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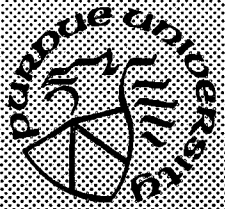
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# **Pulse Width Modulated Control of Robotic Manipulators**

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# PULSE WIDTH MODULATED CONTROL OF ROBOTIC MANIPULATORS

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## Abstract

In this paper we propose a practical discontinuous feedback control scheme for the regulation of joint positions of robotic manipulators. A robust on-off switching control strategy based on a pulse-width-modulation (PWM) feedback scheme is proposed for the joint torques. The discontinuous PWM controller design is carried out on the basis of a suitable controller designed for an average model which is of continuous nature. Simulations of the closed loop performance of the proposed control scheme are presented for a two-link robotic manipulator.

**Keywords:** Pulse-Width-Modulation, Robotic Manipulators, Nonlinear Systems, Variable Structure Systems.

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## 1. INTRODUCTION

In this article a Pulse-Width-Modulated (PWM) feedback control scheme is proposed for the regulation of robot joint positions. The PWM switched control strategy specifies the control torque to the system corresponding to one of two fixed feedback control laws. The discontinuous (on-off) stabilizing controller is determined from a continuous controller which is designed for the nonlinear average model of the PWM controlled system. The average model, used here for design purposes, is obtained by formally imposing an infinite sampling frequency assumption on the PWM actuator. It is shown, in full generality, that the obtained average model coincides with the original plant dynamics, when feedback controlled by a suitable convex combination of the fixed available feedback control laws. The smooth duty ratio function components serve as the averaging parameters for the referred convex combination (See Filippov [1]) and constitute the control inputs for the average model. The infinite sampling frequency idealization on the controlled system establishes exactly the same relationship between the average and the actual PWM controlled responses as the one existing between the actual chattering motions and the Ideal Sliding Dynamics associated with Variable Structure Control Schemes undergoing sliding regimes (Utkin [2]). In the context of single-input single-output systems (SISO), the average (piece-wise smooth) controlled system model has been shown to capture all the essential qualitative features (i.e., stability) of the actual (discontinuous) PWM controlled system, provided a sufficiently high sampling frequency is used (See Sira-Ramirez [3]-[5]). The actual (discontinuous) PWM controller design is shown to be easily obtained from the average closed loop stabilizing design.

PWM controlled systems have been studied in the past in the context of linear systems by Skoog [6], Skoog and Blankenship [7], by La Cava et al [8], Chen and Wu [9] and Gelig and Churliov [10]. For a rather complete survey of the many early contributions available in this field, the reader is referred to Sira-Ramirez [11] and the references therein. The definite relationships between PWM controlled responses for nonlinear single-input single-output systems and their underlying sliding regimes were presented in [3]-[5]. Applications of these fundamental connections to the design of regulating controllers for DC-to-DC Power Converters were given in Sira-Ramirez [12]-[13] and in Sira-Ramirez and Ilic [14]. Some aerospace applications of PWM control can also be found in [11].

PWM control has a considerable appeal in the practical implementation of control systems using digital computers and digital logic circuits. Infact a number of integrated circuits (IC's) may be developed for PWM control systems without requiring the Digital to Analog conversion and complicated analog power amplifiers. This simplicity can lead to considerable cost savings because in a servo system the PWM voltage pulses generated by the control computer can be fed directly to the base of each power transistors. The power transistors would be fully on or fully off (assuming ideal switching

conditions), then the servo motor voltages could be controlled between  $\pm V_{\max}$  resulting in output motor torques of  $\pm \tau_{\max}$ . The beauty of the PWM control system is that the entire system could be implemented on an IC, with the exception of the power transistors, the diodes and a small amount of switching electronics. It is conceivable with further developments in the PWM control theory and IC's for this type of control, a servo system hardware maybe packaged into the size of a matchbox, and as a result it could be located next to the servo motor.

In this article, an extension of the basic SISO results for nonlinear PWM controlled systems is developed for multivariable nonlinear PWM systems. An application example is furnished for the PWM feedback control of angular positions of robotic manipulators.

In Section 2 we consider the relevant features of nonlinear multivariable PWM controlled systems. In particular, we derive the average model and point out its relevance to the actual discontinuous stabilizing design. In Section 3 we derive the PWM controller for a two link robotic manipulator and discuss the simulation results. In Section 4 we propose new directions for further research.

## 2. MULTIVARIABLE PWM CONTROL OF NONLINEAR SYSTEMS

### 2.1. An Average Approach to the Analysis and Design of Multivariable PWM Controlled Nonlinear Systems.

In this section a technique is developed for the analysis of nonlinear systems regulated by means of Pulse Width Modulating the actuators. This technique is based on an obtained average model of the PWM controlled system. This model is useful both as an analysis and as a design tool for a large class of nonlinear systems discontinuously controlled by Pulse Width Modulating of the actuators.

Consider the nonlinear PWM controlled plant:

$$\frac{dx(t)}{dt} = f(x(t)) + G(x(t))u(t); \quad \text{with } u(t) = \text{PWM}[\mu(t)] \quad (2.1)$$

where,

$$u_i(t) = \text{PWM}[\mu_i(t)] = \begin{cases} u_i^+(x(t)) & \text{for } t_k \leq t < t_k + \mu_i(t_k)T \\ u_i^-(x(t)) & \text{for } t_k + \mu_i(t_k)T \leq t < t_k + T \end{cases} \quad (2.2)$$

$i = 1, 2, \dots, m$

and,  $t_{k+1} = t_k + T$ , where  $T$  is the PWM frequency.

Here,  $x(t)$  is the  $n$ -dimensional plant state vector,  $u(t)$  represents an  $m$ -dimensional input vector of discontinuous control functions constituted by fixed multivariable

feedback control laws  $u^+(x(t))$  and  $u^-(x(t))$ . The feedback control components  $u_i(t)$  ( $i = 1, 2, \dots, m$ ) are switched *at most once* from  $u_i^+(x(t))$  to  $u_i^-(x(t))$  within the inter-sampling interval  $[t_k, t_{k+1}]$ , here addressed as the *duty cycle*, of fixed duration  $T$ . The sampling frequency  $1/T$  is assumed to be the same amongst all controllers ( $i = 1 \dots m$ ) (extension to the multi-rate option will not be treated here). The instants at which the switching occur in the inter-sampling interval  $[t_k, t_{k+1}]$  are determined, for each control, by the corresponding component value of the multivariable *duty ratio function*  $\mu(t)$  obtained at the sampling instants  $t_k$ , i.e. by  $\mu(t_k)$ . The components of the duty ratio functions  $\mu(t)$  are continuous piece-wise smooth functions of time. The duty ratio functions are limited between 0 and 1; this way the corresponding activation of the feedback control result in physically meaningful PWM duty ratios which are applied to the actuators. The duty ratio function  $\mu(t)$  will be generated by means of a feedback arrangement whose detailed specification will be presented in section 2.3. The vector function  $f(\cdot)$  and the column vectors  $g_i(\cdot)$  ( $i = 1, 2, \dots, m$ ), of the matrix  $G(x)$ , are considered to be  $n$ -dimensional smooth vector fields. The  $n \times m$  matrix  $G(x)$  is assumed to be locally full rank. We refer to the discontinuous control scheme described by (2.1) - (2.2) as the actual PWM controlled system.

### Definition 2.1

The average PWM system is defined as the smooth dynamical system obtained from (2.1) - (2.2) when the sampling frequency of the PWM actuator,  $1/T$ , grows to infinity.

The next proposition establishes that the above defined average system is easily obtained from the discontinuous system (2.1), by just formally replacing the discontinuous control input vector  $u(t)$  by a continuous piece-wise smooth control vector in which each component is a convex combination of the fixed available feedback control laws  $u_i^+(x(t))$ ,  $u_i^-(x(t))$ . The duty ratio function components  $\mu_i$ , associated with the input vector components are seen to serve as the averaging parameter in the above mentioned convex combination.

### Proposition 2.2

The average PWM system is represented by:

$$\frac{dz(t)}{dt} = f(z(t)) + \sum_{i=1}^m g_i(z(t))[\mu_i(t)u_i^+(z(t)) + (1-\mu_i(t))u_i^-(z(t))] \quad (2.3)$$

### Proof:

We can rewrite the differential equation for the controlled plant in (2.1) - (2.2) as an equivalent integral-difference equation relating the values of the state vector  $x(t)$  at subsequent sampling instants:

$$\begin{aligned} \mathbf{x}(t_k + T) = & \mathbf{x}(t_k) + \int_{t_k}^{t_k+T} \mathbf{f}(\mathbf{x}(\sigma))d\sigma + \sum_{i=1}^m \int_{t_k}^{t_k+\mu_i(t_k)T} \mathbf{g}_i(\mathbf{x}(\sigma))\mathbf{u}_i^+(\mathbf{x}(\sigma))d\sigma \\ & + \sum_{i=1}^m \int_{t_k+\mu_i(t_k)T}^{t_k+T} \mathbf{g}_i(\mathbf{x}(\sigma))\mathbf{u}_i^-(\mathbf{x}(\sigma))d\sigma \end{aligned}$$

Notice that this expression involves no approximation whatsoever.

Subtracting  $\mathbf{x}(t_k)$  from  $\mathbf{x}(t_k+T)$ , dividing by  $T$  and taking the limit as the sample period  $T$  goes to zero, while letting the sampling instant  $t_k$  adopt an arbitrary value  $t$  of the time coordinate, one obtains:

$$\begin{aligned} \lim_{T \rightarrow 0, t_k \rightarrow t} \left[ \frac{\mathbf{x}(t_k+T) - \mathbf{x}(t_k)}{T} \right] &= \frac{d\mathbf{x}(t)}{dt} = \\ &= \lim_{T \rightarrow 0, t_k \rightarrow t} \left[ \frac{\int_{t_k}^{t_k+T} \mathbf{f}(\mathbf{x}(\sigma))d\sigma}{T} \right] + \\ & \lim_{T \rightarrow 0, t_k \rightarrow t} \left[ \frac{\sum_{i=1}^m \int_{t_k}^{t_k+\mu_i(t_k)T} \mathbf{g}_i(\mathbf{x}(\sigma))\mathbf{u}_i^+(\mathbf{x}(\sigma))d\sigma}{T} \right] + \\ & \lim_{T \rightarrow 0, t_k \rightarrow t} \left[ \frac{\sum_{i=1}^m \int_{t_k+\mu_i(t_k)T}^{t_k+T} \mathbf{g}_i(\mathbf{x}(\sigma))\mathbf{u}_i^-(\mathbf{x}(\sigma))d\sigma}{T} \right] \\ &= \mathbf{f}(\mathbf{x}(t)) + \sum_{i=1}^m \mathbf{g}_i(\mathbf{x}(t)) [\mu_i(t)\mathbf{u}_i^+(\mathbf{x}(t)) + (1-\mu_i(t))\mathbf{u}_i^-(\mathbf{x}(t))] \end{aligned} \quad (2.3b)$$

i.e., one obtains the time derivative of the state vector  $\mathbf{x}(t)$ , at each instant of time  $t$ , when the sampling frequency  $1/T$  is assumed to be infinitely large. For the purpose of clarity, we denote by  $\mathbf{z}(t)$  the averaged value of the state vector  $\mathbf{x}(t)$ , and  $\mathbf{x}(t)$  is obtained as the solution of (2.3b). Thus the average system behaviour is given by:

$$\frac{d\mathbf{z}(t)}{dt} = \mathbf{f}(\mathbf{z}(t)) + \sum_{i=1}^m \mathbf{g}_i(\mathbf{z}(t))[\mu_i(t)\mathbf{u}_i^+(\mathbf{z}(t)) + (1-\mu_i(t))\mathbf{u}_i^-(\mathbf{z}(t))]$$

□

Therefore the average PWM control vector  $\mathbf{v}_i(t)$  is given as

$$\mathbf{v}_i(t) = \mu_i(t)\mathbf{u}_i^+(\mathbf{z}(t)) + (1 - \mu_i(t))\mathbf{u}_i^-(\mathbf{z}(t))$$

$$i = 1, 2, \dots, m$$

## 2.2. Properties of Actual and Averaged PWM Controlled Systems.

Notice that the average model (2.3) is highly reminiscent to the *ideal sliding dynamics* obtained by application of the Method of the Equivalent Control (Utkin [2]) to the underlying multi-input Variable Structure System described by (2.1) - (2.2). As a matter of fact, the obtained system of differential equations for the plant dynamics in (2.3), constitutes a Filippov type of geometric average model (See Filippov [1]). Such models are extensively used to describe the idealized smooth behavior (i.e., the average behavior) of sliding regimes occurring in the intersection of a finite number of discontinuous surfaces.

Instead of further relating actual trajectories of PWM controlled systems with non-ideal sliding regimes taking place on integral manifolds of the average PWM model (See Sira-Ramirez [3] - [5] for further details), we shall derive some essential properties of the actual discontinuous responses of (2.1) - (2.2) in relation to the idealized response of the average model (2.3). In particular, we show that the integral equation related to the discontinuous model (2.1) - (2.2) is a second order regular perturbation - in the sampling period  $T$  - of the corresponding integral equation of the average model (2.3). This means that as the sampling frequency goes to infinity, the trajectories of (2.1) - (2.2) tend *continuously* toward those of the average model (2.3). First, we introduce some notation and state some basic lemmas:

Notice that system (2.1) - (2.2) can be written as:

$$\frac{dx(t)}{dt} = \left[ f(x(t)) + \sum_{i=1}^m g_i(x(t))u_i^-(x(t)) \right] + \sum_{i=1}^m w_i g_i(x(t))[u_i^+(x(t)) - u_i^-(x(t))]$$

with

$$w_i(t) = \text{PWM}[\mu_i(t)] = \begin{cases} 1 & \text{for } t_k \leq t < t_k + \mu_i(t_k)T \\ 0 & \text{for } t_k + \mu_i(t_k)T \leq t < t_k + T \end{cases} \quad (2.4)$$

$$i = 1, 2, \dots, m$$

Hence, without any loss of generality, and with a slight abuse of notation, one may assume that the controlled system is of the form:

$$\frac{dx(t)}{dt} = f(x(t)) + G(x(t))w \quad (2.5)$$

with the control vector  $w$  given, componentwise, by (2.4).



**Lemma 2.3**

The average system obtained by infinite sampling frequency in (2.4), (2.5) is given by:

$$\frac{dz(t)}{dt} = f(z(t)) + g(z(t))\mu(t) \quad (2.6)$$

where  $\mu(t)$  is the multivariable duty ratio function.

**Proof:** This is obvious upon letting  $u_i^+(x) = 1$ , and,  $u_i^-(x) = 0$ , for all  $i$ 's in Proposition 2.2.

**Lemma 2.4**

Let  $g(t)$  and  $\mu(t)$  be piecewise smooth functions of  $t$  defined in the closed interval  $[0, T]$  with  $0 < \mu(t) < 1$  for all  $t$ , then, for any  $T$  such that  $0 < \tau < T$ , we have:

$$\int_0^T g(\sigma)\mu(\sigma)d\sigma = \int_0^{\mu(\tau)T} g(\sigma)d\sigma + O(T^2) \quad (2.9)$$

**Proof:**

Expanding  $g(\sigma)$  using a Taylor series about  $\sigma = 0$ , one obtains the following expression for the integral in the right hand side of the above equality (2.9):

$$\begin{aligned} \int_0^{\mu(\tau)T} g(\sigma)d\sigma &= \int_0^{\mu(\tau)T} [g(0) + \left\{ \frac{dg(\sigma)}{d\sigma} \Big|_{\sigma=0} \right\} \sigma + \text{h.o.t}] d\sigma \\ &= g(0)\mu(\tau)T + \frac{1}{2} \left\{ \frac{dg(\sigma)}{d\sigma} \Big|_{\sigma=0} \right\} [\mu(\tau)T]^2 + \text{h.o.t} \end{aligned}$$

where h.o.t are higher order terms.

Using a similar expansion for the integral in the left hand side of (2.4), yields:

$$\begin{aligned} \int_0^T g(\sigma)\mu(\sigma)d\sigma &= \int_0^T [g(0)\mu(0) + \left\{ \frac{d[g(\sigma)\mu(\sigma)]}{d\sigma} \Big|_{\sigma=0} \right\} \sigma + \text{h.o.t}] d\sigma \\ &= g(0)\mu(0)T + \frac{1}{2} \left\{ \frac{d[g(\sigma)\mu(\sigma)]}{d\sigma} \Big|_{\sigma=0} \right\} T^2 + \text{h.o.t} \end{aligned}$$

the result follows.

**Remark** Notice that the above lemma also holds valid when  $g(\cdot)$  is a map taking values in  $R^n$ .

Consider the PWM controlled system given by (2.1) and (2.2), described as an equivalent integral-difference equation, relating the values of the state vector  $x(t)$  at subsequent sampling instants:

$$x(t_k + T) = x(t_k) + \int_{t_k}^{t_k+T} f(x(\sigma))d\sigma + \sum_{i=1}^m \int_{t_k}^{t_k+\mu_i(t_k)T} g_i(x(\sigma))d\sigma \quad (2.10)$$

Similarly, if one rewrites (2.3) as the corresponding integral-difference equation, relating the state  $z(t)$  at times  $t_k$  and  $t_k+T$ , one obtains the following expression:

$$z(t_k + T) = z(t_k) + \int_{t_k}^{t_k+T} f(z(\sigma))d\sigma + \sum_{i=1}^m \int_{t_k}^{t_k+T} g_i(z(\sigma))\mu_i(\sigma)d\sigma \quad (2.11)$$

Under the assumption of identical initial states  $x(t_k)$  and  $z(t_k)$ , a comparison of (2.10) and (2.11) with use of Lemma 2.4 on the last integral of (2.10) reveals that the integral equation for the actual PWM controlled system only differs by a second order term in  $T$  from the integral equation describing the average PWM controlled model (2.11). It is easy to see that this statement remains valid whatever the length of time  $T$  or the number of sampling intervals under consideration. Hence, (2.10) is a *regular second order perturbation* of (2.11) in terms of the parameter  $T$ . It follows, from well known results in the theory of perturbations of integral equations (see Miller [15, pp. 273-285]), that as the sampling period  $T$  decreases to zero, the solution of the actual PWM controlled system *continuously converges* toward the solution of the average PWM model in a global manner (see also Tikhonov et al [16, pp. 180-185] for the same basic result in the context of differential equations). We have thus proved the following theorem.

#### Theorem 2.4

Under the assumption of identical initial conditions, the solutions of the actual PWM controlled system given by (2.1) and (2.2) globally and continuously converge toward the solutions of the average PWM system as the sampling frequency  $1/T$  tends to infinity.

**Remark:** According to the above result, the actual PWM controlled trajectories approximate arbitrarily close to the trajectories of the average PWM system as the sampling frequency gets higher and higher. This corresponds to the very same manner in which the actual sliding regimes approximate arbitrarily close the Ideal Sliding Dynamics, as the imperfections of the switching element are made to tend to become non-existent ([2]). The average model (2.3), hence, captures all the essential features of the actual PWM controlled system given by (2.1) and (2.2), provided a high sampling frequency is used for the Pulse Width Modulation of the actuator. An estimate of the required sampling frequency which guarantees a prespecified error bound between the actual and the average PWM responses was presented in [11].

Notice furthermore that the result of theorem 2.4 is completely independent of the manner in which the duty ratio function is synthesized. Whether the time function  $\mu(t)$

representing the duty ratio function is an open loop or a closed loop signal is immaterial to the approximating properties of the actual PWM controlled system trajectories with respect to those of the average PWM responses as T tends to zero.

### 2.3. A Stabilizing Feedback PWM Controller Design Scheme based on Average PWM Model Stabilization.

In this section we propose a general design procedure for the stabilization of an actual PWM controlled system such as the one given by (2.1) and (2.2). The procedure proposes to consider first the problem of stabilization of the average PWM system by means of an appropriate continuous multivariable duty ratio function specification as a feedback control law. For this task, any of the available nonlinear controller design methods (Lyapunov design, Approximate Linearization, Extended Linearization, Pseudo-linearization or Feedback Linearization) can be utilized. The multivariable duty ratio synthesis may be based on either output tracking error measurements or state feedback strategies. The stabilizing design for the average PWM system model is then translated into an actual (i.e., discontinuous) PWM controller design whose average closed loop model precisely coincides with the average closed loop controlled system. This discontinuous design procedure is essential whenever the available external control inputs are of the fixed amplitude type, such as in the case of ON-OFF systems (actuators delivering fixed positive and negative torques, as in some robotic manipulators or as in jet-controlled satellites), and a stabilizing continuous controller design is either well known or easy to derive.

Consider the PWM system (2.1) - (2.2), reduced to the form (2.4) - (2.5). Suppose, furthermore, that, without regard for the constrained nature of the control signals  $\mu_i(t)$  to the discrete set  $\{0,1\}$ , the following feedback control scheme is known stabilize the average PWM controlled system:

$$\frac{dz(t)}{dt} = f(z(t)) + \sum_{i=1}^m g_i(z(t))[\mu_i(t)]$$

$$y(t) = h(z(t))$$

$$e(t) = y_d(t) - y(t)$$

$$\frac{d\xi}{dt} = \zeta(\xi(t)) + \Gamma(\xi(t))e(t)$$

$$\mu(t) = \eta(\xi(t), e(t)) \quad (2.12)$$

The question would be now: how to translate this continuous stabilizing feedback controller design into a suitable ON-OFF controller of PWM nature which retains the stabilizing features of the given design and at the same time approximates arbitrarily close to the continuous responses of the closed loop system for arbitrary initial states?.

The components of  $\mu(t)$  must be properly limited between 0 and 1, by a limiting operation. This limiting process is essential for the PWM of the actuator to be such that the signal  $\mu(t)$  has a physically meaningful duty ratio specification. We consider  $\mu(t)$  as a *computed duty ratio function* and will denote it from now on as  $\bar{\mu}(t)$ . It is further assumed that the limiting operation on the given  $\mu(t)$  does not destroy the stability of the closed loop system (2.12). The control signal  $\mu(t)$  is obtained as the output of the dynamical compensator characterized by the q-dimensional state vector  $\xi(t)$ . Such a dynamical compensator is driven by the p-dimensional tracking error vector  $e(t)$ , obtained as the difference between the desired p-dimensional output response  $y_d(t)$  and the actual plant output signal  $y(t)$ . The vector  $\zeta(\cdot)$  and the column vectors  $\gamma_i(\cdot)$  ( $i = 1, 2, \dots, p$ ) of the matrix  $\Gamma(\cdot)$ , are considered to be smooth vector fields in their respective spaces. The output map of the compensator  $\eta(\xi, e)$  and that of the plant,  $h(z)$ , are assumed to be smooth maps of dimension m and p, respectively.

On the basis of the fundamental approximating properties of the actual PWM controlled responses with respect to the average PWM controlled trajectories, under sufficiently high sampling frequency assumptions, the feedback control scheme for the actual PWM system is easily obtained from (2.12) as (See Figure 1):

$$\frac{dx(t)}{dt} = f(x(t)) + \sum_{i=1}^m g_i(x(t))u_i(t)$$

$$y(t) = h(x(t))$$

$$e(t) = y_d(t) - y(t)$$

$$\frac{d\xi}{dt} = \zeta(\xi(t)) + \Gamma(\xi(t))e(t)$$

$$\bar{\mu}(t) = \eta(\xi(t), e(t))$$

$$u_i(t) = \text{PWM}[\mu(t)]\mu_i(t) = \inf\{1, \sup\{\bar{\mu}_i(t), 0\}\}; \quad i = 1, 2, \dots, m$$

$$u_i(t) = \text{PWM}[\mu_i(t)] = \begin{cases} 1 & \text{for } t_k \leq t < t_k + \mu_i(t_k)T \\ 0 & \text{for } t_k + \mu_i(t_k)T \leq t < t_k + T \end{cases}$$

$$t_{k+1} = t_k + T \tag{2.13}$$

Thus the design procedure consists of the following steps.

(1) Obtain the average model (2.3) for the PWM controlled system (2.1) - (2.2) by formally substituting the discontinuous control input to the original system by a convex combination, of the form (2.4), of the available feedback control laws. Alternatively, transform the original system to one in which the control variables take values in the discrete set  $\{0, 1\}$  as in (2.4) - (2.5). Such a transformation of the input space coordinates should reflect the fact that only two extreme feedback control inputs are available

(say  $u_i^+(x)$  and  $u_i^-(x)$ ;  $i = 1, 2, \dots, m$ ) on each control channel. Obtain then the average model by simply substituting the components of the vector of switch position function by the corresponding components of the multivariable duty ratio as in (2.6).

(2) Design a nonlinear (static or dynamic) stabilizing feedback control scheme, as it was done in (2.12), for the average PWM controlled system taking as input functions, the components of the duty ratio function vector. Such a design must be carried under the constraint of having each duty ratio function component limited between 0 and 1.

(3) Obtain the actual (discontinuous) PWM stabilizing feedback control in the form of (2.13) directly from the derived stable average PWM closed loop system (2.12). This entitles us to use sufficiently high sampling frequency for the PWM of the actuators. The reader is referred to [11], for developments leading to a (rather conservative) estimate of the sampling frequency in terms of Lipschitz constant of the drift vector field and a constant bound on the input channel matrix. Otherwise, one can always resort to extensive simulations (or trial and error) until a satisfactory PWM sampling frequency which produces a desired agreement between the actual and the average PWM controlled responses is found.

### 3. APPLICATIONS TO THE PWM CONTROL OF ROBOTIC MANIPULATORS

In this section we apply the design procedure outlined in the previous section to the PWM regulation of a two joint robotic manipulator. The pulsed control strategy will consist of a PWM scheme based on utilization of positive and negative pulsed torques to the joints (no zero torque is assumed to be available, but the above results and the control scheme can be easily extended to the ON-OFF-ON cases). We use an already existing stabilizing feedback control scheme presented by Baumann in [17] and we suitably modified it to interpret it as an average PWM closed loop system and thus obtain a duty ratio synthesis in a continuous feedback manner. From here, it is easy to obtain the actual PWM controlled system.

Consider the following state space model of a two link robotic manipulator [17] such as the one shown in Figure 2:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= \frac{1}{1+\sin^2 x_3} [x_4^2 \sin x_3 + 2x_2 x_4 \sin x_3 - g \cos(x_1 + x_3) - 2g \cos x_1] + \\ &\quad \frac{1+\cos x_3}{1+\sin^2 x_3} [x_2^2 \sin x_3 + g \cos(x_1 + x_3)] + \frac{1}{1+\sin^2 x_3} u_1 - \frac{1+\cos x_3}{1+\sin^2 x_3} u_2 \end{aligned}$$

$$\begin{aligned} \frac{dx_3}{dt} &= x_4 \\ \frac{dx_4}{dt} &= -\frac{1+\cos x_3}{1+\sin^2 x_3} [x_4^2 \sin x_3 + 2x_2 x_4 \sin x_3 - g \cos(x_1+x_3) - 2g \cos x_1] + \\ &\quad \frac{3+2\cos x_3}{1+\sin^2 x_3} [-x_2^2 \sin x_3 - g \cos(x_1+x_3)] - \frac{1+\cos x_3}{1+\sin^2 x_3} u_1 + \frac{3+2\cos x_3}{1+\sin^2 x_3} u_2 \end{aligned} \quad (3.1)$$

where,  $g$  is the acceleration due to the gravity,  $x_1$  and  $x_3$  are the angular positions of the two joints while  $x_2$  and  $x_4$  are the corresponding angular velocities and the  $u$ 's are the input torques. All masses and lengths are assumed to be unity.

A state-scheduled continuous multivariable stabilizing feedback controller design, based on Extended Linearization, is given by (See [17]):

$$u_1 = [2 \cos x_1 + \cos(x_1 + x_3)]g - 32(1+\cos x_3)x_2 - 8(1+\cos x_3)x_4 + 16(1+\cos x_3)(x_{1d} - x_1) + (21+22\cos x_3)(x_{2d} - x_3)$$

$$u_2 = g \cos(x_1 + x_3) - 32x_2 - 8x_4 + 16(x_{1d} - x_1) + (23 - \cos x_3)(x_{2d} - x_3) \quad (3.2)$$

where,  $x_{1d}$  and  $x_{2d}$  represent the desired angular positions of joints 1 and 2, respectively; they represent the reference inputs to the controller.

Let us assume that the control actions  $u_1$  and  $u_2$  are piecewise constant torques of magnitude  $\pm u_{1\max}$  and  $\pm u_{2\max}$  for each joint. The given continuous controller design (3.2) may still be utilized on an ON-OFF PWM switching strategy as suggested in the previous section. For this, one simply replaces the control inputs  $u_1$  and  $u_2$  in the system model (3.1) by

$$u_1 = u_{1\max}(2w_1 - 1); \quad u_2 = u_{2\max}(2w_2 - 1) \quad (3.3)$$

and considers the signals  $w_1$  and  $w_2$  (as switch position functions) constrained to the discrete set  $\{0,1\}$ . This reflects the fact that only torques of the form  $\pm u_{1\max}$  and  $\pm u_{2\max}$  are available from the actuators of each joint of the manipulator. The average model of such a PWM controlled system would be simply obtained by directly replacing the discontinuous signals  $w_1$  and  $w_2$  by the duty ratio functions  $\mu_1$  and  $\mu_2$ , respectively, as it was done in Lemma 2.3. The duty ratio functions can now be obtained from the continuous controller (3.2) as:

$$\mu_i = \inf \{1, \sup \{0, \bar{\mu}_i(t)\}\}; \quad i = 1, 2 \quad (3.4)$$

with:

$$\bar{\mu}_1 = 0.5 \{1 + [2 \cos x_1 + \cos(x_1 + x_3)]g - 32(1 + \cos x_3)x_2 - 8(1 + \cos x_3)x_4 + 16(1 + \cos x_3)(x_{1d} - x_1) + (21 + 22 \cos x_3)(x_{2d} - x_3)\}$$

$$\bar{\mu}_2 = 0.5 \{1 + g \cos(x_1 + x_3) - 32x_2 - 8x_4 + 16(x_{1d} - x_1) + (23 - \cos x_3)(x_{2d} - x_3)\} \quad (3.5)$$

The duty ratio functions are sampled at a sufficiently high rate and their sampled values generate a torque pulse width to be used during the next intersampling interval. The actual trajectories undergo discontinuities in the velocity variables while the position variables are smoothed out by the natural integration for the system dynamics.

The phase plane behavior, for each link, of the average PWM controlled manipulator is portrayed in Figures 3 and 4, respectively. The corresponding time responses for the average PWM regulated link angular positions  $x_1$  and  $x_3$ , converging to desired final positions ( $\frac{\pi}{2}$  rad and 0 rad, respectively), are depicted in Figures 5 and 6. The joint velocities of the average system are given in Figures 7 and 8. The frequency of the PWM of the actuator was set at 500 samples per second. The actual PWM controlled responses can be obtained from simulations on an analog computer. We tried to simulate the the actual system on a VAX 11/780 running UNIX v4.4 using the IMSL subroutine IVPBS which utilizes a sixth order Runge-Kutta integration routine. We found the simulation to be extremely sensitive to the integration step size employed in the IVPBS subroutine. It should be noted that the IVPBS integration routine expects the equation of the system to be continuous however our controller is discontinuous as a result of which round-off error builds up, for our system this resulted in about 3% error in the joint positions. It is our belief that better agreement between the responses of the actual and the average PWM controlled system can be obtained in real applications (or when simulations are carried out using an analog computer).

#### 4. CONCLUSIONS

A systematic method has been proposed for the design of multivariable nonlinear feedback regulators of discontinuous nature such as those needed in systems which include PWM of the actuators. The method is based on the feedback stabilization of a multiinput continuous average model easily derived from the original discontinuous system model in a manner similar to the Filippov average model of systems in sliding mode. The approximation characteristics of the responses of the actual discontinuous design, with respect to those of the average design, improve as the sampling frequency of the PWM actuator is increased. The results obtained in this article make available a wide variety of known continuous feedback design techniques for the case of PWM regulated systems. Indeed, if a stabilizing continuous feedback design is available, the corresponding approximating PWM controller can be synthesized by considering simple modifications of the continuous feedback control signals. The resulting feedback scheme is identified with the closed loop specification of the multivariable duty ratio

function for the average PWM model. The discontinuous PWM controller design is then easily derived from this straightforward identification process.

The theoretical results were applied to the PWM feedback stabilization of a two link robotic manipulator via a known state-scheduled controller design based on Extended Linearization. Simulations were performed on both the average and the actual PWM controlled system models. Round off errors of about 3% ,with respect to the average system, were encountered in the actual system response because of the discontinuous nature of the controlled system, response. These errors were due to the nature of the numerical integration schemes used in the simulations.

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## FIGURE CAPTIONS

- Figure 1. Actual Nonlinear PWM Feedback Control Scheme.
- Figure 2. Two Link Robotic Manipulator.
- Figure 3. Phase plane behavior of Link 1 for the Average PWM Controlled Manipulator.
- Figure 4. Phase plane behavior of Link 2 for the Average PWM Controlled Manipulator.
- Figure 5. Position of Link 1 for the Average PWM Controlled Manipulator.
- Figure 6. Position of Link 2 for the Average PWM Controlled Manipulator.
- Figure 7. Velocity of Link 1 for the Average PWM controlled Manipulator.
- Figure 8. Velocity of Link 2 for the Average PWM controlled Manipulator.

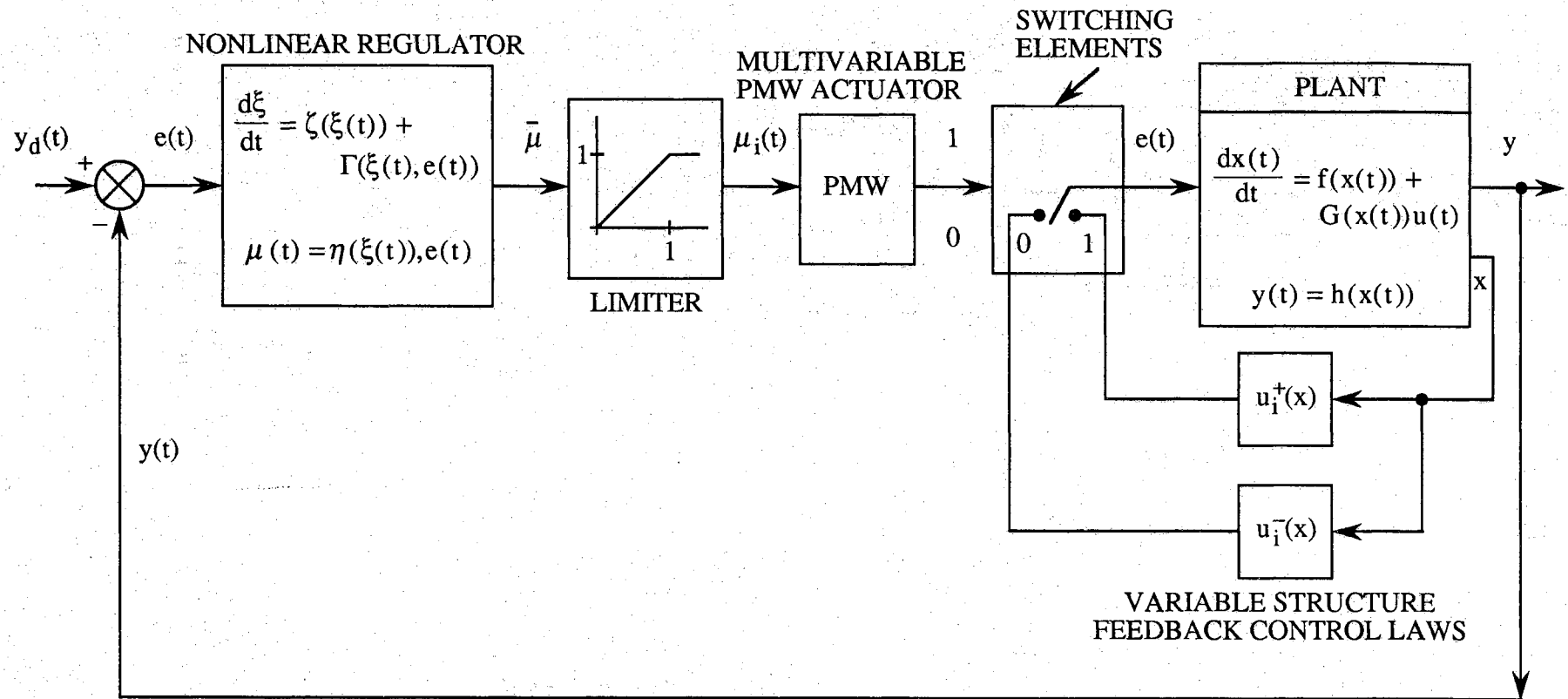


Figure 1. Actual Nonlinear PWM Feedback Control Scheme.

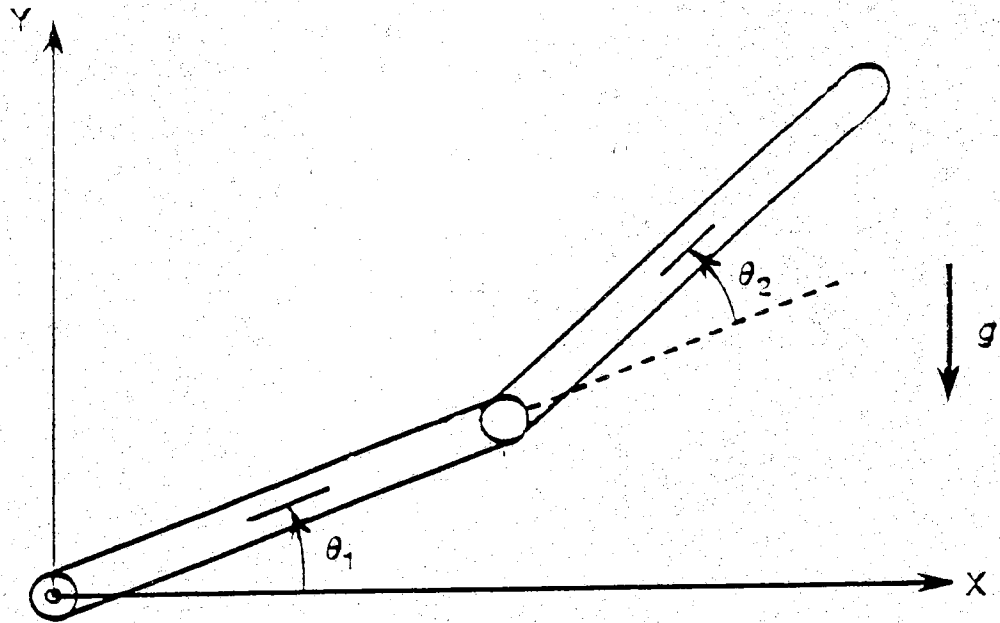


Figure 2. Two Link Robotic Manipulator.

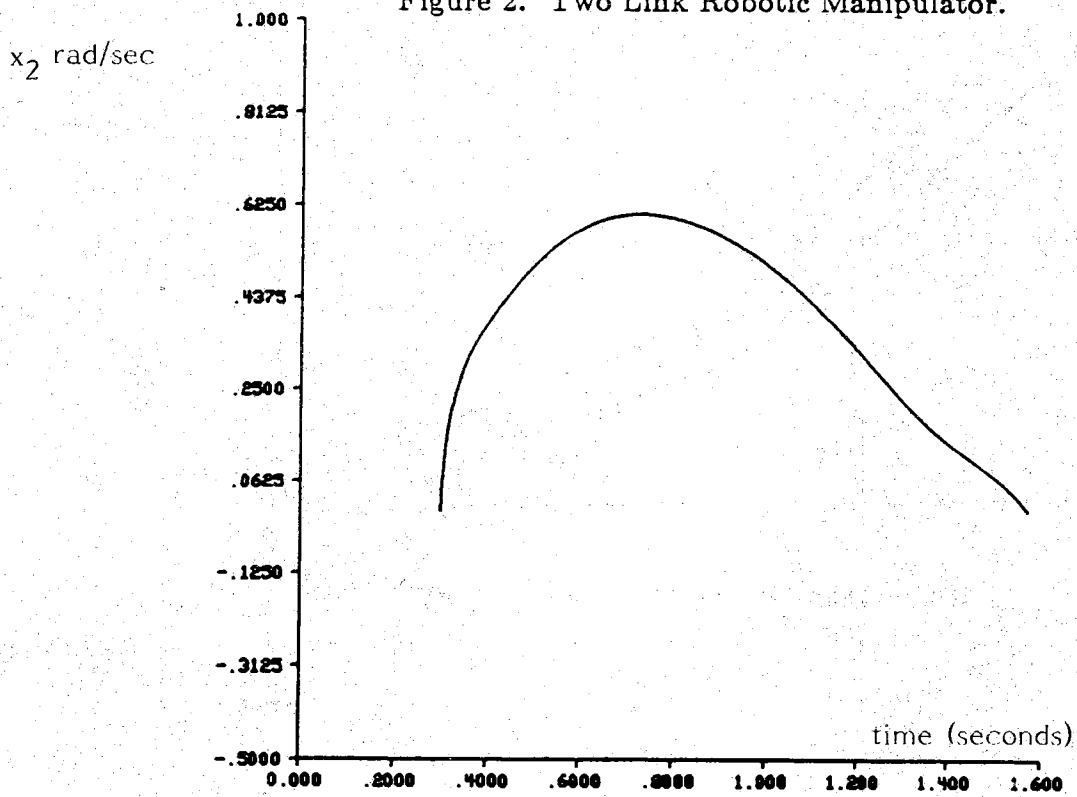


Figure 3. Phase plane behavior of Link 1 for the Average PWM Controlled Manipulator.

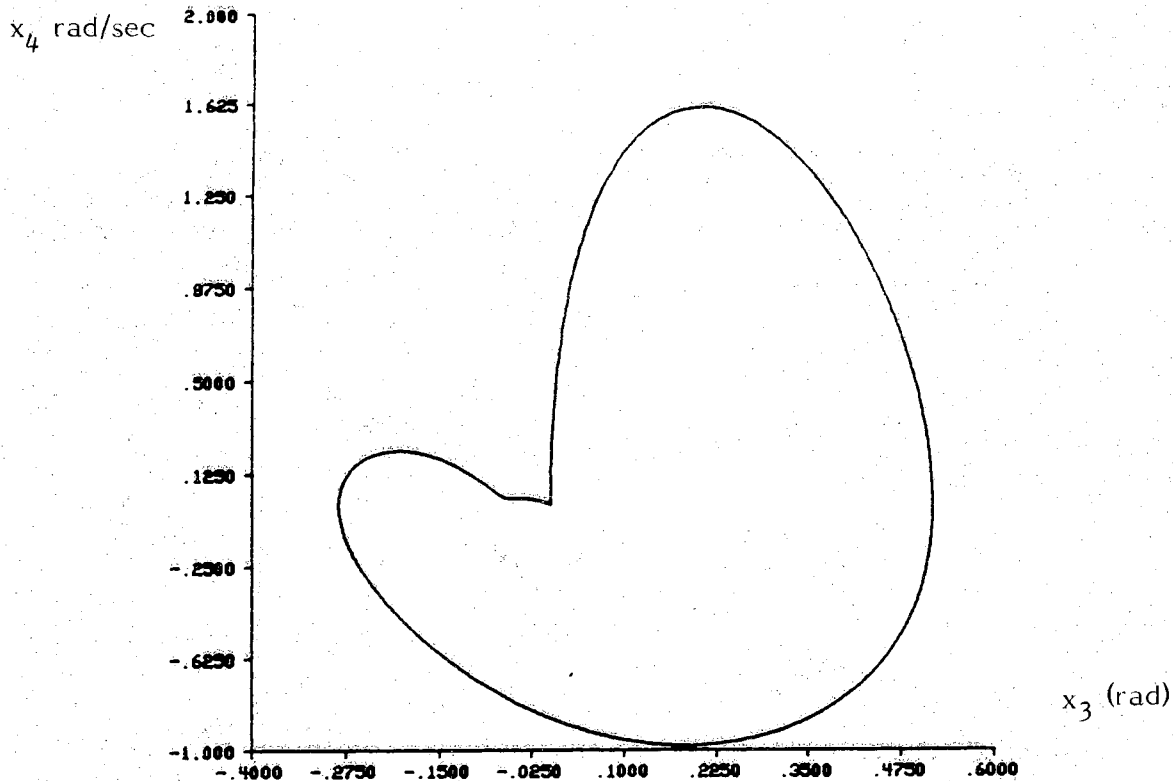


Figure 4. Phase plane behavior of Link 2 for the Average PWM Controlled Manipulator.

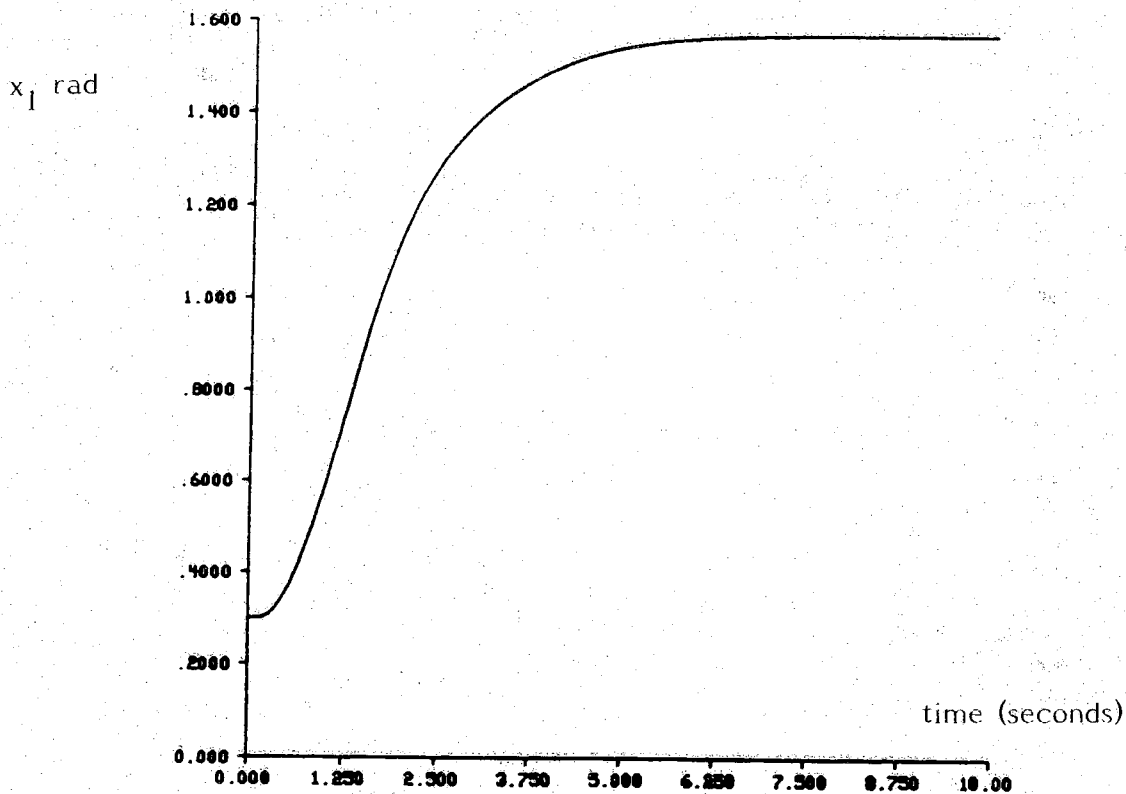


Figure 5. Position of Link 1 for the Average PWM Controlled Manipulator.

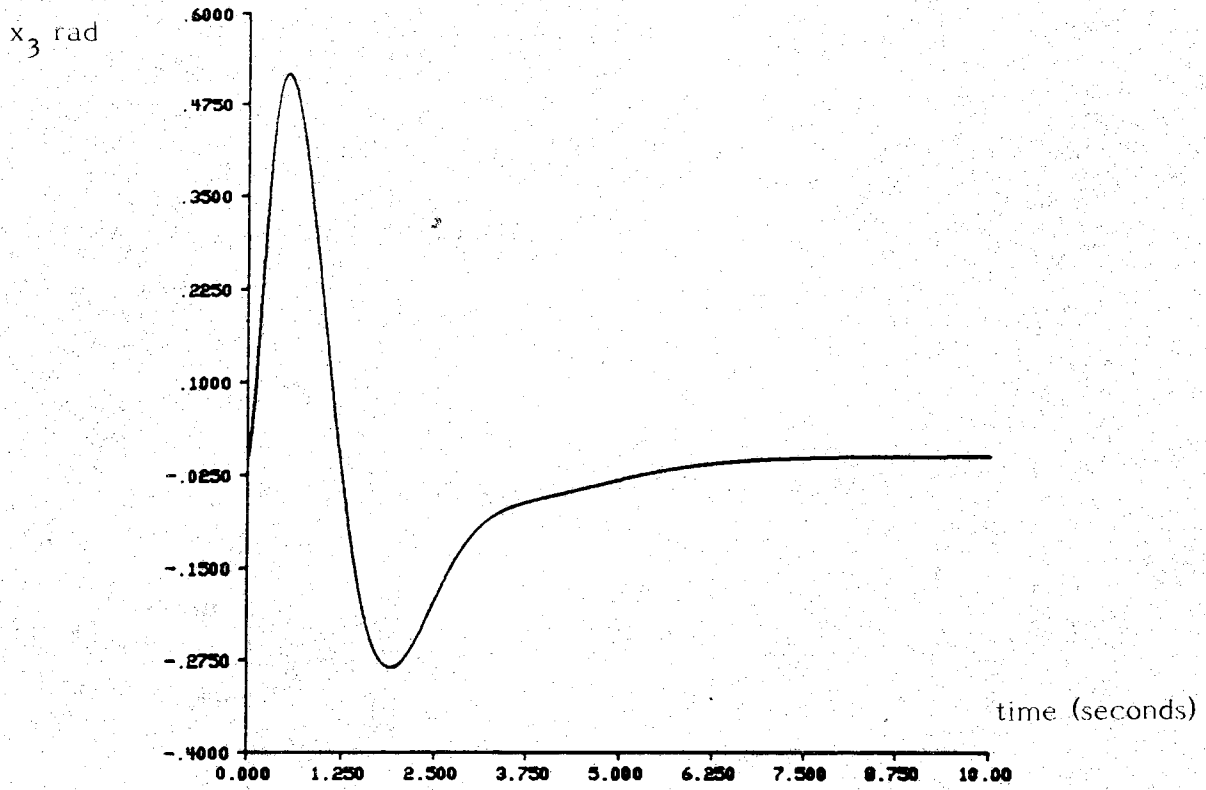


Figure 6. Position of Link 2 for the Average PWM Controlled Manipulator.

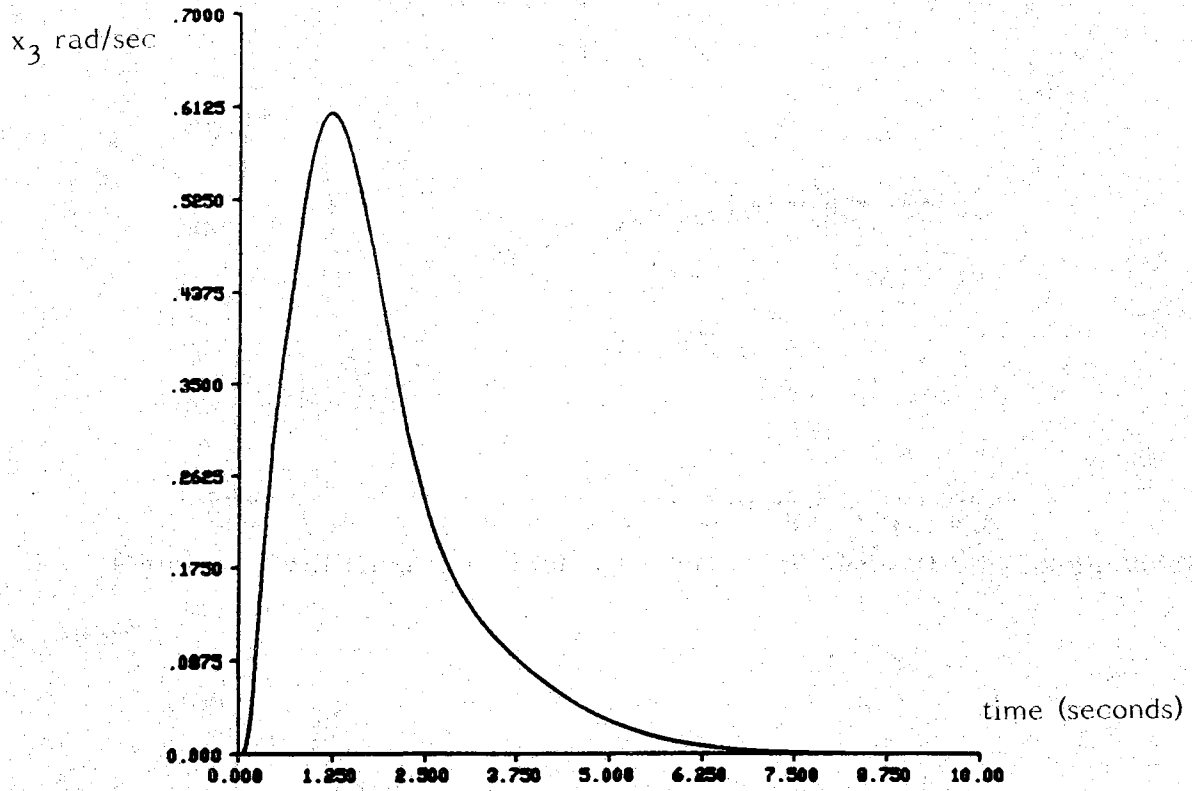


Figure 7. Velocity of Link 1 for the Average PWM controlled Manipulator.

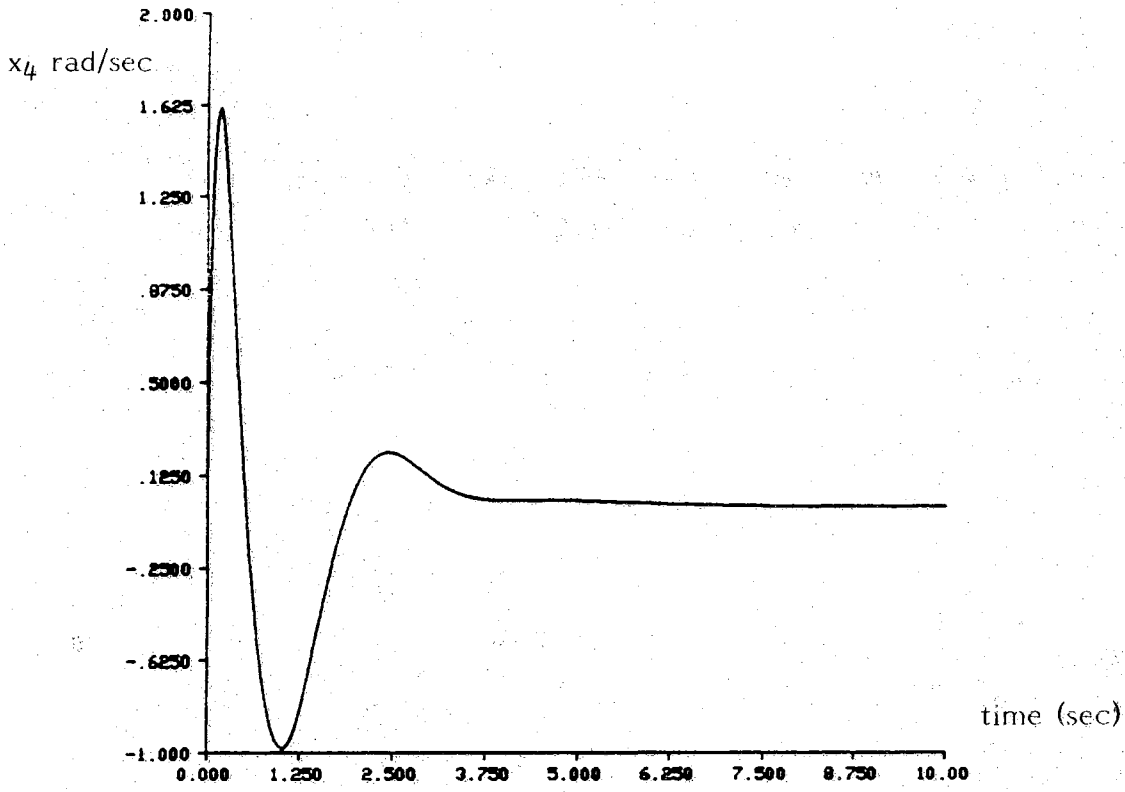


Figure 8. Velocity of Link 2 for the Average PWM controlled Manipulator.