Pure-Radiation News Function in General Relativity

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The Bondi-Sachs analysis of empty-space gravitational fields is extended to include the presence of a pure-radiation stress-energy tensor, $T_{\mu\nu}=\sigma k_\mu k_\nu$. A pure-radiation news function is defined and shown to contribute to the change of the mass aspect in a way analogous to the gravitational news functions. It is suggested that this type of analysis may prove useful in the study of bodies simultaneously emitting high-frequency incoherent and low-frequency coherent gravitational radiation.

IT has been noted by Rao[1] that, starting from any empty-space solution to the gravitational field equations for a cylindrically symmetric field with both states of polarization present,[2] it is always possible to construct a class of solutions to the nonempty-space equations for a stress-energy tensor of the form $\sigma k_\mu k_\nu$, where k_μ is the null propagation vector for the gravitational radiation and σ is of the form $f(u)/\rho$, where $f(u)$ is an arbitrary function of the retarded time $u=t-\rho$. This shows the typical cylindrical $1/\rho$ falloff on a null hypersurface $u=$const. We shall show that this result can be carried over, with appropriate modifications, to the more realistic asymptotically spherical radiation fields discussed by Bondi, van der Burg, and Metzner[3] and by Sachs.[4]

Suppose we have a metric meeting the appropriate boundary and coordinate conditions at infinity, so that it may be analyzed by the method of Bondi *et al.* and Sachs. Then the metric may be put into the form[4]

$$ds^2=(Ve^{2\beta}/r)du^2-2e^{2\beta}dudr+r^2h_{AB}(dx^A-U^Adu)(dx^B-U^Bdu)\,, \quad (1)$$

with

$$2h_{AB}dx^Adx^B=(e^{2\gamma}+e^{2\delta})d\theta^2+4\sin\theta\sinh(\gamma-\delta)d\theta d\varphi+(\sin\theta)^2(e^{-2\gamma}+e^{-2\delta})d\varphi^2. \quad (2)$$

Suppose we modify the empty-space field equations $R_{\mu\nu}=0$, discussed by Bondi *et al.*[3] and Sachs,[4] by the introduction of a right-hand side representing the stress-energy tensor of pure radiation[5]:

$$R_{\mu\nu}=-\sigma k_\mu k_\nu. \quad (3)$$

Here $k_\mu=u_{,\mu}$, where u is the null coordinate of the Bondi-Sachs analysis. We introduce a null tetrad[4] with k_μ as one of its real null vectors, m_μ as the other, and t_μ the complex null vector representing two orthogonal spacelike directions (given k_μ, the others are not uniquely fixed, of course). The relations among the tetrad vectors are given by

$$k_\mu k^\mu=m_\mu m^\mu=t_\mu t^\mu=\bar{t}_\mu\bar{t}^\mu=t_\mu m^\mu=k_\mu t^\mu=0\,,\quad k_\mu m^\mu=t_\mu\bar{t}^\mu=1\,. \quad (4)$$

Since only m^μ has a nonzero projection onto k_ν, we see that the only nonvanishing projection of Eq. (3) onto the tetrad will be

$$R_{\mu\nu}m^\mu m^\nu=-\sigma\,, \quad (5)$$

one of the supplementary conditions.[3,4] But, as pointed out by Sachs,[4] if all the field equations except $R_{\mu\nu}m^\mu m^\nu$ vanish, it follows from the Bianchi identities that

$$\partial(r^2R_{\mu\nu}m^\mu m^\nu)/\partial r=0. \quad (6)$$

This means that we can proceed in solving the field equations just as in the empty-space case,[3,4] integrating the four hypersurface equations and the two standard equations by specification of the initial value of $f(r,\theta,\varphi)=\frac{1}{2}[(\gamma+i\delta)(1-i)]_{u=u_0}$ and introduction of the mass and dipole aspect functions $M(u_0,\theta,\varphi)$, $N(u_0,\theta,\varphi)$ (N combines two real functions into one complex one) and the two gravitational news functions, which are also combined into one complex function $\partial c(u,\theta,\varphi)/\partial u$. It is only the final step of the procedure—the solution of the supplementary conditions—that presents any difference from the empty-space case. Integration of Eq. (6) shows that

$$R_{\mu\nu}m^\mu m^\nu=-h(u,\theta,\varphi)/r^2\,, \quad (7)$$

so that the form of σ must be $\sigma=h(u,\theta,\varphi)/r^2$, showing the $1/r^2$ falloff typical of spherical radiation. We are thus led to introduce a new arbitrary function $h(u,\theta,\varphi)$ by the integration of the supplementary condition (5). Since this function enters into the determination of thy evolution of the mass aspect $\partial M(u,\theta,\varphi)/\partial u$ in a wae analogous to the square of the news function,[4]

$$\partial M/\partial u=-|\partial c/\partial u|^2+\tfrac{1}{2}(\partial/\partial u)\times\{(\sin\theta)^{-1}R[\bar{\Delta}(\Psi\sin\theta)]\}-h(u,\theta,\varphi)\,, \quad (8)$$

it seems appropriate to call $h(u,\theta,\varphi)$ the pure-radiation news function. If the mass is to decrease as retarded

[1] J. K. Rao, Sardar Proc. Nat. Inst. Sci. India (to appear).
[2] J. J. Stachel [J. Math. Phys. **7**, 1321 (1966)] presents a discussion of such fields with earlier references.
[3] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. (London) **A269**, 21 (1962).
[4] R. K. Sachs, Proc. Roy. Soc. (London) **A270**, 103 (1962). We follow the notation of this paper.
[5] R. C. Tolman, P. Ehrenfest, and B. Podolsky, Phys. Rev. **37**, 602 (1931); A. Lichnerowicz, Ann. Matematica Pura Applicata **50**, 1 (1960). By "pure radiation" we always mean a directed flow of radiation, which may be composed of one coherent null radiation field, or the sum of a number of mutually incoherent wave packets ("quanta") of radiation which may be characterized by one null propagation vector k_μ. This is to be contrasted with disordered, nonflowing black-body radiation. See R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, New York, 1934), Chap. VIII.

time increases, clearly h must be chosen as a non-negative function. Since the mass aspect function enters into the remaining two supplementary conditions which determine the evolution of the dipole aspect functions $N(u,\theta,\varphi)$, the pure-radiation news function contributes indirectly to the evolution of the dipole aspect as well. One case of the above analysis has been well known for some time. This is Vaidya's radiating Schwarzschild metric,[6] where no gravitational radiation is emitted (as is to be expected for a spherically symmetric metric), but all the mass loss is carried away by the pure-radiation stress-energy tensor.

In general, the pure-radiation stress-energy tensor will not correspond to the coherent radiation field of any source, but rather represents an incoherent radiation flow of any null rest-mass type. However, as Robinson has shown,[7] if k_μ represents a shear-free geodesic congruence, the stress-energy tensor may be regarded as generated by a (coherent) null electromagnetic field. The result of Goldberg and Sachs[8] shows that such a null electromagnetic field can only be associated with an algebraically special gravitational field. So, for any realistic model of a radiating system (which will have a gravitational field of type I), the pure radiation field must be of the incoherent radiation type. Since the pure-radiation news function enters the field equations via the pure-radiation stress-energy tensor, which is already a quadratic function of the field variables or, more generally, an average over the sum of a number of stress-energy tensors which are such quadratic functions of mutually incoherent fields, it is not surprising that h enters linearly, and not quadratically, into Eq. (8). The positive definiteness of the energy for the fields involved should assure that h is a non-negative function.

Isaacson[9] has shown how the pure-radiation stress-energy tensor may be interpreted as the approximate effect of high-frequency gravitational radiation propagating in the background metric that it helps to create via Eq. (3). Since the high-frequency, incoherent, gravitational-bremsstrahlung radiation due to collisions within an astrophysical object may be expected under certain circumstances to be equal in importance or more important than the longer-wavelength coherent gravitational radiation associated with the oscillations of the body as a whole,[10,11] the method outlined in this paper might prove useful in the approximate treatment of both types of radiation by the Bondi-Sachs techniques. In this approximation, in the region far enough outside the sources for the Bondi-Sachs approach to apply, the mass loss due to incoherent high-frequency gravitational radiation simply adds on to the mass loss due to coherent gravitational radiation. The former may usually be computed to sufficient accuracy by treating the object as a spherically symmetric collection of particles, while our results show that the latter may be computed by considering only the coherent nonspherically symmetric motion of the body as a whole.

It appears likely that, by taking gravitational news functions which vary stochastically around some smooth functions, one could derive both the coherent news functions and the pure-radiation news function by taking appropriate averages. Even better approximations to the simultaneous treatment of the coherent and incoherent radiation might be possible with this approach. We have carried out such a treatment for the Vaidya metric, and hope to return to this problem.

If one starts from a given empty-space solution to the field equations, it must be remembered that the family of null hypersurfaces $u=\text{const}$ is not uniquely determined by the Bondi-Sachs conditions, but is fixed up to transformations generated by the Bondi-Metzner-Sachs group.[4] Thus, one has a choice, within the elements of this group, along which null hypersurface of the empty-space solution one wishes to propagate the pure radiation. Once this choice is made, of course, Eq. (3) singles out a preferred family of null hypersurfaces.

[6] P. C. Vaidya, Nature **171**, 260 (1953). See J. Plebanski and J. Stachel [J. Math. Phys. **9**, 269 (1968)] for derivation and further references.

[7] I. Robinson, J. Math. Phys. **2**, 290 (1961).

[8] J. N. Goldberg and R. K. Sachs, Acta Physiol. Polon. **22**, 13 (1962).

[9] R. Isaacson, Phys. Rev. **166**, 1263 (1968); **166**, 1272 (1968).

[10] S. Weinberg, Phys. Rev. **140**, B516 (1965).

[11] M. Carmeli, Phys. Rev. **158**, 1243 (1967).