF''(P)\Psi\cdot\boldsymbol{W}+\tfrac{1}{2}F''(P)\boldsymbol{W}^2+\eta g(P)+\eta rg'(P)\Psi+\eta g'(P)\boldsymbol{W}+\mathrm{O}(|r|^2\|\boldsymbol{W}\|_\omega\\&\qquad+|r|\cdot\|\boldsymbol{W}\|_\omega^2+\|\boldsymbol{W}\|_\omega^3+|r|^4+|\eta|\cdot|r|^2+|\eta|\cdot\|\boldsymbol{W}\|_\omega^2).\end{aligned}\tag{7.4}$$

Put

$$\begin{aligned}W(D_1,D_2,\eta)=\{&\boldsymbol{W}\in C((-r^*,r^*);E^\perp);\ \|\boldsymbol{W}(r)\|_\omega\le D_1(|r|^2+|\eta|),\ \|\boldsymbol{W}(r)\\&-\boldsymbol{W}(r')\|_\omega\le D_2(|r|+|r'|+|\eta|)|r-r'|\},\end{aligned}$$

and suppose $\boldsymbol{W}\in W(D_1,D_2,\eta)$. Then, from (7.4) and the oddness of eigenfunctions, we obtain

$$\begin{aligned}&\langle\mathcal{L}(S(x;r)+\boldsymbol{W})+\eta g(S(x;r)+\boldsymbol{W}),\Psi^*\rangle_{L^2}=-r+\mathrm{O}(|r|^2+\|\boldsymbol{W}\|_\omega^2+|\eta|^2)=-r+\mathrm{O}(|r|^2+|\eta|^2),\\&\langle\mathcal{L}(S(x;r)+\boldsymbol{W})+\eta g(S(x;r)+\boldsymbol{W}),\Phi^*\rangle_{L^2}=\mathrm{O}(|r|^2+\|\boldsymbol{W}\|_\omega^2+|\eta|^2)=\mathrm{O}(|r|^2+|\eta|^2).\end{aligned}$$

Therefore, it follows from (7.2) and (7.3) that

$$\dot{l}=J(r,\boldsymbol{W})=r+\mathrm{O}(|r|^2+|\eta|^2),\qquad \dot{r}=\tilde{K}(r,\boldsymbol{W})=\mathrm{O}(|r|^2+|\eta|^2).$$

Thus, we have the equations of $(r,\boldsymbol{W})$ and l as

$$\dot{r}=\tilde{K}(r,\boldsymbol{W}),\qquad \boldsymbol{W}_t=L\boldsymbol{W}+\tilde{G}(r,\boldsymbol{W}),\qquad \dot{l}=J(r,\boldsymbol{W}),\tag{7.5}$$

where by (7.1) and (7.4) $\tilde{G}$ is given as

$$\begin{aligned}\tilde{G}(r,\boldsymbol{W})&=R(\mathcal{L}(S(x;r)+\boldsymbol{W})+\eta g(S(x;r)+\boldsymbol{W})+\dot{l}(P_x+r\Psi_x+\boldsymbol{W}_x)-\dot{r}\Psi)-L\boldsymbol{W}\\&=\mathrm{O}(|r|^2+\|\boldsymbol{W}\|_\omega^2+|\eta|)+J(r,\boldsymbol{W})R(r\Psi_x+\boldsymbol{W}_x)=\mathrm{O}(|r|^2+|\eta|),\\\|\tilde{G}(r,\boldsymbol{W})-\tilde{G}(r',\boldsymbol{W}')\|&\le\mathrm{O}(|r|+|r'|+|\eta|)(|r-r'|+\|\boldsymbol{W}-\boldsymbol{W}'\|_\omega).\end{aligned}$$

Then, in a quite similar way to Ei [1], we can show the existence of a function $\sigma(r;\eta)\in W(D_1,D_2,\eta)$ for appropriate constants D_1 and D_2 such that $(r,\sigma(r;\eta))$ is a positively and exponentially attractive local invariant manifold for Eq. (7.5). Since the solution $\boldsymbol{U}$ is represented as $\boldsymbol{U}=\tau(l)\{S(x;r)+\boldsymbol{W}\}$ by the solution $(r,\boldsymbol{W},l)$ of (7.5), we have

$$\|\boldsymbol{U}(t)-\tau(l(t))S(\cdot;r(t))\|\le C\|\boldsymbol{W}\|_\omega\le C(|r(t)|^2+|\eta|).$$

7.2. Proof of Theorem 4.2

Define the function $\zeta^\dagger(r;\eta)(\cdot)\in E^\perp$ by the unique solution of

$$0=L\zeta^\dagger+\tfrac{1}{2}r^2F''(P)\Psi^2+\eta g(P)+r^2\Psi_x.\tag{7.6}$$

Note $\Psi_x\in E^\perp$ because of the symmetry of Ψ_x. It is easily seen that $\zeta^\dagger$ is given by $\zeta^\dagger=r^2\zeta_1+\eta\zeta_2$, where ζ_j $(j=1,2)$ are functions defined in Section 4. In a similar way to the proof of Theorem 4.1, we have the following proposition.

Proposition 7.2. *There exists a neighborhood $\mathcal{U} \subset X$ of $M(r^*)$ such that any $\boldsymbol{U} \in \mathcal{U}$ is represented as*

$$\boldsymbol{U} = \tau(l)\{S(x;r) + \zeta^{\dagger}(r;\eta) + \boldsymbol{W}\}$$

by uniquely determined $l \in \boldsymbol{R}$, r with $|r| < r^*$ and $\boldsymbol{W} \in E^{\perp}$.

Substituting $\boldsymbol{U} = \tau(l)\{S(x;r) + \zeta^{\dagger}(r,\eta) + \boldsymbol{W}\}$ into (4.2), we have similar to (7.1)

$$\begin{aligned}
&-\dot{l}(P_x + r\Psi_x + \zeta_x^{\dagger} + \boldsymbol{W}_x) + \dot{r}(\Psi + \zeta_r^{\dagger}) + \boldsymbol{W}_t \\
&\quad = L\boldsymbol{W} - rP_x + L\zeta^{\dagger} + \tfrac{1}{2}r^2F''(P)\Psi^2 + rF''(P)\Psi\cdot\zeta^{\dagger} + \tfrac{1}{6}r^3F'''(P)\Psi^3 + \eta g(P)\eta rg'(P)\Psi \\
&\qquad + \mathrm{O}(|\zeta^{\dagger}|^2 + |r|^4 + |\eta|(|\zeta^{\dagger}| + |\boldsymbol{W}|) + |r||\boldsymbol{W}| + |\boldsymbol{W}|^2).
\end{aligned}$$

The estimates (4.4) on $\dot{l}$, $\dot{r}$ and (7.6) with the estimate $\|\zeta^{\dagger}(r,\eta)\|_{\omega} \le \mathrm{O}(|r|^2 + |\eta|)$ show

$$\begin{aligned}
\boldsymbol{W}_t &= L\boldsymbol{W} - rP_x + rF''(P)\Psi\cdot\zeta^{\dagger} + \tfrac{1}{6}r^3F'''(P)\Psi^3 + \eta rg'(P)\Psi - r^2\Psi_x + \dot{l}(P_x + r\Psi_x + \zeta_x^{\dagger} + \boldsymbol{W}_x) \\
&\quad - \dot{r}(\Psi + \zeta_r^{\dagger}) + \mathrm{O}(|\zeta^{\dagger}|^2 + |r|^4 + |\eta|(|\zeta^{\dagger}| + |\boldsymbol{W}|) + |r||\boldsymbol{W}| + |\boldsymbol{W}|^2) \\
&= L\boldsymbol{W} + \mathrm{O}(|r|^2 + |\eta|^2)P_x + \mathrm{O}(|r|(|r|^2 + |\eta|^2))\Psi_x + \dot{l}\boldsymbol{W}_x - \dot{r}\Psi + rF''(P)\Psi\cdot\zeta^{\dagger} + \tfrac{1}{6}r^3F'''(P)\Psi^3 \\
&\quad + \eta rg'(P)\Psi + r\zeta_x^{\dagger} + \mathrm{O}(|r|^4 + |\eta|^2 + |\eta|(|\zeta^{\dagger}| + |\boldsymbol{W}|) + |r||\boldsymbol{W}| + |\boldsymbol{W}|^2) \\
&= L\boldsymbol{W} + \mathrm{O}(|r|^2 + |\eta|^2)P_x + \mathrm{O}(|r|(|r|^2 + |\eta|^2))\Psi_x + \dot{l}\boldsymbol{W}_x - \dot{r}\Psi + \mathrm{O}(|r|^3 + |\eta|^{3/2} + |r||\boldsymbol{W}| + |\boldsymbol{W}|^2).
\end{aligned} \tag{7.7}$$

Operating the projection R on the above equation, we see

$$\boldsymbol{W}_t = L\boldsymbol{W} + \dot{l}R\boldsymbol{W}_x + \mathrm{O}(|r|^3 + |\eta|^{3/2} + |r||\boldsymbol{W}| + |\boldsymbol{W}|^2). \tag{7.8}$$

Following the proof of Theorem 4.1, we can show the existence of an exponentially attractive local invariant manifold given by $\boldsymbol{W} = \sigma^{\dagger}(r;\eta)$ with $\|\sigma^{\dagger}(r;\eta)\|_{\omega} \le \mathrm{O}(|r|^3 + |\eta|^{3/2})$. Hence by taking the inner product of (7.7) and Ψ^* in X with the estimate $\boldsymbol{W} = \sigma^{\dagger} = \mathrm{O}(|r|^3 + |\eta|^{3/2})$, we obtain (4.6).

On the other hand, by taking the inner product of (7.7) and Φ^*, we have

$$\dot{r} = r\langle F''(P)\Psi\cdot\zeta^{\dagger}, \Phi^*\rangle_{L^2} + \tfrac{1}{6}r^3\langle F'''(P)\Psi^3, \Phi^*\rangle_{L^2} + \eta r\langle g'(P)\Psi, \Phi^*\rangle_{L^2} + r\langle \zeta_x^{\dagger}, \Phi^*\rangle_{L^2} + \mathrm{O}(|r|^4 + |\eta|^2). \tag{7.9}$$

Substituting the representation $\zeta^{\dagger} = r^2\zeta_1 + \eta\zeta_2$ into (7.9), we obtain the theorem.

7.3. Proof of Corollary 4.1

Quite similar to the previous subsections, one finds that there exists a neighborhood $\mathcal{U} \subset X$ of $M(r^*)$ such that for any $\boldsymbol{U} \in \mathcal{U}$ is represented as

$$\boldsymbol{U} = \tau(l)\{P(x;\eta) + r\Psi(x) + r^2\zeta_1 + \boldsymbol{W}\}$$

by uniquely determined $l \in \boldsymbol{R}$, $|r| < r^*$ and $\boldsymbol{W} \in E^{\perp}$. Since $P(x;\eta) = P(x) + \eta P_1(x) + \eta^2 P_2(x;\eta)$ satisfy $\mathcal{L}(P(x;\eta)) + \eta g(P(x;\eta)) \equiv 0$ for any small η and $P_1 \in E^{\perp}$, $P_1 = \zeta_2$ holds. Therefore, we also see similarly that there exists an exponentially attractive local invariant manifold given by $\boldsymbol{W} = \sigma^*(r;\eta)$ with $\|\sigma^*(r;\eta)\|_{\omega} \le \mathrm{O}(|r|^3 + |\eta|^{3/2})$ and $\sigma^*(0;\eta)(x) \equiv 0$. Furthermore, $\sigma^*(\cdot;\eta)$ is belonging to $W(D_1, D_2, \eta)$ as in the proof of Theorem 4.1, which leads to $\sigma_r^*(0;\eta) \le \mathrm{O}(|\eta|)$. Thus, we find that $\sigma^*(r;\eta) = C\eta rp(x) + \mathrm{O}(r^2)$ for a constant

C and a function $p(x)$. Hence, there exist functions $\sigma^l(r;\eta)$ and $\sigma^r(r;\eta)$ with $|\sigma^l(r;\eta)| \le \mathrm{O}(|r|^3 + |\eta|^{3/2})$ and $|\sigma^r(r;\eta)| \le \mathrm{O}(|r|^4 + |\eta|^2)$ such that $\sigma^l(0;\eta) = \sigma^r(0;\eta) = 0$ and $\sigma^l_r(0;\eta) = \mathrm{O}(|\eta|)$, $\sigma^r_r(0;\eta) = \mathrm{O}(|\eta|^2)$ and they give the equation along the local invariant manifold by

$$\dot{l} = r + \sigma^l(r;\eta) \equiv \hat{H}(r;\eta), \qquad \dot{r} = K(r;\eta) + \sigma^r(r;\eta) \equiv \hat{K}(r;\eta). \tag{7.10}$$

Here, we used

$$\sigma^l(r;\eta) = \langle \mathrm{O}(\zeta^\dagger + \sigma^*) + \mathrm{O}((r\Psi + \zeta^\dagger + \sigma^*)^2), \Phi^* \rangle_{L^2} \quad \text{and}$$
$$\sigma^r(r;\eta) = \langle \mathrm{O}((\zeta^\dagger + \sigma^*)^2) + \mathrm{O}((r\Psi + \zeta^\dagger + \sigma^*)^2), \Psi^* \rangle_{L^2}$$

for the estimates of σ^l and σ^r. Note $r = 0$ is an equilibrium of (7.10), which corresponds to the standing pulse solution $P(x;\eta)$. Hence, the linearized stability of $P(x;\eta)$ is determined by $\hat{K}'(0;\eta) = M_2\eta + \mathrm{O}(\eta^2) \equiv \lambda(\eta)$ and the solution $\boldsymbol{U}(t)$ is also represented by

$$\boldsymbol{U}(t,x) = P(x - l(t);\eta) + r(t)\Psi(x - l(t)) + r^2(t)\zeta_1(x - l(t)) + \sigma^*(r(t);\eta)(x - l(t)).$$

This is expanded for small $l(t)$ and $r(t)$ as

$$P(x;\eta) - l(t)P_x(x;\eta) + r(t)\Psi(x) + r(t)\sigma^*_r(0;\eta)(x) + \mathrm{O}(|l(t)|^2 + |r(t)|^2) \tag{7.11}$$

by using the estimate $\|\sigma^*(r;\eta)\|_\omega \le \mathrm{O}(|r\eta|)$.

Now, let us linearize (7.10) with respect to $r = 0$ as

$$\begin{pmatrix} \dot{l} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} 0 & 1 + \sigma^l_r(0;\eta) \\ 0 & M_2\eta + \sigma^r_r(0;\eta) \end{pmatrix} \begin{pmatrix} l \\ r \end{pmatrix}. \tag{7.12}$$

Let

$$A = \begin{pmatrix} 0 & 1 + \hat{\lambda}_1(\eta) \\ 0 & \hat{\lambda}_2(\eta) \end{pmatrix},$$

where $\hat{\lambda}_1(\eta) = 1 + \sigma^l_r(0;\eta) = 1 + \mathrm{O}(|\eta|)$ and $\hat{\lambda}_2(\eta) = \lambda(\eta) = M_2\eta + \sigma^r_r(0;\eta) = M_2\eta + \mathrm{O}(|\eta|^2)$. Eigenvalues of A are 0 and $\lambda(\eta)$ with associated eigenvectors $\boldsymbol{\phi}_1 = (1, 0)$ and $\boldsymbol{\phi}_2 = (\hat{\lambda}_1(\eta), \hat{\lambda}_2(\eta))$, respectively. Hence, one of the solutions of (7.10) is given by

$$\begin{pmatrix} l(t) \\ r(t) \end{pmatrix} = \varepsilon\, \mathrm{e}^{\lambda(\eta)t}\, \boldsymbol{\phi}_2 + \mathrm{O}(\varepsilon^2)$$

for sufficiently small $\varepsilon > 0$. Substituting above $l(t)$ and $r(t)$ into (7.11), we have

$$\begin{aligned} \boldsymbol{U}(t,x) &= P(x;\eta) - \varepsilon\hat{\lambda}_1(\eta)\,\mathrm{e}^{\lambda(\eta)t}\, P_x(x;\eta) + \varepsilon\hat{\lambda}_2(\eta)\,\mathrm{e}^{\lambda(\eta)t}\,\Psi(x) + \varepsilon\hat{\lambda}_2(\eta)\,\mathrm{e}^{\lambda(\eta)t}\,\sigma^*_r(0;\eta)(x) + \mathrm{O}(\varepsilon^2) \\ &= P(x;\eta) - \varepsilon\,\mathrm{e}^{\lambda(\eta)t}\{\hat{\lambda}_1(\eta)P_x(x;\eta) - \hat{\lambda}_2(\eta)\Psi(x) - \hat{\lambda}_2(\eta)\sigma^*_r(0;\eta)(x)\} + \mathrm{O}(|\varepsilon|^2). \end{aligned} \tag{7.13}$$

Since $\boldsymbol{U}(t,x)$ of (7.13) satisfies $\boldsymbol{U}_t = \mathcal{L}(\boldsymbol{U}) + \eta g(\boldsymbol{U}) = \varepsilon\,\mathrm{e}^{\lambda(\eta)t}\, L(\eta)\{\hat{\lambda}_1(\eta)P_x(x;\eta) - \hat{\lambda}_2(\eta)\Psi(x) - \hat{\lambda}_2(\eta)\sigma^*_r(0;\eta)(x)\} + \mathrm{O}(\varepsilon^2)$, which leads to as $\varepsilon \downarrow 0$

$$\begin{aligned} &\lambda(\eta)\{\hat{\lambda}_1(\eta)P_x(x;\eta) - \hat{\lambda}_2(\eta)\Psi(x) - \hat{\lambda}_2(\eta)\sigma^*_r(0;\eta)(x)\} \\ &\quad = L(\eta)\{\hat{\lambda}_1(\eta)P_x(x;\eta) - \hat{\lambda}_2(\eta)\Psi(x) - \hat{\lambda}_2(\eta)\sigma^*_r(0;\eta)(x)\}. \end{aligned}$$

Thus, the proof is completed.

7.4. Proof of Theorem 5.1

The proof of Theorem 5.1 is quite similar to the proof of Theorem 2.1 in [1]. Hence, we just give the outline of the proof.

Let $X = \{L^2(\boldsymbol{R})\}^N$ and $L(h) = \mathcal{L}'(P(x;h))$, and let $L^*(h)$ be the adjoint operator of $L(h)$. Then, we have the following two propositions which can be proved in a quite similar manner to [1], so we omit the details.

Proposition 7.3. *There exist positive constants C and h^* such that for h with $h > h^*$, the operator $L(h)$ has four eigenvalues $\{\lambda_j(h)\}_{j=1,\dots,4}$ with $|\lambda_j(h)| \le C\delta(h)$. Eigenvalues with geometrical multiplicities are repeated as many times as their multiplicities indicate. Other spectra of $L(h)$ are in the left-hand side of $z = -\rho_0$ for a positive constant ρ_0.*

Let $E(h)$ be the generalized eigenspace corresponding to eigenvalues $\{\lambda_j(h)\}_{j=1,\dots,4}$. Note that $\dim E(h) = 4$. The adjoint operator $L^*(h)$ has also similar four eigenvalues $\{\lambda_j^*(h)\}_{j=1,\dots,4}$ with $|\lambda_j^*(h)| \le C\delta(h)$. Let $E^*(h)$ be the generalized eigenspace corresponding to eigenvalues $\{\lambda_j^*(h)\}_{j=1,\dots,4}$.

Proposition 7.4. *$E(h)$ and $E^*(h)$ are spanned by four functions $\{\boldsymbol{\phi}_j(h)(\cdot)\}$, $\{\boldsymbol{\psi}_j(h)(\cdot)\}$ and $\{\boldsymbol{\phi}_j^*(h)(\cdot)\}$, $\{\boldsymbol{\psi}_j^*(h)(\cdot)\}$ $(j = 1, 2)$, respectively, such that for $j = 1, 2$,*

$$\boldsymbol{\phi}_j(h)(x) = P_x(x - h_j) + \mathrm{O}(\delta), \tag{7.14}$$

$$\boldsymbol{\psi}_j(h)(x) = \Psi(x - h_j) + \mathrm{O}(\delta), \tag{7.15}$$

$$\boldsymbol{\phi}_j^*(h)(x) = \Phi^*(x - h_j) + \mathrm{O}(\delta), \tag{7.16}$$

$$\boldsymbol{\psi}_j^*(h)(x) = \Psi^*(x - h_j) + \mathrm{O}(\delta), \tag{7.17}$$

$$\langle \boldsymbol{\phi}_j(h), \boldsymbol{\phi}_k^*(h)\rangle_{L^2} = 0, \quad j \ne k, \tag{7.18}$$

$$\langle \boldsymbol{\phi}_j(h), \boldsymbol{\phi}_j^*(h)\rangle_{L^2} = 1 \tag{7.19}$$

hold, where $\delta = \delta(\boldsymbol{h})$, $h_1 = 0$ and $h_2 = h$, and $\mathrm{O}(\delta)$ means here $\|\mathrm{O}(\delta)\|_{H^2} \le C\delta$.

Now, we fix $h^* > 0$ large enough such that Propositions 7.3 and 7.4 hold.

Let operators $Q(h)$ and $R(h)$ be the projections from X to $E(h)$ and $R(h) = \mathrm{Id} - Q(h)$, respectively, where Id is the identity on X. Let $E^\perp(h) = R(h)X$. Note that $E^\perp(h)$ is characterized by

$$E^\perp(h) = \{\boldsymbol{U} \in X; \langle \boldsymbol{U}, \boldsymbol{\phi}_j^*(h)\rangle_{L^2} = \langle \boldsymbol{U}, \boldsymbol{\psi}_j^*(h)\rangle_{L^2} = 0 \ \ (j = 1, 2)\}.$$

Fix $\hat{h}$ with $\hat{h} > h^*$ arbitrarily and put $\delta^* = \delta(h^*)$, $\hat{\delta} = \delta(\hat{h})$. Then, we can show that there exists a map $\Pi(h)$ homeomorphic from $E^\perp(\hat{h})$ to $E^\perp(h)$ for $h > \hat{h}$ similar to [1].

Fix $\rho_1 > 0$ and define $H(\hat{h}, \rho_1) = \{h; \hat{h} < h < \hat{h} + \rho_1\}$, $\mathcal{M} = \mathcal{M}(\hat{h}, \rho_1) = \{\tau(l)S(\cdot, h, \boldsymbol{r}); l \in \boldsymbol{R}, h \in H(\hat{h}, \rho_1), |\boldsymbol{r}| < r^*\}$. Then we note [1] that there exist a positive constant C_1 depending only on ρ_1 and independent of $\hat{h}$ with $\hat{h} > h^*$ for sufficiently large h^* such that for any $\boldsymbol{h} \in H(\hat{h}, \rho_1)$ the map $\Pi(h)$ satisfies

$$\|\Pi(h)\|, \|\Pi^{-1}(h)\|, \left\|\frac{\partial}{\partial h}\Pi(h)\right\| \le C_1, \qquad \|\Pi(h)\|_\infty, \|\Pi^{-1}(h)\|_\infty, \left\|\frac{\partial}{\partial h}\Pi(h)\right\|_\infty \le C_1,$$

where $\|\cdot\|_\infty$ is an operator norm with respect to the sup-norm $\|\cdot\|_\infty$ on $\boldsymbol{R}^1$.

Let $A = L(\hat{h})$ and X^ω be the space with the norm $\|\cdot\|_\omega$ defined by the fractional power A^ω of A for $\omega \in [0, 1)$. Hereafter, we fix ω in $3/4 < \omega < 1$ such that X^ω is imbedded into $BU^1(\boldsymbol{R})$ [3].

We have the following proposition (e.g. [1]).

Proposition 7.5. *There exists a neighborhood* $U = U(\hat{h}, \rho_1)$ *of* $\mathcal{M}(\hat{h}, \rho_1)$ *in* X^ω *such that any* $\boldsymbol{U} \in U$ *is represented by*

$$\boldsymbol{U} = \tau(l)\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\}$$

for $l \in \boldsymbol{R}$, $\boldsymbol{h} \in H(\hat{h}, \rho_1)$ *and* $\boldsymbol{W} \in E^\perp(\hat{h})$.

We transform Eq. (4.2) of $\boldsymbol{U}$ to that of $(\boldsymbol{W}, l, h, \boldsymbol{r})$ by

$$\boldsymbol{W}(t, x) = \tau(l)\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\}$$

for $l \in \boldsymbol{R}$, $h \in H(\hat{h}, \rho_1)$, $|\boldsymbol{r}| < r^*$ and $\boldsymbol{W} \in E^\perp(\hat{h})$. Since $\tau'(l) = -\tau(l)(\partial/\partial x)$ holds, we have

$$\begin{aligned}\boldsymbol{U}_t &= \dot{l}\tau'(l)\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\} + \tau(l)\left(\frac{\partial}{\partial(h, \boldsymbol{r})}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\}(\dot{h}, \dot{\boldsymbol{r}}) + \Pi(h)\boldsymbol{W}_t\right)\\ &= \tau(l)\left(-\dot{l}\frac{\partial}{\partial x}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\} + \frac{\partial}{\partial(h, \boldsymbol{r})}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\}(\dot{h}, \dot{\boldsymbol{r}}) + \Pi(h)\boldsymbol{W}_t\right) \quad \text{and}\\ \mathcal{L}(\boldsymbol{U}) + \eta g(\boldsymbol{U}) &= \mathcal{L}(\tau(l)\boldsymbol{V}) + \eta g(\tau(l)\boldsymbol{V}) = \tau(l)(\mathcal{L}(\boldsymbol{V}) + \eta g(\boldsymbol{V})),\end{aligned}$$

where $\boldsymbol{V} = S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}$. Hence, it follows that

$$-\dot{l}\frac{\partial}{\partial x}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\} + \frac{\partial}{\partial(h, \boldsymbol{r})}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\}(\dot{h}, \dot{\boldsymbol{r}}) + \Pi(h)\boldsymbol{W}_t = \mathcal{L}(\boldsymbol{V}) + \eta g(\boldsymbol{V}),$$

and that

$$\begin{aligned}&Q(h)\left[-\dot{l}\frac{\partial}{\partial x}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\} + \frac{\partial}{\partial(h, \boldsymbol{r})}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\}(\dot{h}, \dot{\boldsymbol{r}})\right]\\ &\quad = Q(h)\mathcal{L}(S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}) + \eta Q(h)g(S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}),\end{aligned} \tag{7.20}$$

$$\begin{aligned}&\Pi^{-1}(h)R(h)\left[-\dot{l}\frac{\partial}{\partial x}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\} + \frac{\partial}{\partial(h, \boldsymbol{r})}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\}(\dot{h}, \dot{\boldsymbol{r}})\right] + \boldsymbol{W}_t\\ &\quad = \Pi^{-1}(h)R(h)\{\mathcal{L}(S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}) + \eta g(S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W})\}.\end{aligned} \tag{7.21}$$

Let $\rho_2 > 0$ and $C_2 > 0$ be constants such that if $|\boldsymbol{r}| < r^*$, $\|\boldsymbol{W}\|_\omega < \rho_2$ and $h \in H(\hat{h}, \rho_1)$, then

$$|\mathcal{L}(S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}) - \mathcal{L}(P(x; h)) - L(h)\Pi(h)\boldsymbol{W}| \le C_2(|\boldsymbol{r}|^2 + |\boldsymbol{W}|^2) \tag{7.22}$$

holds. We note that ρ_2 is taken to be independent of $\hat{h}$ and depending only on ρ_1.

Put

$$\begin{aligned}W(\hat{h}, \rho_1, D_1, D_2, \eta) = \{&\boldsymbol{W}(\cdot) \in C(H(\hat{h}, \rho_1) \times (-r^*, r^*); E^\perp(\hat{h}) \cap X^\omega); \|\boldsymbol{W}(h, \boldsymbol{r})\|_\omega\\ &\le D_1\Delta_1(h, \boldsymbol{r}, \eta), \|\boldsymbol{W}(h, \boldsymbol{r}) - \boldsymbol{W}(h', \boldsymbol{r}')\| \le D_2(\delta(h) + \delta(h')\\ &+ |\boldsymbol{r}| + |\boldsymbol{r}'| + |\eta|)(|h - h'| + |\boldsymbol{r} - \boldsymbol{r}'|)\},\end{aligned}$$

where $\Delta_1(h, \boldsymbol{r}, \eta) = \delta(h) + |\boldsymbol{r}|^2 + |\eta|$. We determine D_1, D_2 later but suppose h^* is large enough and r^*, $|\eta|$ are small enough so as to $D_1\Delta_1(h, \boldsymbol{r}, \eta) < \rho_2$ for $h \in H(\hat{h}, \rho_1)$ with $\hat{h} > h^*$ and $|\boldsymbol{r}| < r^*$. If $\boldsymbol{W} \in W(\hat{h}, \rho_1, D_1, D_2, \eta)$, then (7.20) yields

$$\dot{l} = J^*(h, \boldsymbol{r}, \boldsymbol{W}) = r_1 - \tilde{H}_1(h) + \mathrm{O}(\delta^2 + |\boldsymbol{r}|^2 + |\eta|^2), \tag{7.23}$$

$$\dot{h} = H^*(h, \boldsymbol{r}, \boldsymbol{W}) = r_2 - r_1 + \tilde{H}_1(h) - \tilde{H}_2(h) + \mathrm{O}(\delta^2 + |\boldsymbol{r}|^2 + |\eta|^2), \tag{7.24}$$

$$\dot{r}_j = K_j^*(h, r_j, \boldsymbol{W}) = K(r_j; \eta) + H_j(h) + \mathrm{O}(\delta^2 + |\boldsymbol{r}|^2 + |\eta|^2), \quad j = 1, 2, \tag{7.25}$$

where $\delta = \delta(h)$ and $\tilde{H}_j(h) = \langle \mathcal{L}(P(\cdot + h_j; h)), \Psi^* \rangle_{L^2}$, $H_j(h) = \langle \mathcal{L}(P(\cdot + h_j; h)), \Phi^* \rangle_{L^2}$ and $h_1 = 0$, $h_2 = h$. Especially,

$$\dot{l}, \dot{h}_j = \mathrm{O}(\varDelta_2), \qquad \dot{r}_j = \mathrm{O}(\varDelta_3). \tag{7.26}$$

Hence, it follows from (7.21) and (7.22) that

$$\boldsymbol{W}_t = A(h)\boldsymbol{W} + G^*(h, \boldsymbol{r}, \boldsymbol{W}) \tag{7.27}$$

with $\|G^*\| = \mathrm{O}(\varDelta_1)$ for $h \in H(\hat{h}, \rho_1,)$ and $\boldsymbol{W} \in W(\hat{h}, \rho_1, D_1, D_2, \eta)$, where

$$\begin{aligned}
&A(h) = \Pi^{-1}(h)L(h)\Pi(h),\\
&G^*(h, \boldsymbol{r}, \boldsymbol{W}) = \Pi^{-1}(h)R(h)\Bigg[\mathcal{L}(S(x; h, \boldsymbol{r})) + L_2(\boldsymbol{W}, \boldsymbol{W}) + \eta g(S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W})\\
&\qquad + J^* \frac{\partial}{\partial x}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\} - \frac{\partial}{\partial(h, \boldsymbol{r})}\{S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\}(H^*, \boldsymbol{K}^*)\Bigg],\\
&L_2(\boldsymbol{W}, \boldsymbol{W}) = L_2(h, \boldsymbol{r}, \boldsymbol{W})(\boldsymbol{W}, \boldsymbol{W}) = \mathcal{L}(S(x; h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}) - \mathcal{L}(S(x; h, \boldsymbol{r})) - L(h)\Pi(h)\boldsymbol{W},\\
&\boldsymbol{K}^* = \boldsymbol{K}^*(h, \boldsymbol{r}, \boldsymbol{W}) = (K_1^*(h, r_1, \boldsymbol{W}), K_2^*(h, r_1, \boldsymbol{W})).
\end{aligned}$$

Then, in quite a similar way to [1], we can show the existence of a function $\sigma(h, \boldsymbol{r}; \eta) \in W(\hat{h}, \rho_1, D_1, D_2, \eta)$ for appropriate constants D_1 and D_2 such that the set $\{(h, \boldsymbol{r}, \sigma(h, \boldsymbol{r}; \eta)); \boldsymbol{h} \in H(\hat{h}, \rho_1), |\boldsymbol{r}| < r^*\}$ is a positively attractive local invariant manifold for the solution $(h, \boldsymbol{r}, \boldsymbol{W})$ of (7.24), (7.25) and (7.27). Hence, by using the similar discussions to the proof of Theorem 2.1 in [1], we can show that there exists a neighborhood U of

$$\bigcup_{\hat{h} > h^*} \{(h, \boldsymbol{r}, 0); h \in H(\hat{h}, \rho_1), |\boldsymbol{r}| < r^*\} = \{(h, \boldsymbol{r}, 0); h > h^*\}$$

such that if $(h(0), \boldsymbol{r}(0), \boldsymbol{W}(0)) \in U$, then the solution of (7.24), (7.25) and (7.27) is attracted exponentially and remains in $\varDelta_1$ neighborhood of the set as long as $h > h^*$ and $|\boldsymbol{r}| < r^*$, that is, $\|\boldsymbol{W}(t)\| \le C\varDelta_1(h(t), \boldsymbol{r}(t), \eta)$. Since the solution $\boldsymbol{U}$ of (4.2) is given by $\boldsymbol{U} = \tau(l)\{S(h, \boldsymbol{r}) + \Pi(h)\boldsymbol{W}\}$ by using the solution $(h, \boldsymbol{r}, \boldsymbol{W})$ of (7.24), (7.25) and (7.27), this reads

$$\|\boldsymbol{U}(t) - \tau(l)\{S(h(t), \boldsymbol{r}(t))\}\| \le \|\Pi(h)\boldsymbol{W}\| \le C\varDelta_1(h(t), \boldsymbol{r}(t), \eta),$$

and the proof is complete.

7.5. Proof of Theorems 5.2 and 5.3 and Corollary 5.1

The proof of Theorem 5.2 is quite similar to the proof of Theorem 4.2 because we have already got the estimates of $\dot{l}$, $\dot{h}$ and $\dot{r}_j$ by (7.23)–(7.25).

Theorem 5.3 can be also proved following the proof of Theorem 2.3 in [1]. Especially, the expression of the constant $\tilde{M}_0$ is given by Proposition 4.5 in [1].

Corollary 5.1 is directly obtained by Theorems 5.2 and 5.3 by taking $r_2 = -r_1 \equiv r$ and $h = 2l$.

8. Discussion

We have first numerically shown that if there is the situation where very slowly one-dimensional travelling pulse solutions exist, very complex spatio-temporal patterns possibly appear in two dimensions. The reason is that such very slowly travelling pulses possess reflection mechanism by which expanding rings split into several pieces when they approach one another. From mathematical viewpoints, we have proven that any travelling pulses bifurcating primarily and super-critically from a standing pulse always reflect when they approach, if their velocity is very slow. The center manifold theory reveals that the interaction of two travelling pulses can be described by four-dimensional system of ODEs. The analysis of the system enables to explain that very slowly travelling pulses reflect one another, as if they were elastic-like particles. Furthermore, the interval $(0, L)$ with the zero-flux boundary conditions is very long, it is shown that the pulse-like solution reflect near $x = 0$ and L as if it were rebounding with two walls (Fig. 5.3). The method which we used here is also applicable to higher dimensional problems. For instance, if there exist travelling spots that move very slowly in two dimensions, they possess the property of reflection [4,5,8]. This will be proved in a forthcoming paper.

Acknowledgements

M. Mimura would like to thank for the supports of the Grand-in-Aid for Scientific Research (A) 12304006 and for Scientific Research on Priority Area (B) 11214101. M. Nagayama is supported by the Grand-in-Aid for Scientific Research 12740062. S.-I. Ei is supported by the Grand-in-Aid for Scientific Research 12640217.

Appendix A. Proof of Proposition 4.2

We will only show the proof for $\Psi(x)$. Since $L\Psi = -P_x$ holds, $\Psi(x)$ satisfies the ODE

$$D\Psi_{xx} + F'(P(x); k_c)\Psi = -P_x.$$

Putting $G(x) = F'(P(x); k_c)$, $\Phi(x) = P_x(x)$ and $W = D\Psi_x$, we have the equivalent ODE

$$\Psi_x = D^{-1}W, \qquad W_x = -G(x)\Psi - \Phi. \tag{A.1}$$

Since $G(x) \to F'(0; k_c) + F''(0; k_c)\,\mathrm{e}^{-\alpha x}\,\boldsymbol{a}$ and $\Phi(x) \to -\alpha\,\mathrm{e}^{-\alpha x}\,\boldsymbol{a}$ as $x \to +\infty$, (A.1) becomes asymptotically

$$\Psi_x = D^{-1}W, \qquad W_x = -G_0\Psi + \alpha\,\mathrm{e}^{-\alpha x}\,\boldsymbol{a}, \tag{A.2}$$

where $G_0 = F'(0; k_c)$. The solution of (A.2) gives the asymptotic form of Ψ and the proof is finished.

In order to obtain the solution of (A.2), we put $(\Psi, W) = \mathrm{e}^{-\alpha x}\,\boldsymbol{V}$ and then (A.2) is written by

$$\boldsymbol{V}_x = \hat{G}_0\boldsymbol{V} + \mathrm{e}^{-\alpha x}\,\hat{\boldsymbol{a}},$$

where

$$\hat{G}_0 = \begin{pmatrix} \mathbf{0} & D^{-1} \\ -G_0 & \mathbf{0} \end{pmatrix}$$

and $\hat{\boldsymbol{a}} = {}^t(\mathbf{0}, \alpha\boldsymbol{a})$. Transforming this equation by $\boldsymbol{V} = \mathrm{e}^{-\alpha x}\,\boldsymbol{v}$, we have

$$\boldsymbol{v}_x = A\boldsymbol{v} + \hat{\boldsymbol{a}}, \tag{A.3}$$

where

$$A=\begin{pmatrix}\alpha I_N & D^{-1}\\ -G_0 & \alpha I_N\end{pmatrix}$$

and I_N is the unit matrix of degree N. Clearly, $\hat{\Phi}={}^t(\boldsymbol{a},-\alpha D\boldsymbol{a})$ satisfies $A\hat{\Phi}=\mathbf{0}$ and $\ker A=\mathrm{span}\{\hat{\Phi}\}$.

On the other hand, $\hat{\Phi}^*={}^t(-\alpha D\boldsymbol{a}^*,\boldsymbol{a}^*)$ satisfies ${}^tA\hat{\Phi}^*=\mathbf{0}$ and $\ker{}^tA=\mathrm{span}\{\hat{\Phi}^*\}$ because $\boldsymbol{a}^*$ satisfies $\alpha^2D\boldsymbol{a}^*+{}^tG_0\boldsymbol{a}^*=\mathbf{0}$. Therefore, the projection $\hat{Q}$ from $\boldsymbol{R}^{2N}$ to $\ker A$ is given by

$$\hat{Q}\boldsymbol{v}=\frac{\langle\boldsymbol{v},\hat{\Phi}^*\rangle}{\langle\hat{\Phi},\hat{\Phi}^*\rangle}\hat{\Phi}=-\frac{\langle\boldsymbol{v},\hat{\Phi}^*\rangle}{2\alpha\langle\boldsymbol{a},D\boldsymbol{a}^*\rangle}\hat{\Phi}.$$

Let $\hat{R}=I_{2N}-\hat{Q}$. Since the eigenvalues of A except 0 are in the left-hand side of imaginary axis, the solution $\boldsymbol{v}$ of (A.3) is given by

$$\boldsymbol{v}(x)=x\hat{Q}\hat{\boldsymbol{a}}+\beta_2\hat{\Phi}+\mathrm{O}(\mathrm{e}^{-\gamma_1x})-A^{-1}\hat{R}\hat{\boldsymbol{a}}=-\frac{\langle\boldsymbol{a},\boldsymbol{a}^*\rangle}{2\langle\boldsymbol{a},D\boldsymbol{a}^*\rangle}x\hat{\Phi}+\beta_2\hat{\Phi}+\mathrm{O}(\mathrm{e}^{-\gamma_1x})-A^{-1}\hat{R}\hat{\boldsymbol{a}}$$

for a constant β and a positive constant γ_1. Hence, we may take

$$\boldsymbol{v}(x)=-\frac{\langle\boldsymbol{a},\boldsymbol{a}^*\rangle}{2\langle\boldsymbol{a},D\boldsymbol{a}^*\rangle}x\hat{\Phi}+\beta\hat{\Phi}-A^{-1}\hat{R}\hat{\boldsymbol{a}}$$

as the asymptotic form of $\boldsymbol{v}(x)$. Let ${}^t(\boldsymbol{b},\boldsymbol{c})=-A^{-1}\hat{R}\hat{\boldsymbol{a}}$. Picking up the first component of $\boldsymbol{v}$, we have

$$\boldsymbol{v}_1(x)=-\frac{\langle\boldsymbol{a},\boldsymbol{a}^*\rangle}{2\langle\boldsymbol{a},D\boldsymbol{a}^*\rangle}x\boldsymbol{a}+\beta\boldsymbol{a}+\boldsymbol{b},$$

where $\boldsymbol{v}={}^t(\boldsymbol{v}_1,\boldsymbol{v}_2)$. Now, $\Psi(x)=\mathrm{e}^{-\alpha x}\,\boldsymbol{v}_1(x)$ and hence $W(x)=D\Psi_x(x)=\mathrm{e}^{-\alpha x}\,D\{\partial_x\boldsymbol{v}_1(x)-\alpha\boldsymbol{v}_1(x)\}$ holds, which means

$$\begin{aligned}\boldsymbol{v}_2(x)=D\{\partial_x\boldsymbol{v}_1(x)-\alpha\boldsymbol{v}_1(x)\}&=\frac{\alpha\langle\boldsymbol{a},\boldsymbol{a}^*\rangle}{2\langle\boldsymbol{a},D\boldsymbol{a}^*\rangle}xD\boldsymbol{a}-\left(\frac{\langle\boldsymbol{a},\boldsymbol{a}^*\rangle}{2\langle\boldsymbol{a},D\boldsymbol{a}^*\rangle}+\alpha\beta\right)D\boldsymbol{a}-\alpha D\boldsymbol{b}\\&=\frac{\alpha\langle\boldsymbol{a},\boldsymbol{a}^*\rangle}{2\langle\boldsymbol{a},D\boldsymbol{a}^*\rangle}xD\boldsymbol{a}-\alpha\beta D\boldsymbol{a}-\left(\frac{\langle\boldsymbol{a},\boldsymbol{a}^*\rangle}{2\langle\boldsymbol{a},D\boldsymbol{a}^*\rangle}D\boldsymbol{a}+\alpha D\boldsymbol{b}\right).\end{aligned}$$

Comparing with the component $\boldsymbol{v}_2$, we have

$$\boldsymbol{c}=-\left(\frac{\langle\boldsymbol{a},\boldsymbol{a}^*\rangle}{2\langle\boldsymbol{a},D\boldsymbol{a}^*\rangle}D\boldsymbol{a}+\alpha D\boldsymbol{b}\right).$$

Substituting this into the equation $A{}^t(\boldsymbol{b},\boldsymbol{c})=-\hat{R}\hat{\boldsymbol{a}}$, we see

$$G_0\boldsymbol{b}+\alpha^2D\boldsymbol{b}=\alpha\left(\boldsymbol{a}-\frac{\langle\boldsymbol{a},\boldsymbol{a}^*\rangle}{\langle\boldsymbol{a},D\boldsymbol{a}^*\rangle}D\boldsymbol{a}\right)\tag{A.4}$$

holds. The matrix $(G_0+\alpha^2D)$ has 0 eigenvalue with the associated eigenvector $\boldsymbol{a}$ and the transposed matrix $({}^tG_0+\alpha^2D)$ has also 0 eigenvalue with the associated eigenvector $\boldsymbol{a}^*$. Since the right-hand side of (A.4) is clearly orthogonal to $\boldsymbol{a}^*$, there exist a vector $\boldsymbol{b}$ satisfying (A.4). The vector $\boldsymbol{b}$ is uniquely determined by the orthogonal condition $\langle{}^t(\boldsymbol{b},\boldsymbol{c}),\hat{\Phi}^*\rangle=0$ as $\boldsymbol{b}=\beta'\boldsymbol{a}+\boldsymbol{b}'$ with $\langle\boldsymbol{b}',\boldsymbol{a}^*\rangle=0$. Thus, the proof is completed.

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