Commun. math. Phys. 11, 214—220 (1969)

Pure Operations and Measurements*

K.-E. HELLWIG and K. KRAUS
Institut für Theoretische Physik der Universität Marburg

Received July 17, 1968

Abstract. States of a quantum system may be influenced by an external intervention. Following HAAG and KASTLER, such a transformation of states is called an operation, and is called pure if it transforms pure states into pure states. Operations are discussed here under the assumption that they are caused by interactions with another system (apparatus), described by a S matrix. Pure operations are then shown to correspond, with one exception, to operators A with norm smaller than one. The Hermitean operators $F = A^*A$ represent quantum effects as defined axiomatically by LUDWIG. The particular case of local operations in quantum field theory is also investigated.

In algebraic quantum theory as proposed by SEGAL [1] and applied to local field theory by HAAG and KASTLER [2], observables are represented by Hermitean elements H of an abstract C^*-algebra $\mathfrak{A}$ with unit element I. States are positive linear functionals φ over $\mathfrak{A}$ with $\varphi(I) = 1$, the expectation value of H in state φ being given by $\varphi(H)$. A state φ is called pure if it cannot be decomposed as $\varphi = \alpha\varphi_1 + (1 - \alpha)\varphi_2$ with $0 < \alpha < 1$ and $\varphi_1 \neq \varphi_2$.

The (Heisenberg) state φ changes if (and only if) the system is influenced externally. In Ref. [2] interventions of this type are called operations. They are called pure operations if they transform pure states into pure states. Pure operations are assumed to be in one-to-one correspondence with elements A of $\mathfrak{A}$ with $\|A\| \leqq 1$, the pure operation corresponding to A transforming a state φ into φ_A defined by $\varphi_A(H) = \frac{\varphi(A^*HA)}{\varphi(A^*A)}$. The quantity $\varphi(A^*A)$ represents the transition probability between states φ and φ_A, and therefore φ_A is defined only if $\varphi(A^*A) \neq 0$.

Any faithful $*$-representation R of $\mathfrak{A}$ by a concrete C^*-algebra $R(\mathfrak{A})$ of operators acting on a Hilbert space $\mathfrak{H}$ may be used to describe the system under question in the usual Hilbert space framework of quantum mechanics, different faithful $*$-representations being physically equivalent [2]. Let us furthermore assume that $\mathfrak{A}$ is primitive, i.e., as possessing an irreducible faithful $*$-representation.

Subsequently, a fixed irreducible faithful $*$-representation R of $\mathfrak{A}$ will be used throughout. We will use the letters $\mathfrak{A}$ for $R(\mathfrak{A})$ and X for $R(X)$, $X \in \mathfrak{A}$, i.e., $\mathfrak{A}$ will be identified with the concrete C^*-algebra $R(\mathfrak{A})$ on

* Supported in part by the Deutsche Forschungsgemeinschaft.

Hilbert space $\mathfrak{H}$. Of course, $R(I) = 1$ (unit operator in $\mathfrak{H}$) for such representations. Then it suffices [2] to consider, instead of arbitrary states φ, the particular subset of states given by $\varphi(X) = \mathrm{Tr}(X W)$ with an arbitrary density matrix W, i.e., a positive Hermitean operator W on $\mathfrak{H}$ with $\mathrm{Tr}\, W = 1$. Pure states in this subset are given by one-dimensional projections $W = P_f = |f\rangle\langle f|$ with $f \in \mathfrak{H}$, $\|f\| = 1$.

Consider now a second system, called apparatus, with state space $\mathfrak{H}'$, and let it interact with the first system, called object, during a finite time interval. Using a suitable interaction picture, the final result of this interaction may be described by a unitary operator $\boldsymbol{S}$ on the product space $\mathfrak{H} \otimes \mathfrak{H}'$. If object and apparatus before the interaction are in states W and W', respectively, the state of the combined system after the interaction becomes $\boldsymbol{S}(W \otimes W')\,\boldsymbol{S}^*$. Assume furthermore that, after the interaction, the observer looks for some property of the apparatus, described by a projection operator Q' on $\mathfrak{H}'$, and selects such instances for which Q' is true. According to the theory of quantum measurement (e.g., LUDWIG [3]) the state of object plus apparatus becomes

$$\boldsymbol{W} = \frac{\hat{\boldsymbol{W}}}{\mathrm{Tr}\,\hat{\boldsymbol{W}}} \quad \text{with} \quad \hat{\boldsymbol{W}} = (1 \otimes Q')\,\boldsymbol{S}(W \otimes W')\,\boldsymbol{S}^*(1 \otimes Q') . \tag{1}$$

If the object is considered separately, its state is

$$\tilde{W} = \mathrm{Tr}'\boldsymbol{W} = \frac{\hat{W}}{\mathrm{Tr}\,\hat{W}} \quad \text{with} \quad \hat{W} = \mathrm{Tr}'\hat{\boldsymbol{W}} . \tag{2}$$

Here Tr' denotes the trace with respect to $\mathfrak{H}'$, i.e., with arbitrary $f, g \in \mathfrak{H}$,

$$(f, \mathrm{Tr}'\boldsymbol{W} g) \underset{\text{def.}}{=} \sum_\nu ((f \otimes f'_\nu), \boldsymbol{W}(g \otimes f'_\nu)) \tag{3}$$

with a complete orthonormal system f'_ν in $\mathfrak{H}'$, which may be chosen arbitrarily. The described process thus transforms the object state W into $\tilde{W}$, and should be considered an operation in the sense of Ref. [2]. The transition probability is $\mathrm{Tr}\,\hat{W} = \mathrm{Tr}\,\hat{\boldsymbol{W}} = \mathrm{Tr}((1 \otimes Q')\,\boldsymbol{S}(W \otimes W')\,\boldsymbol{S}^*)$, because the latter expression represents the probability that the apparatus has the property Q' after the interaction.

The transformation $W \to \hat{W}$ may also be described directly in the object Hilbert space $\mathfrak{H}$. With the spectral decomposition $W' = \sum_i c_i P_{g'_i}$, $c_i > 0$, $\sum_i c_i = 1$, and a complete orthonormal system f'_k in the subspace $Q'\mathfrak{H}'$ of $\mathfrak{H}'$, define operators A_{ki} on $\mathfrak{H}$ by

$$(f, A_{ki} g) \underset{\text{def.}}{=} ((f \otimes f'_k), \boldsymbol{S}(g \otimes g'_i)) \text{ for arbitrary } f, g \in \mathfrak{H} . \tag{4}$$

This definition implies immediately

$$\|A_{ki}\| \leqq 1 , \tag{5}$$

and a straightforward calculation from (1) and (2) (cf. Appendix 1) yields

$$\hat{W} = \sum_{ki} c_i A_{ki} W A_{ki}^* . \tag{6}$$

Finally, $\operatorname{Tr} \hat{W} = \operatorname{Tr} \left(\sum_{ki} c_i A_{ki}^* A_{ki} W \right) \leqq 1$ for all W leads to

$$F \underset{\text{def.}}{=} \sum_{ki} c_i A_{ki}^* A_{ki} \leqq 1 . \tag{7}$$

The transition probability $\operatorname{Tr} \hat{W}$ is then given by $\operatorname{Tr}(F W)$ as well.

General operations of the type (6), with A_{ki} restricted by (5), (7), and perhaps some further conditions, will be the subject of future investigations. Here we will confine ourselves to the particular case of pure operations. In this case, Eq. (6) for $W = P_f = |f\rangle \langle f|$ with an arbitrary unit vector $f \in \mathfrak{H}$ must imply $\hat{W} = \lambda P_g$ with $g = g(f)$ and $\lambda \leqq 1$. Since

$$A_{ki} P_f A_{ki}^* = |A_{ki} f\rangle \langle A_{ki} f| ,$$

this holds if and only if

$$A_{ki} f = \alpha_{ki}(f)\, g(f) \tag{8}$$

with $g(f)$ independent of i and k.

Eq. (8) may be evaluated by means of the following

Lemma. *For a finite or infinite sequence of linear operators* $A_\nu \neq 0$ *with* $A_\nu f = \alpha_\nu(f)\, g(f)$ *there are two alternatives*:

(i) $A_\nu = \alpha_\nu B$ *with a fixed operator* B.

(ii) $A_\nu = |g\rangle \langle f_\nu|$ *with* g *fixed,* f_ν *arbitrary and not all* f_ν *parallel.*

This lemma is proved in Appendix 2. Thus according to Eq. (8), pure operations can be divided into two classes.

Pure operations of the first kind correspond to alternative (i) of the lemma, $A_{ki} = \alpha_{ki} B$. Eq. (7) implies $\sum_{ki} c_i |\alpha_{ki}|^2 B^* B \leqq 1$, and accordingly $A \underset{\text{def.}}{=} \left(\sum_{ki} c_i |\alpha_{ki}|^2 \right)^{1/2} B$ satisfies $\|A\| \leqq 1$. By (6) and (7),

$$\hat{W} = A W A^* \tag{9}$$

and

$$F = A^* A . \tag{10}$$

This case resembles the definition of pure operations given by HAAG and KASTLER.

Pure operations of the second kind, corresponding to alternative (ii) of the lemma, are described by $A_{ki} = |g\rangle \langle f_{ki}|$, with at least two linearly independent vectors f_{ki}. In Appendix 3 we shall give an example for W', Q', and S, which via (1) and (2) yield a pure operation of this type. Therefore, within our approach, pure operations of the second kind cannot be excluded a priori. They can not, however, occur in quantum field theory. As shown below, A_{ki} in this case has to belong to some local algebra of observables. Therefore, according to MISRA [4], it can not be a completely continuous operator like $|g\rangle \langle f_{ki}|$.

According to Eq. (9), pure operations of the first kind correspond to operators A with norm smaller than one. We shall show now that any such operator may be considered as describing a pure operation. More precisely, there is a Hilbert space $\mathfrak{H}'$, a density matrix W', a projection Q', and an operator $\boldsymbol{S}$, such that Eqs. (1) and (2) lead to (9). If there were no physical restrictions for the possible states and properties of an apparatus and its interactions with the object, then pure operations (9) with an arbitrary A actually could be performed. The proof of this assertion will be most simple if we assume a two-dimensional state space $\mathfrak{H}'$ and $W' = Q' = P_{g'_1}$ and construct a suitable unitary operator $\boldsymbol{S}$. Choose a second unit vector g'_2 in $\mathfrak{H}'$ orthogonal to g'_1. Then $\mathfrak{H} \otimes \mathfrak{H}'$ may be decomposed as $\mathfrak{H}_1 \oplus \mathfrak{H}_2$, with $\mathfrak{H}_1 = \mathfrak{H} \otimes g'_1$ and $\mathfrak{H}_2 = \mathfrak{H} \otimes g'_2$ both isomorphic to $\mathfrak{H}$, such that they may be canonically identified with $\mathfrak{H}$, leading to $\mathfrak{H} \otimes \mathfrak{H}' = \mathfrak{H} \oplus \mathfrak{H}$. The operator

$$\boldsymbol{S} = \begin{pmatrix} A & (1 - A\,A^*)^{1/2} \\ (1 - A^*\,A)^{1/2} & A^* \end{pmatrix} \tag{11}$$

in $\mathfrak{H} \oplus \mathfrak{H}$ is obviously a unitary one (RIESZ-NAGY [5]). Since g'_i and f'_k, as defined above, now coincide with g'_1, definition (4) with (11) yields one single operator $A_{11} = A$, because

$$(f, A_{11} g) = ((f \otimes g'_1), \boldsymbol{S}(g \otimes g'_1)) = (f, A g)$$

for all $f, g \in \mathfrak{H}$. Therefore (6) leads to (9), q.e.d.

At this point, some remarks on the operators F defined by (7) or, for a particular case, by (10), seem appropriate. An axiomatic approach to quantum theories different from the algebraic one has been developed by LUDWIG [6]. This approach starts from so-called effects, which are produced by the quantum objects on suitable macroscopic devices. The operator F, defined by (7), describes exactly such an effect, if in Eq. (1) W' is a realizable state and Q' a macroscopic property of a *macroscopic apparatus* [3], and if $\boldsymbol{S}$ corresponds to a *physical* interaction. If we do not take into account these physical restrictions on W', Q', and $\boldsymbol{S}$, as we have done so far, any Hermitean operator F with $0 \leqq F \leqq 1$ represents such an effect. Because then, according to the discussion above, $F^{1/2} = A$ will describe an operation, and Eq. (10) leads back to F. (Of course, the correspondence between effects and operations, or even pure operations, is not one-to-one: Many different operations lead to the same effect.) A more physical approach should take into account the above-mentioned physical restrictions on W', Q', and $\boldsymbol{S}$, and then should determine the classes of pure operations A and effects F which are not only mathematically possible, but actually realizable.

Some restrictions of this type, at least for S, may easily be formulated in algebraic quantum field theory, which assigns local algebras of observables $\mathfrak{A}_C$ to all finite space-time regions C. Particular cases are the so-called Haag fields, for which the $\mathfrak{A}_C$ are von Neumann algebras. A very natural requirement for a *local* operation, caused by interaction of an apparatus with the field in the region C, is the assumption

$$S \in \mathfrak{A}_C \otimes \mathfrak{L}(\mathfrak{H}') , \tag{12}$$

with $\mathfrak{L}(\mathfrak{H}')$ denoting the algebra of all bounded operators on $\mathfrak{H}'$. Consider an arbitrary operator $T \in \mathfrak{A}'_C$. Then $T \otimes 1$ commutes with S (DIXMIER [7]), and therefore (4) leads to $(f, A_{ki} T g) = ((f \otimes f'_k), S(Tg \otimes g'_i)) = ((f \otimes f'_k), S(T \otimes 1)(g \otimes g'_i)) = ((f \otimes f'_k), (T \otimes 1) S(g \otimes g'_i)) = (T^* f, A_{ki} g) = (f, T A_{ki} g)$. This implies $A_{ki} \in \mathfrak{A}''_C = \mathfrak{A}_C$, and (7) yields $F \in \mathfrak{A}_C$. With (12), therefore, local operations correspond to operators $A_{ki} \in \mathfrak{A}_C$, local pure operations to $A \in \mathfrak{A}_C$, and local effects to $F \in \mathfrak{A}_C$.

Moreover, for arbitrary $A \in \mathfrak{A}_C$ the operator S defined by (11) satisfies the requirement (12) [7]. Therefore any $A \in \mathfrak{A}_C$ with $\|A\| \leqq 1$ and any Hermitean $F \in \mathfrak{A}_C$ with $0 \leqq F \leqq 1$ may again be interpreted as describing a local pure operation and a local effect, respectively, if no other physical restrictions than (12) are imposed.

Acknowledgement. We are very grateful to Prof. R. HAAG for useful comments, and to Mr. M. SCHLUROFF for a critical reading of the manuscript. One of us (K.K.) wants to express his gratitude to Prof. H. J. BORCHERS for a stimulating discussion. Our cooperation was made possible by financial support from the Deutsche Forschungsgemeinschaft.

Appendix 1: Proof of Eq. (6)

By (1) and (3),

$$(f, \hat{W} g) = \sum_{\nu} ((f \otimes f'_\nu), (1 \otimes Q') S(W \otimes W') S^* (1 \otimes Q')(g \otimes f'_\nu)) . \tag{A 1}$$

Complete the orthonormal basis f'_k in $Q'\mathfrak{H}'$ to a basis f'_ν in $\mathfrak{H}'$, and use these f'_ν in (A 1). Likewise, complete the eigenvectors g'_i of W$'$ to a second basis g'_μ in $\mathfrak{H}'$. Choose any basis g_λ in $\mathfrak{H}$, and insert as intermediate states in (A 1) the basis $g_\lambda \otimes g'_\mu$ of $\mathfrak{H} \otimes \mathfrak{H}'$ at the appropriate places. This yields

$$\begin{aligned}(f, \hat{W} g) = &\sum_{\nu\lambda\mu\varrho\sigma} ((f \otimes f'_\nu), (1 \otimes Q') S(g_\lambda \otimes g'_\mu)) \\ &\cdot ((g_\lambda \otimes g'_\mu), (W \otimes W')(g_\varrho \otimes g'_\sigma))((g_\varrho \otimes g'_\sigma), S^*(1 \otimes Q')(g \otimes f'_\nu)) \\ = &\sum_{k\lambda\varrho i} ((f \otimes f'_k), S(g_\lambda \otimes g'_i)) c_i (g_\lambda, W g_\varrho)((g_\varrho \otimes g'_i), S^*(g \otimes f'_k)) \\ = &\sum_{ki\lambda\varrho} c_i (f, A_{ki} g_\lambda)(g_\lambda, W g_\varrho)(g_\varrho, A^*_{ki} g) ,\end{aligned}$$

the last step following from definition (4) of A_{ki}. The result is exactly Eq. (6).

Appendix 2: Proof of the Lemma

Lemma. *A finite or infinite sequence of bounded linear operators $A_\nu \neq 0$ on Hilbert space $\mathfrak{H}$ with $A_\nu f = \alpha_\nu(f)\, g(f)$ satisfies $A_\nu = \alpha_\nu B$ with a fixed operator B, or $A_\nu = |g\rangle\langle f_\nu|$ with arbitrary vectors f_ν and g.*
(Note that $A_\nu = |g\rangle\langle f_\nu|$ with $f_\nu = \alpha_\nu^* f$ may also be written $A_\nu = \alpha_\nu |g\rangle\langle f| = \alpha_\nu B$, which means that these possibilities are mutually exclusive only if some f_ν are linearly independent.)

Proof. Denote by $\mathfrak{N}_\nu$ the subspace of $\mathfrak{H}$ with $A_\nu f = 0$ for all $f \in \mathfrak{N}_\nu$, and by $\mathfrak{L}_\nu$ its orthogonal complement $\mathfrak{H} \ominus \mathfrak{N}_\nu$. $\mathfrak{L}_\nu$ is one-dimensional if and only if $A_\nu = |g_\nu\rangle\langle f_\nu|$. We prove the lemma by complete induction with respect to the number n of operators A_ν. For $n = 1$ the lemma is trivial. Suppose its validity for n operators $A_1 \ldots A_n$. Consider A_n and A_{n+1}.

(i) Let the dimension of $\mathfrak{L}_n$ be one, i.e., $A_n = |g\rangle\langle f_n|$, and assume the dimension of $\mathfrak{L}_{n+1}$ to be greater than one. Then there is at least one f orthogonal to f_n with $A_{n+1} f \neq 0$. Moreover, $A_{n+1} f$ parallel to g for all such f would imply $A_{n+1} = |g\rangle\langle f_{n+1}|$, since $A_{n+1} f_n$, too, must be parallel to $A_n f_n = \|f_n\|^2 g$. Therefore, a vector f orthogonal to f_n exists with $A_{n+1} f$ not parallel to g. Then $A_{n+1}(f + f_n)$ is not parallel to $A_n(f + f_n) = \|f_n\|^2 g$, which is a contradiction. The same conclusion is valid for n and $n + 1$ interchanged. Therefore, $\mathfrak{L}_n$ one-dimensional is equivalent to $\mathfrak{L}_{n+1}$ one-dimensional.

(ii) Let $\mathfrak{L}_n$ be one-dimensional, i.e., $A_n = |g\rangle\langle f_n|$. Then $A_{n+1} = |g_{n+1}\rangle\langle h_{n+1}|$, and $A_{n+1} f$ parallel to $A_n f$ obviously implies $g_{n+1} = \alpha g$, or $A_{n+1} = |g\rangle\langle f_{n+1}|$ with $f_{n+1} = \alpha^* h_{n+1}$.

(iii) Let $\mathfrak{L}_n$ be more than one-dimensional. Suppose there is a vector $f \in \mathfrak{N}_n$ with $A_{n+1} f = g \neq 0$. For $h \in \mathfrak{L}_n$, $A_n(h + f) = A_n h$ has to be proportional to $A_{n+1}(h + f) = A_{n+1} h + g$. If $A_{n+1} h \neq 0$, we have $A_{n+1} h = \beta A_n h$ with $\beta \neq 0$, and $A_n h$ turns out to be proportional to g. If $A_{n+1} h = 0$, the latter is obvious. This implies $A_n h$ parallel to g for all $h \in \mathfrak{L}_n$, and thus for all $h \in \mathfrak{H}$, i.e., $A_n = |g\rangle\langle f_n|$. This contradiction leads to $\mathfrak{N}_n \subseteq \mathfrak{N}_{n+1}$. Interchange of A_n and A_{n+1} yields the opposite inclusion, hence $\mathfrak{N}_n = \mathfrak{N}_{n+1} \underset{\text{def.}}{=} \mathfrak{N}$.

Take linearly independent f and g in $\mathfrak{L} \underset{\text{def.}}{=} \mathfrak{H} \ominus \mathfrak{N}$. Then $A_n f$ and $A_n g$ are linearly independent, too, and $A_{n+1} f = \alpha A_n f$, $A_{n+1} g = \alpha' A_n g$ with $\alpha, \alpha' \neq 0$. But $A_{n+1}(f + g) = \alpha \left(A_n f + \frac{\alpha'}{\alpha} A_n g\right)$ parallel to $A_n(f + g) = A_n f + A_n g$ implies $\alpha = \alpha'$. Thus $A_{n+1} f = \alpha A_n f$ for all $f \in \mathfrak{L}$, i.e., $A_{n+1} = \alpha A_n = \alpha \alpha_n B = \alpha_{n+1} B$. This completes the proof.

Appendix 3: An Example for Pure Operations of Second Kind

Consider state spaces $\mathfrak{H}$ of countably many, and $\mathfrak{H}'$ of three dimensions. With two different orthonormal systems f_1', f_2', f_3' and g_1', g_2', g_3' in $\mathfrak{H}'$, define $Q' = P_{f_1'} + P_{f_2'}$, $W' = c_1 P_{g_1'} + c_2 P_{g_2'}$, $P' = P_{g_1'} + P_{g_2'}$. Furthermore, choose three arbitrary unit vectors f_1, f_2, g in $\mathfrak{H}$ and a unitary two-by-two matrix u_{ki}, and define $\boldsymbol{f}_k = \sum_{i=1}^{2} u_{ki}(f_i \otimes g'_i)$, $k = 1, 2$. The vectors $\boldsymbol{f}_k$ are orthonormal and belong to the subspace $(1 \otimes P')(\mathfrak{H} \otimes \mathfrak{H}')$ of $\mathfrak{H} \otimes \mathfrak{H}'$.

Then the projection operators

$$\boldsymbol{P}_1 = P_{\boldsymbol{f}_1} + P_{\boldsymbol{f}_2}, \qquad \boldsymbol{P}_2 = (1 \otimes P') - \boldsymbol{P}_1, \qquad \boldsymbol{P}_3 = 1 \otimes (1 - P') = 1 \otimes P_{f'_3}$$

decompose $\mathfrak{H} \otimes \mathfrak{H}'$ into three orthogonal subspaces, the first being two-dimensional, the second and third of infinite dimension. The same holds true for

$$\tilde{\boldsymbol{P}}_1 = P_g \otimes Q', \quad \tilde{\boldsymbol{P}}_2 = 1 \otimes (1 - Q') = 1 \otimes P_{g'_3}, \quad \tilde{\boldsymbol{P}}_3 = (1 - P_g) \otimes Q'.$$

Consequently, there is a unitary operator $\boldsymbol{S}$ with $\boldsymbol{S}\boldsymbol{P}_\alpha \boldsymbol{S}^* = \tilde{\boldsymbol{P}}_\alpha$, $\alpha = 1, 2, 3$, which in addition may be assumed to satisfy $\boldsymbol{S}\boldsymbol{f}_k = g \otimes f'_k$, $k = 1, 2$. Then

$$(1 \otimes Q')\, \boldsymbol{S}\boldsymbol{P}_2 = (1 \otimes Q')\, \tilde{\boldsymbol{P}}_2 \boldsymbol{S} = (1 \otimes Q')\,(1 \otimes (1 - Q'))\, \boldsymbol{S} = 0\,,$$

$$\boldsymbol{S}\boldsymbol{P}_1 = \boldsymbol{S}\boldsymbol{P}_1\boldsymbol{P}_1 = \tilde{\boldsymbol{P}}_1\boldsymbol{S}\boldsymbol{P}_1 = (P_g \otimes Q')\, \boldsymbol{S}\boldsymbol{P}_1 = (1 \otimes Q')\, \boldsymbol{S}\boldsymbol{P}_1\,,$$

and therefore

$$(1 \otimes Q')\, \boldsymbol{S}(1 \otimes P') = (1 \otimes Q')\, \boldsymbol{S}(\boldsymbol{P}_1 + \boldsymbol{P}_2) = \boldsymbol{S}\boldsymbol{P}_1 = \boldsymbol{S} \sum_{j=1}^{2} |\boldsymbol{f}_j\rangle\langle \boldsymbol{f}_j|\,.$$

Thus, finally,

$$\begin{aligned}(f, A_{ki}h) &= ((f \otimes f'_k), \boldsymbol{S}(h \otimes g'_i)) = ((f \otimes f'_k), (1 \otimes Q')\, \boldsymbol{S}(1 \otimes P')\,(h \otimes g'_i)) \\ &= \sum_j ((f \otimes f'_k)\, \boldsymbol{S}\boldsymbol{f}_j)\,(\boldsymbol{f}_j, (h \otimes g'_i)) \\ &= \sum_j ((f \otimes f'_k), (g \otimes f'_j))\,(\boldsymbol{f}_j, (h \otimes g'_i)) \\ &= (f, g)\,(\boldsymbol{f}_k, (h \otimes g'_i)) = (f, g) \Big(\sum_l u_{kl}(f_l \otimes g'_l), (h \otimes g'_i)\Big) \\ &= (f, g)\,(u_{ki} f_i, h)\end{aligned}$$

or

$$A_{ki} = |g\rangle\langle f_{ki}| \quad \text{with} \quad f_{ki} = u_{ki} f_i\,.$$

If f_1 and f_2 are not parallel, these operators A_{ki} belong to a pure operation of the second kind.

References

1. SEGAL, I. E.: Ann. Math. **48**, 930 (1947).
2. HAAG, R., and D. KASTLER: J. Math. Phys. **5**, 848 (1964).
3. LUDWIG, G.: In: Werner Heisenberg und die Physik unserer Zeit, S. 150. Braunschweig: F. Vieweg 1961.
4. MISRA, B.: Helv. Phys. Acta **38**, 189 (1965).
5. RIESZ, F., and B. SZ.-NAGY: Functional analysis. Appendix of the 3rd Ed. New York: F. Ungar 1960.
6. LUDWIG, G.: Commun. Math. Phys. **9**, 1 (1968), and further work quoted there.
7. DIXMIER, J.: Les algèbres d'opérateurs dans l'espace Hilbertien (algèbres de von Neumann), Chap. I, § 2, No. 4. Paris: Gauthier-Villars 1957.

K. KRAUS
Inst. f. Theoret. Physik der Universität
3550 Marburg (Lahn), Renthof 7