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PURITY OF THE STRATIFICATION BY NEWTON POLYGONS

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Contents

1. Introduction

In this paper we prove four theorems: one on surface singularities, two on F-crystals, and one on moduli of p-divisible groups. The reason we put together these results in one paper is that the proofs, as given here, show how these theorems are related. Let us first describe our results.

Let $(S,0)$ be a normal surface singularity over an algebraically closed field of characteristic p. Let $\tilde{S} \to S$ be a resolution of singularities. Our first result is Theorem 3.2:

(1) *Any $\mathbb{Q}_p$-cohomology class on the link of the singularity extends to the resolution, more precisely*

$$H^1_{\acute{e}t}(S \setminus \{0\}, \mathbb{Q}_p) \cong H^1_{\acute{e}t}(\tilde{S}, \mathbb{Q}_p).$$

The interest in this lies in the fact that we can work with $\mathbb{Q}_p$-coefficients in characteristic p. The formula can be seen as a weak form of purity with $\mathbb{Q}_p$-coefficients; when we consider $\mathbb{Q}_\ell$-coefficients the same result holds and follows in a straightforward manner from purity for étale cohomology. The result also holds for singularities of mixed characteristic; see Remark 3.13.

For the following two statements, let S be a scheme of characteristic $p > 0$ and let $(\mathcal{E}, F)$ be a nondegenerate F-crystal over S.

For any geometric point $\bar{s}$ of S we can define the Newton polygon of $(\mathcal{E}, F)$ at $\bar{s}$; see 2.12. We say that $(\mathcal{E}, F)$ is *isoclinic* if all slopes of the Newton polygon are equal, in all points of S. Our investigations show that the following working

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hypothesis is reasonable: if $\pi_1(S) = \{1\}$, then an isoclinic F-crystal $\mathcal{E}$ is isogenous to a constant F-crystal. We prove a result in this direction when $S = \operatorname{Spec} A$, and A is a complete local Noetherian ring with algebraically closed residue field k. See Proposition 2.15 and Remark 2.18. This result implies the Isogeny Theorem 2.17 for p-divisible groups:

(2) *If G over A is a p-divisible group, G is isoclinic, and A is normal, then G is isogenous to a constant p-divisible group $G_0 \times \operatorname{Spec} A$.*

Grothendieck showed that under a specialization $s \to s_0$ the Newton polygon of $\mathcal{E}$ at s_0 lies on or above the Newton polygon at s. Katz analyzed and completed this result by proving that the locus where the Newton polygon is bounded from below is closed in S. Our central result, which we call the Purity Theorem 4.1, says:

(3) *If the Newton polygon jumps somewhere, then it jumps already in codimension one.*

As an application of (2) and (3) we study a problem on local moduli of simple p-divisible groups: describe all deformations which do not change the Newton polygon. By the Isogeny Theorem (2) we know that local deformations are obtained by isogenies. Because of this, and because a global moduli space does not exist, we look at the moduli space T of isogenies $\varphi : H \to X$, where H is a fixed simple p-divisible group and where φ has a fixed degree; see Subsection 5.9. We prove that T is a *catalogue*: a classifying space which contains all objects that we want to consider over an algebraically closed field (Proposition 5.10). Our main theorem in Section 5 is that T is irreducible. We obtain two corollaries:

(4a) *A simple local-local p-divisible group can be deformed into a p-divisible group with the same Newton polygon and with $a = 1$ (Corollary 5.12).*

(4b) *Grothendieck's conjecture holds for a simple local-local p-divisible group (Corollary 5.13).*

We quickly recall Grothendieck's conjecture alluded to above; see [12, page 150]. Let X be a p-divisible group and let D denote its formal deformation space. Let δ be the Newton polygon of the covariant Dieudonné module of X. Let β be a Newton polygon, having the same endpoints as δ. If β occurs in the family over D, then β lies on or below δ; see discussion preceding (3). Grothendieck's conjecture is that the converse holds: if β lies on or below δ, then $U_\beta = \{d \in D$ where the Newton polygon is $\beta\}$ is not empty. We would like to understand better the stratification $D = \coprod U_\beta$ of D. For example: what is the dimension of U_β? Our result (3) gives an estimate. If $U_\beta \neq \emptyset$, then

$$\dim(U_\beta) \geq \dim(D) - c,$$

where

$$c = \text{maximum length } k \text{ of a chain,}$$
$$\rho = \beta_0 \succ \beta_1 \succ \cdots \succ \beta_k = \beta,\ \beta_i \neq \beta_{i+1}.$$

The symbol ρ is the generic Newton polygon (see Lemma 5.15) and the relation $\beta_i \succ \beta_{i+1}$ means β_{i+1} lies on or above β_i.

Generally speaking the Purity Theorem can be used to bound codimensions of Newton polygon strata. This can be applied to the F-crystal defined by the ith crystalline cohomology of a family of varieties over S; for example, the universal family of hypersurfaces of a given degree in $\mathbb{P}^n$.

The Purity Theorem also gives the correct lower bound for the dimension of the Newton polygon strata in the moduli spaces of principally polarized abelian varieties $A_{g,1,n} \otimes \mathbb{F}_p$ ($n \geq 3$, $p \nmid n$). Li and Oort [18] have proven that the supersingular stratum $S_g \subset A_{g,1,n} \otimes \mathbb{F}_p$ has the correct dimension $[g^2/4]$. It follows that any irreducible component of a Newton polygon stratum $U_\beta \subset A_{g,1,n} \otimes \mathbb{F}_p$ whose closure meets S_g has the correct dimension (apply the Purity Theorem to bound the codimension of S_g in $\overline{U_\beta}$). For more information on this, compare [29].

We turn to a discussion of the proof of the Purity Theorem. Let us analyze the situation where S is normal, local, two-dimensional with closed point 0, and where $\mathcal{E}$ has constant Newton polygon over $S \setminus \{0\}$. We consider the smallest slope λ which occurs in the generic fiber. The part of $\mathcal{E}|_{S\setminus\{0\}}$ of slope λ gives us an isoclinic F-crystal $\mathcal{E}'$ over $S \setminus \{0\}$, which is determined by a certain monodromy representation of $\pi_1(S \setminus \{0\})$. By taking a suitable exterior power we may assume that $\mathcal{E}'$ has rank 1 and that the representation is abelian. Here the result (1) applies and hence $\mathcal{E}'$ extends to $\mathcal{F}$ over $\tilde{S}$. We apply the extension theorem on homomorphisms of F-crystals [8] to obtain a nonzero map $\mathcal{F}|_{0'} \to \mathcal{E}|_{0'}$ for some point $0'$ of $\tilde{S}$ lying over 0. From this we conclude that λ occurs in the Newton polygon of $\mathcal{E}$ over 0 as desired.

The proof of (1) has four ingredients. We use algebraization [2] to make the singularity algebraic. By performing an alteration [7], we may assume our singularity is obtained by contracting a connected union of irreducible components E of the special fiber of a family of stable curves $X \to \operatorname{Spec} k[t]$. Next, patching à la Harbater and Raynaud produces a global class over $X \setminus E$ from a class on the link of the singularity. The final step is to use [8] to extend a class in the 1st étale cohomology group of the generic fiber of X to a class over X. We can also deal with the mixed characteristic case; here we use the result of Tate [35] on extensions of homomorphisms of Barsotti-Tate groups (see Remark 3.13).

2. Results on F-crystals

2.1. Conventions. In this section S denotes a connected scheme of characteristic p. We use the term *crystal* to mean a crystal of finite locally free $\mathcal{O}_{cris}$-modules. See [3, page 226]. Here $\mathcal{O}_{cris}$ denotes the structure sheaf on the category $\mathrm{CRIS}(S/\operatorname{Spec}\mathbb{Z}_p)$ (big crystalline site of S). If $T \to S$ is a morphism, then we use $\mathcal{E}|_T$ to denote the pullback of $\mathcal{E}$ to $\mathrm{CRIS}(T/\operatorname{Spec}\mathbb{Z}_p)$. For a crystal $\mathcal{E}$, we denote by $\mathcal{E}^{(n)}$ the pullback of $\mathcal{E}$ by the nth iterate of the Frobenius endomorphism of S. An *F-crystal over S* is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a crystal over S and $F : \mathcal{E}^{(1)} \to \mathcal{E}$ is a morphism of crystals. We usually denote an F-crystal by $\mathcal{E}$, the map F being understood. Recall that $\mathcal{E}$ is a *nondegenerate F-crystal* if the kernel and cokernel of F are annihilated by some power of p. See [32, 3.1.1]. All F-crystals in this paper will be nondegenerate.

Let $\lambda \in \mathbb{Q}_{\geq 0}$ be a nonnegative rational integer. Write $\lambda = b/a$ in lowest terms. We will say that an F-crystal $\mathcal{E}$ is *divisible by* λ, if there exists a morphism of F-crystals $\varphi : \mathcal{E}^{(a)} \to \mathcal{E}$ such that $F^a = p^{a\lambda}\varphi$.

A perfect scheme S in characteristic p is a scheme such that the Frobenius map $(-)^p : \mathcal{O}_S \to \mathcal{O}_S$ is an isomorphism. A crystal over a perfect scheme S is simply given by a finite locally free sheaf of $W(\mathcal{O}_S)$-modules.

2.2. Definition of isoclinic F-crystals. Let k be an algebraically closed field of characteristic p. Let W be the Witt ring of k, with Frobenius map σ. Recall that a nondegenerate F-crystal over $\operatorname{Spec} k$ corresponds to a finite free W-module M with a σ-linear endomorphism $F : M \to M$ such that $M/F(M)$ has finite length. We briefly describe the Newton polygon attached to (M, F); see [16, (1.3)]. Let L be the fraction field of W. The vector space $M \otimes L$ has a slope decomposition $M \otimes L = \bigoplus_{\lambda \in \mathbb{Q}} V_\lambda$. This decomposition is characterized by the following properties: (1) it is F-stable, (2) each V_λ has a basis of elements e such that $F^n(e) = p^{n\lambda} e$ for n sufficiently divisible. The slope λ occurs in the Newton polygon of (M, F) with multiplicity $\dim V_\lambda$, and they are organized in increasing order. We say that (M, F) is *isoclinic of slope* λ if $V_{\lambda'} = 0$ for $\lambda' \neq \lambda$.

Let S be a scheme of characteristic p. Let $\mathcal{E}$ be a nondegenerate F-crystal on S. We say that $\mathcal{E}$ is *isoclinic of slope* λ over S if the pullback of $\mathcal{E}$ to every geometric point of S is isoclinic of slope λ.

2.3. The standard F-crystal of slope λ. Let $\lambda \in \mathbb{Q}_{\geq 0}$ be a nonnegative rational number. Write $\lambda = b/a$ in lowest terms (note $0 = 0/1$). We define the *standard F-isocrystal $\mathcal{E}_\lambda$ of slope* λ over $\operatorname{Spec} \mathbb{F}_p$ to be the crystal $\mathcal{O}^a_{cris}$ with basis $\{e_1, \ldots, e_a\}$, and with F acting by the formulae: $F(e_i) = e_{i+1}$, for $i = 1, \ldots, a-1$, and $F(e_a) = p^{a\lambda} e_1$. Note that $\mathcal{E}_\lambda$ is divisible by λ, with $\varphi_\lambda : \mathcal{E}_\lambda^{(a)} \to \mathcal{E}_\lambda$ given by $\varphi_\lambda(e_i) = e_i$. We recall that

$$D_\lambda := \operatorname{End}(\mathcal{E}_\lambda|_{\operatorname{Spec} \bar{\mathbb{F}}_p}) \otimes \mathbb{Q}$$

is the division algebra over $\mathbb{Q}_p$ whose invariant λ is the class of λ in $\mathbb{Q}/\mathbb{Z}$. We remark that all endomorphisms of $\mathcal{E}_\lambda$ are realized over $\mathbb{F}_{p^a}$.

2.4. Divisible isoclinic F-crystals. Let S be a scheme of characteristic p, and let $\mathcal{E}$ be an isoclinic F-crystal of slope λ over S which is λ-divisible. Denote by $\varphi : \mathcal{E}^{(a)} \to \mathcal{E}$ the map such that $F^a = p^{a\lambda}\varphi$ which is supposed to exist by our definition of divisibility. Since $(\mathcal{E}, F)$ is isoclinic of slope λ, we see that φ is an isomorphism, i.e., the pair $(\mathcal{E}, \varphi)$ is a "unit-root F^a-crystal".

Let us consider the sheaves $\mathcal{H}_n$ on the étale site of S given by

$$\Gamma(T, \mathcal{H}_n) = \Gamma(\mathrm{CRIS}(T/\operatorname{Spec} \mathbb{Z}_p), (\mathcal{E}/p^n\mathcal{E}))^{\varphi=1}.$$

This is a locally constant sheaf of finite free $\mathbb{Z}/p^n\mathbb{Z}$-modules of rank equal to $a(\operatorname{rk} \mathcal{E})$. This follows from the results of [4, Section 2.4] in the following way. Let $\mathcal{E}(\varphi)$ denote the crystal $\mathcal{E}(\varphi) := \mathcal{E} \oplus \mathcal{E}^{(1)} \oplus \ldots \oplus \mathcal{E}^{(a-1)}$, with F given by the matrix

$$\begin{pmatrix} 0 & 0 & \ldots & \varphi \\ \mathrm{id} & 0 & \ldots & 0 \\ \ldots & & & \\ \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \mathrm{id} & 0 \end{pmatrix} : \mathcal{E}(\varphi)^{(1)} = \mathcal{E}^{(1)} \oplus \ldots \oplus \mathcal{E}^{(a)} \longrightarrow \mathcal{E}(\varphi) = \mathcal{E} \oplus \ldots \oplus \mathcal{E}^{(a-1)}.$$

Then $\mathcal{E}(\varphi)$ is a unit root F-crystal, and the sheaf $\mathcal{H}_n$ is clearly canonically isomorphic to the sheaf $H\big(\mathcal{E}(\varphi)/p^n\big)$ defined in [4, page 205]. The desired result then follows from the results of [4], in particular the remarks at the bottom of page 205 and the discussion directly preceding Proposition 2.4.4.

We can identify $\mathcal{H}_n$ with the sheaf $\mathcal{H}om((\mathcal{E}_\lambda, F, \varphi_\lambda), (\mathcal{E}/p^n, F, \varphi))$ of homomorphisms $\alpha : \mathcal{E}_\lambda \to \mathcal{E}/p^n$ of truncated F-crystals which are compatible with φ_λ and φ, i.e., $\alpha^{(a)} \circ \varphi_\lambda = \varphi \circ \alpha$. Indeed, given such an α we have $\alpha(e_1) \in \Gamma(\mathcal{E}/p^n\mathcal{E})^{\varphi=1}$.

The reverse construction is left to the reader. Thus $\mathcal{H}_n$ is canonically a sheaf of right modules over the sheaf of endomorphisms of $(\mathcal{E}_\lambda/p^n, F)$ compatible with φ_λ:

$$\mathcal{D}_{\lambda,n} := \mathcal{E}nd\big((\mathcal{E}_\lambda/p^n, F, \varphi_\lambda)\big).$$

For the same reasons as above, the sheaves $\mathcal{D}_{\lambda,n}$ are finite locally constant sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-algebras over $S_{\acute{e}t}$. In fact, each $\mathcal{H}_n$ is finite locally free over $\mathcal{D}_{\lambda,n}$. Thus the system $\{\mathcal{H}_n\} = \{\mathcal{H}om((\mathcal{E}_\lambda, F, \varphi_\lambda), (\mathcal{E}/p^n, F, \varphi))\}$ is a finite locally free sheaf of right $\{\mathcal{D}_{\lambda,n}\}$-modules over $S_{\acute{e}t}$, in other words it is a *lisse* sheaf. We abbreviate: $\mathcal{H}om(\mathcal{E}_\lambda, \mathcal{E}) := \{\mathcal{H}_n\}$ and $\mathcal{D}_\lambda = \{\mathcal{D}_{\lambda,n}\}$.

Note that $\Gamma\big(\operatorname{Spec}\bar{\mathbb{F}}_p, \mathcal{D}_{\lambda,n}\big) = \mathcal{O}_\lambda/p^n\mathcal{O}_\lambda$, where $\mathcal{O}_\lambda \subset D_\lambda$ is the maximal order. Hence $\Gamma\big(\operatorname{Spec}\bar{\mathbb{F}}_p, \mathcal{D}_\lambda\big) = \mathcal{O}_\lambda$. We also note that if $S \to \operatorname{Spec}\mathbb{F}_p$ factors through $\operatorname{Spec}\mathbb{F}_{p^a}$, then $\mathcal{D}_\lambda$ is in fact a constant sheaf (i.e., all $\mathcal{D}_{\lambda,n}$ are constant).

We leave it to the reader to establish the following result. It generalizes the case of slope 0 done in [4], and follows from this case by the discussion above.

2.5. Proposition. *The functor*

$$\mathcal{E} \longmapsto \mathcal{H}\mathrm{om}(\mathcal{E}_\lambda, \mathcal{E})$$

establishes an equivalence between the category of λ-divisible F-crystals isoclinic of slope λ endowed with a choice of φ and the category of lisse right $\mathcal{D}_\lambda$-modules on $S_{\acute{e}t}$.

2.6. There is a small snag to this proposition. The map φ may not be uniquely determined by $\mathcal{E}$ and its Frobenius map F. However, any two choices φ, φ' differ by a map which is annihilated by $p^{a\lambda}$, namely $p^{a\lambda}(\varphi - \varphi') = p^{a\lambda}\varphi - p^{a\lambda}\varphi' = F^a - F^a = 0$. For good schemes (good means no p-torsion in the universal PD-envelopes, e.g., regular schemes and perfect schemes) there is at most one φ. In general however, the categories of λ-divisible λ-isoclinic F-crystals and of lisse right $\mathcal{D}_\lambda$-modules over S will still be equivalent *up to isogeny* by the above.

In particular, such F-crystals up to isogeny correspond bijectively to continuous twisted Galois representations

$$\rho : \pi_1(S, \bar{s}) \longrightarrow \mathrm{GL}_d(D_\lambda).$$

Here twisted means that $\pi_1(S)$ acts nontrivially on the coefficients D_λ as in 2.3. The integer d equals $(\operatorname{rk}\mathcal{E})/a$. The idea of considering these representations is not new, and we could have used other references above. See for example [11] and [15, pp. 142–146]; as we saw above, in [4] we find the case of slope zero over any base.

2.7. Study of F-crystals over a field. Suppose that S is the spectrum of a field K. Choose a Cohen ring Λ for K, and let $\sigma : \Lambda \to \Lambda$ be a lift of Frobenius on K. Let K^{pf} be a perfect closure of K. We remark that under the identification $K^{\mathrm{pf}} = \varinjlim(K \to K \to \dots)$ we obtain

$$W(K^{\mathrm{pf}}) = \varinjlim(\Lambda \xrightarrow{\sigma} \Lambda \xrightarrow{\sigma} \dots).$$

Thus we get an injection $\Lambda \to W(K^{\mathrm{pf}})$ compatible with σ.

We recall (see for example [4, Proposition 1.3.3]) that an F-crystal $\mathcal{E}$ over K is given by a triple (M, ∇, F) over Λ. (Thus M is a finite free Λ-module, ∇ is a topologically quasi-nilpotent continuous connection, and F is a horizontal σ-linear self-map of M.) The pullback $\mathcal{E}|_{\operatorname{Spec}K^{\mathrm{pf}}}$ of $\mathcal{E}$ to K^{pf} corresponds to the pair $(M \otimes_\Lambda W(K^{\mathrm{pf}}), F \otimes \sigma)$. The *slopes* of the F-crystal $\mathcal{E}$ are simply defined as the slopes of the pullback of $\mathcal{E}$ to $\operatorname{Spec}K^{\mathrm{pf}}$, i.e., the slopes of $(M \otimes_\Lambda W(K^{\mathrm{pf}}), F \otimes \sigma)$.

Suppose these slopes are $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_s$ with multiplicities $A_1, \ldots, A_s$. We claim that the result of [16, Corollary 2.6.3] holds in this situation.

2.8. Claim. *(M, ∇, F) is isogenous to an F-crystal (M', ∇', F') which admits a filtration*

$$0 \subset (M'_1, \nabla', F') \subset (M'_2, \nabla', F') \subset \ldots \subset (M'_s, \nabla', F') = (M', \nabla', F')$$

in which M'_i/M'_{i-1} is a free Λ-module such that the resulting F-crystal $(M'_i/M'_{i-1}, \nabla', F')$ is divisible by λ_i, has rank A_i and is isoclinic of slope λ_i.

Proof. We claim the results 2.4.2 (Newton-Hodge filtration), 2.6.1 (Isogeny Theorem) and 2.6.2 of [16] in our situation. For the proof of 2.4.2, just set $A_0 = K$ and $A_\infty = \Lambda$, and the proof given in [16] goes through with only minor modifications. For 2.6.1 the proof is exactly the same as for the case of a perfect field given in [16], i.e., the module M' works. The proofs of 2.6.2 and 2.6.3 go through without changes. □

2.9. Note that the F-crystals $\mathcal{E}_i$ corresponding to $(M'_i/M'_{i-1}, \nabla', F')$ are well defined up to isogeny by $\mathcal{E}$. We define the *Galois representation associated to $\mathcal{E}$ in slope λ_i* to be the representation

$$\rho : \mathrm{Gal}(K^{sep}/K) \longrightarrow \mathrm{GL}_{d_i}(D_{\lambda_i})$$

associated to the λ_i-divisible λ_i-isoclinic F-crystal $\mathcal{E}_i$ over $\operatorname{Spec} K$; see 2.6.

2.10. Proposition. *Let R be a discrete valuation ring of characteristic p with fraction field K. Let $\mathcal{E}$ be an F-crystal over R. Suppose that at the two points of* $\operatorname{Spec} R$ *the Newton polygons of $\mathcal{E}$ coincide. Then the Galois representations in every slope λ associated to $\mathcal{E}$ over K are unramified (i.e., they come from representations of $\pi_1(\operatorname{Spec} R)$).*

Proof. We replace R by its strict henselization. We have to show that the representations are trivial.

Let us denote by $R^\wedge$ the completion of R, and let $K^\wedge$ be the fraction field of $R^\wedge$. Since ([33, Ch. II, section 3, Cor. 4]) the map of absolute Galois groups

$$\mathrm{Gal}_{K^\wedge} \longrightarrow \mathrm{Gal}_K$$

is surjective, it suffices to handle the case where R is complete.

In this case we have $R = k[[t]]$ with k separably closed. Let $\bar{k}$ be an algebraic closure of k. Consider the ring extension $k[[t]] \subset \bar{k}[[t]]$. It induces an extension of fraction fields $K \subset L$. Again the map

$$\mathrm{Gal}_L \longrightarrow \mathrm{Gal}_K$$

is surjective. See [5, Lemma 3.2.5]. We sketch the proof. The union R' of the henselian local rings $k^{p^{-n}}[[t]]$ is a henselian discrete valuation ring whose completion is $\bar{k}[[t]]$. Hence, by the first part of the proof of the proposition we have that $\mathrm{Gal}_L \to \mathrm{Gal}_{K'}$ is surjective. After this, we simply observe that $K \subset K'$ is purely inseparable. Thus we may assume that $R = k[[t]]$ with k algebraically closed.

In this case the result follows from [16, Theorem 2.7.4], which states that over $k((t))^{\mathrm{pf}}$ the F-crystal $\mathcal{E}$ becomes isogenous to a constant F-crystal, and therefore has trivial associated Galois representations. □

2.11. *Remark.* In fact, we can be a little more precise. If R is henselian with residue field k, then we have $\pi_1(\operatorname{Spec} R) \cong \operatorname{Gal}(k^{sep}/k)$. The resulting representations

$$\operatorname{Gal}(k^{sep}/k) \longrightarrow \operatorname{GL}_{d_i}(D_{\lambda_i})$$

are in fact the Galois representations associated to the F-crystal $\mathcal{E}|_{\operatorname{Spec} k}$ in slope λ_i. To prove this we may replace R by $k^{\mathrm{pf}}[[t]]$, where k^{pf} is the perfect closure of k. From [16] we see that the slope filtration over the fraction field comes from a slope filtration over $k^{\mathrm{pf}}[[t]]$, with ith piece a λ_i-divisible F-crystal. From this the result follows. We will not use this result.

2.12. Before we can state the next proposition we introduce some notation. We remark that the result we have in mind can be proved in much greater generality. However, since the authors do not see a use for this at present, we have decided to avoid the added technicalities by formulating a less general result.

Let A be an excellent domain, with fraction field K of characteristic p. We write the perfect closure K^{pf} of K as the union of its finite extensions K_α of K. Let A_α denote the integral closure of A in K_α. The union $A^{\mathrm{pf}} = \bigcup A_\alpha$ is a perfect ring. Write $S = \operatorname{Spec} A$, $S_\alpha = \operatorname{Spec} A_\alpha$ and $S^{\mathrm{pf}} = \operatorname{Spec} A^{\mathrm{pf}}$. Thus $S^{\mathrm{pf}} = \varprojlim S_\alpha$. All the S_α are excellent integral schemes, and S^{pf} is integral as well.

A *nice* scheme T over S^{pf} is any scheme obtained as follows: For some $\alpha_\circ$ we are given a modification $T_{\alpha_\circ} \to S_{\alpha_\circ}$, and for $\alpha > \alpha_\circ$ the scheme T_α is the normalization of $T_{\alpha_\circ}$ in the field K_α. All the transition morphisms $T_\alpha \to T_{\alpha'}$ are affine, hence the scheme $T = \varprojlim T_\alpha$ exists (and it dominates S^{pf}). We remark that the schemes T_α are modifications of S_α, they are proper over $S = \operatorname{Spec} A$, integral, excellent and have function field K_α. Thus any nice scheme T is integral, perfect and has function field K^{pf}.

Note that if $X \to T$ is a morphism of finite presentation, then there exists a β and a morphism of finite presentation $X_\beta \to T_\beta$ such that $X \cong X_\beta \times_{T_\beta} T$. In case $X \to T$ is (finite) étale, we can choose $X_\beta \to T_\beta$ to be (finite) étale. Similarly for proper. See EGA IV 8.8 & 8.10.5.

Suppose that $\varphi : X \to T$ is proper, of finite presentation, and that $\eta \in X$ is a point mapping to the generic point of T inducing an isomorphism $\kappa(\eta) = R(T) = K^{\mathrm{pf}}$. Then there exists a nice scheme $T' \to S$ and a morphism $T' \to X$ over S such that the generic point of T' maps to η.

To prove this, write $X = X_\beta \times_{T_\beta} T$ as above, with X_β proper over T_β. The point η maps to a point $\eta_\beta \in X_\beta$. Note that $K \subset \kappa(\eta_\beta) \subset \kappa(\eta) = K^{\mathrm{pf}}$. Thus there exists a γ such that $\kappa(\eta_\beta) \subset K_\gamma$ (use that X_β is of finite type over T_β). In other words, the point η_γ in $X_\gamma = X_\beta \times_{T_\beta} T_\gamma$ will satisfy $\kappa(\eta_\gamma) = K_\gamma = R(T_\gamma)$. Let $T'_\gamma \subset X_\gamma$ be the schematic closure of $\{\eta_\gamma\}$. Then the composition $T'_\gamma \to T_\gamma \to S_\gamma$ is a modification. Let T'_δ be the normalization of T'_γ in K_δ for $\delta > \gamma$. Then there are maps $T'_\delta \to T_\delta$ by the universal properties of normalizations, and we obtain the desired morphism

$$T' = \varprojlim\, T'_\delta \longrightarrow \varprojlim\, X_\gamma \times_{T_\gamma} T_\delta = X.$$

2.13. Proposition. *Let $A \subset K$, S, S_α, etc. be as above. Let $\mathcal{E}$ be a nondegenerate F-crystal over S^{pf}. Let λ be the smallest slope of $\mathcal{E}|_{\operatorname{Spec} K^{\mathrm{pf}}}$. Then there exists a nice morphism $\pi : T \to S^{\mathrm{pf}}$ such that $\mathcal{E}|_T$ is isogenous to an F-crystal which is divisible by λ.*

Proof. Since S^{pf} is perfect, the F-crystal $\mathcal{E}$ is simply a finite locally free sheaf of $W(\mathcal{O}_{S^{\mathrm{pf}}})$-modules endowed with a Frobenius linear endomorphism $F : \mathcal{E} \to \mathcal{E}$. Let us use the suggestive notation $\mathcal{E} \otimes W(K^{\mathrm{pf}})$ to denote the $W(K^{\mathrm{pf}})$-module which corresponds to the pull back $\mathcal{E}|_{\operatorname{Spec} K^{\mathrm{pf}}}$. By the Isogeny Theorem [16, 2.6.1], there exists an isogeny

$$N \longrightarrow \mathcal{E} \otimes W(K^{\mathrm{pf}})$$

such that N is divisible by λ. Note that for some $r \in \mathbb{N}$:

$$(*) \qquad p^r \mathcal{E} \otimes W(K^{\mathrm{pf}}) \subset N \subset \mathcal{E} \otimes W(K^{\mathrm{pf}}).$$

We claim that, for any submodule N as in $(*)$, there exist (1) a nice morphism $\pi : T \to S^{\mathrm{pf}}$, (2) a finite locally free $W(\mathcal{O}_T)$-module $\mathcal{E}'$, and (3) a $W(\mathcal{O}_T)$-linear injection

$$\mathcal{E}' \longrightarrow \pi^*\mathcal{E} := \pi^{-1}\mathcal{E} \otimes_{W(\pi^{-1}\mathcal{O})} W(\mathcal{O}_T)$$

such that $\mathcal{E}' \otimes W(K^{\mathrm{pf}}) = N \subset \pi^*\mathcal{E} \otimes W(K^{\mathrm{pf}}) = \mathcal{E} \otimes W(K^{\mathrm{pf}})$ and such that $\pi^*\mathcal{E}/\mathcal{E}'$ has a finite filtration whose graded is a locally free $\mathcal{O}_T$-module.

We prove this by induction on r. Let $N' := N + p^{r-1}\mathcal{E} \otimes W(K^{\mathrm{pf}})$. By the induction hypothesis we obtain $\pi : T \to S^{\mathrm{pf}}$, $\mathcal{E}' \to \pi^*\mathcal{E}$ such that $\mathcal{E}' \otimes W(K^{\mathrm{pf}}) = N'$. Note that $pN' \subset N \subset N'$. Hence, N is the inverse image of a subvector space $V \subset \mathcal{E}' \otimes K^{\mathrm{pf}}$ (under the map $\mathcal{E}' \otimes W(K^{\mathrm{pf}}) \to \mathcal{E}' \otimes K^{\mathrm{pf}}$). Say $\dim V = m$. The sheaf of $\mathcal{O}_T$-modules $\mathcal{E}'/p\mathcal{E}' = \mathcal{E}' \otimes_{W(\mathcal{O}_T)} \mathcal{O}_T$ is finite locally free. Hence the Grassmannian $X = \mathbb{G}\mathrm{r}(m, \mathcal{E}'/p\mathcal{E}')$ parameterizing rank m locally direct summands is of finite presentation over T. The subvector space $V \subset \mathcal{E}' \otimes K^{\mathrm{pf}}$ determines a point $\eta \in X$ with $\kappa(\eta) = K^{\mathrm{pf}}$. Therefore, by the discussion preceding the proposition, we can find a nice scheme $T' \to S^{\mathrm{pf}}$ and a morphism $T' \to X$ mapping the generic point of T' to η. Letting $\tau : T' \to T$ be the induced map, this means that V extends to a locally direct summand $\mathcal{V} \subset \tau^*(\mathcal{E}'/p\mathcal{E}')$. Set $\mathcal{E}''$ equal to the inverse image of $\mathcal{V}$ under the map $\tau^*\mathcal{E}' \to \tau^*(\mathcal{E}'/p\mathcal{E}')$. Then $\mathcal{E}''$ is finite locally free over $W(\mathcal{O}_{T'})$, and $\mathcal{E}'' \otimes W(K^{\mathrm{pf}}) = N$. The quotient $(\pi \circ \tau)^*\mathcal{E}/\mathcal{E}''$ has a filtration $0 \subset \tau^*\mathcal{E}'/\mathcal{E}'' \subset \tau^*\pi^*\mathcal{E}/\mathcal{E}''$ with quotients $\tau^*\mathcal{E}'/\mathcal{E}''$, locally free by construction, and $\tau^*(\pi^*\mathcal{E}/\mathcal{E}')$, having a suitable filtration by the induction hypothesis. This proves the claim.

We apply the claim to the λ-divisible subcrystal N of $\mathcal{E} \otimes W(K^{\mathrm{pf}})$. We obtain $\pi : T \to S^{\mathrm{pf}}$ and $\mathcal{E}' \subset \pi^*\mathcal{E}$. Since $F(N) \subset N$, the condition on $\pi^*\mathcal{E}/\mathcal{E}'$ insures that $F(\mathcal{E}') \subset \mathcal{E}'$ (use that T is integral and perfect). The fact that N is divisible by λ means that $F^a(N) \subset p^{a\lambda}N$. Again the condition on $\pi^*\mathcal{E}/\mathcal{E}'$ insures that this implies $F^a(\mathcal{E}') \subset p^{a\lambda}\mathcal{E}'$. Since $W(\mathcal{O}_T)$ does not have p-torsion, we can form the map $\varphi = p^{-a\lambda}F^a$ on $\mathcal{E}'$ as desired. □

2.14. Let us consider a Noetherian complete local ring A of characteristic p with algebraically closed residue field k. Let $\mathcal{E}$ be an isoclinic F-crystal over A of slope λ. According to 2.9 we obtain a Galois representation (associated to $\mathcal{E}|_{\operatorname{Spec}\kappa(\mathfrak{p})}$ in slope λ)

$$\rho_{\mathfrak{p}} : \mathrm{Gal}(\kappa(\mathfrak{p})^{sep}/\kappa(\mathfrak{p})) \longrightarrow \mathrm{GL}_d(D_\lambda),$$

where $\mathfrak{p} \subset A$ is any prime ideal of A. (Note that our assumptions on A imply that $\bar{\mathbb{F}}_p \subset A$, so that we get actual representations, not twisted ones.)

2.15. Proposition. *The Galois representation $\rho_{\mathfrak{p}}$ is trivial.*

Proof. We may replace A by $A/\mathfrak{p}$. Then $\mathfrak{p} = (0)$ and $K = \kappa(\mathfrak{p})$ is the fraction field of the excellent domain A. We use the notation introduced in Subsection 2.12. According to Proposition 2.13 we can find a nice morphism $\pi : T \to \operatorname{Spec} A^{\mathrm{pf}}$ such that $\pi^*\mathcal{E}$ becomes isogenous to a λ-divisible F-crystal $\mathcal{E}'$ over T. By 2.6 this isocrystal determines a continuous representation

$$\rho : \pi_1(T) \longrightarrow \mathrm{GL}_d(D_\lambda).$$

When we restrict ρ to $\pi_1(\operatorname{Spec} K^{\mathrm{pf}})$, we recover the representation $\rho_{\mathfrak{p}}$.

Let us write $0 \in \operatorname{Spec} A$ for the closed point of $\operatorname{Spec} A$. Let T_0 denote the fiber over 0 of the morphism $T \to \operatorname{Spec} A$. Obviously, $\mathcal{E}'|_{T_0}$ is isogenous to the constant F-crystal with fiber $\mathcal{E}|_0$. We conclude that, for any geometric point $\bar{t}$ of T_0, the composition

$$\pi_1(T_0, \bar{t}) \longrightarrow \pi_1(T, \bar{t}) \xrightarrow{\rho} \mathrm{GL}_d(D_\lambda)$$

is trivial. We claim that this implies that ρ (and so $\rho_{\mathfrak{p}}$) is trivial.

We have to prove the following statement. Any finite étale covering $X \to T$ such that $X \times_T T_0$ is a trivial covering of T_0 is itself trivial. By the discussion in 2.12 there exists a diagram

$$\begin{array}{ccc} X & \longrightarrow & X_\alpha \\ \downarrow & & \downarrow \\ T & \longrightarrow & T_\alpha \\ \downarrow & & \downarrow \\ \operatorname{Spec} A^{\mathrm{pf}} & \longrightarrow & \operatorname{Spec} A_\alpha \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A \end{array}$$

such that $X = X_\alpha \times_{T_\alpha} T$. We may assume that $X_\alpha \to T_\alpha$ is finite étale. Since $T_0 = \varprojlim (T_\gamma)_0$, we may assume, after increasing α, that $X_\alpha \times_{T_\alpha} (T_\alpha)_0$ is a trivial covering of $(T_\alpha)_0$.

At this point we can use Zariski's theorem on formal functions. The morphism $X_\alpha \to \operatorname{Spec} A$ is proper being the composition of the finite morphism $X_\alpha \to T_\alpha$ and the proper morphism $T_\alpha \to \operatorname{Spec} A$; see 2.12. Thus the map on connected components $\pi_0(X_\alpha) \to \pi_0((X_\alpha)_0)$ is bijective. The triviality of $(X_\alpha)_0$ over $(T_\alpha)_0$ implies that $\pi_0(X_\alpha)$ has at least d elements, if the degree of $X \to T$ is d. Thus $X \to T$ is trivial. $\square$

2.16. *Remark.* We mention a generalization of this proposition that can be proved with the same methods. Assume that $\mathcal{E}$ is a nondegenerate F-crystal over $\operatorname{Spec} A$ whose Newton polygon is constant (i.e., the Newton polygon is the same in all points of $\operatorname{Spec} A$). Then the Galois representations in *all slopes* associated to the fibers of $\mathcal{E}$ in the points of $\operatorname{Spec} A$ are trivial. We remark that the case $\dim A = 1$ of this generalization follows from Proposition 2.10. To prove this more general statement one has to strengthen Proposition 2.13 to a theorem like the filtration theorem of [16].

2.17. Corollary (Isogeny Theorem)**.** *Let A be a Noetherian complete local domain with algebraically closed residue field, normal and with field of fractions K of characteristic p. Let G be a p-divisible group over* $\operatorname{Spec} A$. *Assume that the covariant crystalline Dieudonné module $\mathbb{D}(G)$ (see* [22]*) is isoclinic over* $\operatorname{Spec} A$. *Then there exists a p-divisible group G_0 over* $\operatorname{Spec} k$ *and an isogeny* $G_0 \times_{\operatorname{Spec} k} \operatorname{Spec} A \to G$ *over A.*

Proof. By Proposition 2.15, the Galois representation associated to $\mathbb{D}(G \otimes K) = \mathbb{D}(G)|_{\operatorname{Spec} K}$ is trivial. This means that there is an isogeny $\mathcal{E}_\lambda^n|_{\operatorname{Spec} K} \to \mathbb{D}(G)$, where $\lambda = a/b$ is the slope of $\mathbb{D}(G)$; see Proposition 2.5 and Subsection 2.6. Since λ comes from a p-divisible group, we have $\lambda \in [0,1] \cap \mathbb{Q}$. This means that $(\mathcal{E}_\lambda)^n$ is isogenous to the Dieudonné module of a p-divisible group G_0 over $\operatorname{Spec} \mathbb{F}_p$. (For example the p-divisible group $(G_{b-a,a})^n$ of [20].) Therefore we have an isogeny $G_0 \otimes K \to G \otimes K$ over the field of fractions K of A. This extends to a morphism of p-divisible groups over A by [8, Introduction]. □

2.18. *Remark.* One would like to have the corollary also for F-crystals. Proposition 2.15 implies that there exists an isogeny $(\mathcal{E}_\lambda)^n|_{\operatorname{Spec} K} \to \mathcal{E}|_{\operatorname{Spec} K}$ (for some λ and n) over the generic point $\operatorname{Spec} K \hookrightarrow \operatorname{Spec} A$. The extension theorem of [8] holds for F-crystals over discrete valuation rings, but we do not know whether an extension theorem over Noetherian normal schemes holds. The reason is that the universal PD-envelopes may behave badly, even for excellent normal rings. However, the corollary should hold in a suitable setting of convergent F-isocrystals.

3. Cohomology of the link of a surface singularity

3.1. Let A be a local complete Noetherian ring, normal of dimension 2 with algebraically closed residue field k. Let $S = \operatorname{Spec} A$, let $0 \in \operatorname{Spec} A$ be the closed point, and let $U = S \setminus \{0\}$. Let us choose a resolution of singularities $\pi : \widetilde{S} \to S$ (see for example [19]), and let $E = \pi^{-1}(0)$ be the exceptional fiber. We identify $\pi^{-1}(U)$ with U:

$$\begin{array}{ccccc} U & \longrightarrow & \widetilde{S} & \longleftarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & S & \longleftarrow & 0 \end{array}$$

3.2. Theorem. *The natural map*

$$H^1_{ét}(\widetilde{S}, \mathbb{Q}_p) \longrightarrow H^1_{ét}(U, \mathbb{Q}_p)$$

is an isomorphism.

Informally speaking this means that any étale $\mathbb{Z}_p$-covering of U extends to an étale covering of $\widetilde{S}$. The map of the theorem is easily seen to be injective. Note that $H^1_{ét}(\widetilde{S}, \mathbb{Q}_p) = H^1_{ét}(E, \mathbb{Q}_p)$.

If k is the complex numbers and the singularity is algebraic, one can find this result in Mumford's paper [26]. More generally, when k is an algebraically closed field of characteristic different from p, the theorem follows easily using standard methods of étale cohomology and the nondegeneracy of the intersection matrix of the components of E. However, the reader can verify that our methods lead to a proof of the theorem in these cases also.

In the rest of this section we will discuss the case where A has characteristic p, i.e., where $k \subset A$. At the end of the section we will indicate how to change the arguments to prove the mixed characteristic $(0,p)$ case.

We fix a nonzero element α of $H^1_{ét}(U, \mathbb{Q}_p)$. It will be viewed as a continuous homomorphism $\pi_1(U) \to \mathbb{Q}_p$. We may assume (by replacing α by a multiple) that the image of α lies in $\mathbb{Z}_p$ and that $\alpha : \pi_1(U) \to \mathbb{Z}_p$ is surjective.

Let $\eta \in \widetilde{S}$ be the generic point of a component of E. Consider the complete local ring $\mathcal{O} = \widehat{\mathcal{O}}_{\widetilde{S},\eta}$ of $\widetilde{S}$ at η. Let K be the field of fractions of $\mathcal{O}$. There is a map $\mathrm{Gal}_K \to \pi_1(U)$ induced by functoriality of the fundamental group from the morphism $\operatorname{Spec} K \to U$.

3.3. Lemma. *If the composition* $\mathrm{Gal}_K \to \pi_1(U) \xrightarrow{\alpha} \mathbb{Q}_p$ *is unramified for every* η *as above, then* α *extends.*

Proof. The assumption implies that each of the finite étale coverings $V_n \to U$ associated to $\alpha \bmod p^n : \pi_1(U) \to \mathbb{Z}_p/p^n\mathbb{Z}_p$ extends to a finite étale covering outside codimension 2. By Zariski–Nagata on purity of branch locus [SGA2, X Thm 3.4], we see they extend to the whole of $\widetilde{S}$. □

3.4. Lemma. *Let* $A \to B$ *be a finite ring homomorphism such that* B *is a normal domain also. If the element* $\alpha \in H^1_{\acute{e}t}(U, \mathbb{Q}_p)$ *extends to an étale cohomology class on the desingularization of* $\operatorname{Spec} B$*, then it extends to* $\widetilde{S}$*.*

Proof. We may dominate the desingularization of $\operatorname{Spec} B$ by a desingularization $\tilde{T}$ which dominates $\widetilde{S}$. Let $\eta \in \widetilde{S}$ be as above, and let $\xi \in \tilde{T}$ be a point of $\tilde{T}$ mapping to η. Such a point exists as $\tilde{T} \to \widetilde{S}$ is proper and dominant, hence surjective.

Let $\mathcal{O}'$ be the complete local ring of $\tilde{T}$ at ξ, and let K' be its field of fractions. We remark that $\mathcal{O} \subset \mathcal{O}'$ is a finite extension of complete discrete valuation rings. The inertia group $I' \subset \mathrm{Gal}_{K'}$ therefore maps onto an open subgroup of the inertia subgroup $I \subset \mathrm{Gal}_K$.

By assumption we know that the composition

$$I' \to I \to \mathrm{Gal}_K \to \pi_1(U) \xrightarrow{\alpha} \mathbb{Q}_p$$

is trivial. Since $\mathbb{Q}_p$ is torsion free this implies that $\alpha|_I$ is trivial. Thus by Lemma 3.3 we are done. □

We are going to use that the singularity S is algebraizable; see [2, Theorem 3.8]. This means that there exist a projective surface X over k and a point $0 \in X$ such that the complete local ring of X at 0 is isomorphic to A. (Recall that we are treating the case $k \subset A$.) After blowing up X, we may assume in addition that there exists a flat morphism $X \to \mathbb{P}^1_k$. By [7], we may perform an alteration:

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & X \\ \downarrow & & \downarrow \\ C & \longrightarrow & \mathbb{P}^1_k. \end{array}$$

Here Y is a projective surface smooth over k, C is a projective curve smooth over k, and $Y \to C$ is a strict semi-stable curve over C smooth over an open part of C. The map $\psi : Y \to X$ is an alteration.

Let E' be a connected component of $\psi^{-1}(0)$; let $P \in C$ be the image of E' in C. Note that E' is a union of irreducible components of Y_P. Also, E' is a normal crossings divisor in Y. We remark that $E' \neq Y_P$. (For example because the class of Y_P maps to the class of a fiber of $X \to \mathbb{P}^1$, but the class of E' maps to zero. Classes are taken in Chow groups modulo algebraic equivalence.)

Let $Y \to Y' \to X$ be the Stein factorization of ψ. Let $0' \in Y'(k)$ be the image of E' in Y'. The extension

$$A \cong \widehat{\mathcal{O}}_{X,0} \subset \widehat{\mathcal{O}}_{Y',0'}$$

is a finite extension of normal local rings. Thus by Lemma 3.4 we have reduced our problem to the case where A is as in Situation 3.5 below.

3.5. Situation. Here $X \to \operatorname{Spec} k[[t]]$ is a flat projective family of curves. The scheme X is regular. The special fiber X_0 is a strict normal crossings divisor in X, and the generic fiber X_K is a smooth curve. Furthermore, $E \subset X_0$ is a connected union of irreducible components, and $E \neq X_0$.

Finally, the ring A is the complete local ring of the scheme obtained from X by collapsing E. More precisely, there exists a modification $X \to X'$, with the following properties: (1) It is an isomorphism outside $E \subset X$. (2) The image of E is a single point $P \in X'$. (3) The complete local ring of X' at P is isomorphic to A.

3.6. Let us interpret the data α in a slightly different manner in this situation. The map $\alpha \bmod p^n : \pi_1(U) \to \mathbb{Z}_p/p^n\mathbb{Z}_p$ corresponds to a finite morphism $T_n \to \operatorname{Spec} A$ with T_n a normal 2-dimensional scheme; T_n is endowed with an action of $\mathbb{Z}/p^n\mathbb{Z}$, such that over U it is a $\mathbb{Z}/p^n\mathbb{Z}$-principal homogeneous space.

Let $X_{/E}$ denote the formal scheme obtained by formally completing X along E. There is a morphism of formal schemes

$$X_{/E} \longrightarrow \operatorname{Spf} A$$

which induces an isomorphism $A \to \Gamma(X_{/E}, \mathcal{O}_{X_{/E}})$. Let $\mathfrak{T}_n$ denote the formal scheme obtained from T_n under base change with $\operatorname{Spf} A \to \operatorname{Spec} A$. We can form the fiber products

$$X_{/E} \times_{\operatorname{Spf} A} \mathfrak{T}_n = X_{/E} \times_{\operatorname{Spec} A} T_n.$$

Finally, we can "formally normalize" these formal schemes to obtain formal schemes $\mathfrak{Z}_n$ finite over $X_{/E}$. ("Formally normalize" means to take the integral closure of the formal affine rings.)

In more down to earth terms one can describe $\mathfrak{Z}_n$ as follows. Let $\mathfrak{U} = \operatorname{Spf} C$ be a formal affine open of $X_{/E}$. There is a ring map $A \to C$ (see above), such that $\mathfrak{m}_A C$ is an ideal of definition of C. Write $T_n = \operatorname{Spec} B_n$. Then the inverse image of $\mathfrak{U}$ in $\mathfrak{Z}_n$ is the formal spectrum of the following algebra:

$$C_n := \text{normalization of } B_n \otimes_A C.$$

The ideal of definition is $\mathfrak{m}_A C_n$; we do not have to complete as $A \to B_n$ is finite and C is excellent, so that C_n is finite over C, hence complete.

A few more remarks. The complete local rings of C at its maximal ideals are regular local rings of dimension two, as X is regular. Hence by [21, (18.H)], the maps $C \to C_n$ are flat. Further, $C \to C_n$ is finite étale outside of $\operatorname{Spec} C/\mathfrak{m}_A C$. Finally, there is a canonical action of $\mathbb{Z}/p^n\mathbb{Z}$ on $\mathfrak{Z}_n$, coming from its action on T_n.

3.7. Let D be the union of the components of X_0 not contained in E, considered as a reduced scheme over $\operatorname{Spec} k$. Let $E \cap D = \{x_1, \ldots, x_r\}$. For each i, the morphism $D \to \operatorname{Spec} k$ is smooth at x_i. Let K_i be the fraction field of the complete local ring $\mathcal{O}_i$ of D at x_i. Note that A maps to $\mathcal{O}_i$, for example since A maps to $\Gamma(X_{/E}, \mathcal{O}_{X_{/E}})$. Also, $A \to \mathcal{O}_i$ is local. Thus we obtain a map $\operatorname{Spec} K_i \to U$ and hence we get a Galois representation

$$\alpha_i : \operatorname{Gal}_{K_i} \longrightarrow \mathbb{Z}_p.$$

Here is another way to obtain these representations. Clearly, there is a morphism $\operatorname{Spf} \mathcal{O}_i \to X_{/E}$. The pullback of the ideal of definition of $X_{/E}$ to $\mathcal{O}_i$ is an ideal of definition of $\mathcal{O}_i$. Let us form the fiber products

$$\mathrm{Spf}\,\mathcal{O}_i \times_{X_{/E}} \mathfrak{Z}_n.$$

These will correspond to generically étale, finite flat ring extensions $\mathcal{O}_i \subset \mathcal{O}_{i,n}$, endowed with an action of $\mathbb{Z}/p^n\mathbb{Z}$. The corresponding system of separable ring extensions $K_i \subset K_{i,n}$ (with $\mathbb{Z}/p^n\mathbb{Z}$ actions) is another avatar of α_i.

3.8. Lemma. *Let D be a projective 1-dimensional scheme over k, let $\{x_1, \ldots, x_r\}$ be a finite set of smooth points of $D \to \mathrm{Spec}\,k$, and let K_i be the fraction field of $\mathcal{O}_i = \widehat{\mathcal{O}}_{D,x_i}$. Furthermore, suppose we are given maps $\alpha_i : \mathrm{Gal}_{K_i} \longrightarrow \mathbb{Z}_p$. Then there exists an element*

$$\alpha_D \in H^1_{ét}(D \setminus \{x_1, \ldots, x_r\}, \mathbb{Z}_p)$$

such that we recover each α_i from this by pulling back via $\mathrm{Spec}\,K_i \to D\setminus\{x_1, \ldots, x_r\}$.

Proof. (Similar results occur in the work of Harbater; see e.g. [13].) Let D' be the union of those irreducible components of D which contain a point x_i; so $\{x_1, \ldots, x_n\} \subset D'$. The map

$$H^1_{ét}(D \setminus \{x_1, \ldots, x_r\}, \mathbb{Z}_p) \longrightarrow H^1_{ét}(D' \setminus \{x_1, \ldots, x_r\}, \mathbb{Z}_p)$$

is surjective; we leave this to the reader as an exercise in étale cohomology. Thus it is sufficient to prove the lemma in case every component of D contains at least one x_i.

Let us write $W = D \setminus \{x_1, \ldots, x_r\}$ and $j : W \to D$ for the inclusion morphism. By the above, we may assume that W is affine. Note that $Rj_*\mathcal{O}_W = j_*\mathcal{O}_W$, and that there exists an exact sequence

$$0 \to \mathcal{O}_D \to j_*\mathcal{O}_W \to \bigoplus K_i/\mathcal{O}_i \to 0.$$

Therefore we obtain the exact sequence

$$H^0(W, \mathcal{O}) \to \bigoplus K_i/\mathcal{O}_i \to H^1_{ét}(D, \mathcal{O}_D) \to 0.$$

On each of these we have the map "Frobenius $-$ identity". The cokernel of this map on $H^0(W, \mathcal{O})$ is equal to $H^1_{ét}(W, \mathbb{Z}/p\mathbb{Z})$ and on $\bigoplus K_i/\mathcal{O}_i$ it is $\bigoplus \mathrm{Hom}(\mathrm{Gal}_{K_i}, \mathbb{Z}/p\mathbb{Z})$. The map on cokernels is surjective as the cokernel of the map "Frobenius $-$ identity" on the finite-dimensional k vector space $H^1_{ét}(D, \mathcal{O}_D)$ is zero; see [SGA7, XXII 1]. Thus the result holds with coefficients $\mathbb{Z}/p\mathbb{Z}$.

Finally, note that the map $H^1_{ét}(W, \mathbb{Z}_p) \to H^1_{ét}(W, \mathbb{Z}/p\mathbb{Z})$ is surjective. [As the p-cohomological dimension of W is ≤ 1.] From this, and the mod p case, the result follows formally. Indeed, suppose that we already have $\alpha_n \in H^1_{ét}(W, \mathbb{Z}_p)$ such that $\alpha_n|_{\mathrm{Gal}_{K_i}} - \alpha_i = p^n\beta_i$ for certain β_i. Then we can find a $\beta \in H^1_{ét}(W, \mathbb{Z}_p)$ which gives back β_i modulo p when we restrict it to Gal_{K_i}. As the next element we take $\alpha_{n+1} = \alpha_n + p^n\beta$. In the end we put $\alpha_D = \lim \alpha_n$. $\square$

3.9. We return to the situation at the end of Subsection 3.7. Choose an element $\alpha_D \in H^1_{ét}(D \setminus \{x_1, \ldots, x_r\}, \mathbb{Z}_p)$ as in the lemma.

For each $m \in \mathbb{N}$ we consider the scheme $X_m = X \otimes k[t]/(t^m)$. The formal scheme $X_{/E} \otimes k[t]/(t^m)$ is simply the formal completion of X_m in E. Let U_m be the open subscheme of X_m whose underlying open set is $X_0 \setminus E = D \setminus \{x_1, \ldots, x_r\}$. Thus we have $(U_m)_{\mathrm{red}} \cong D\setminus\{x_1, \ldots, x_r\}$. By the topological invariance of étale cohomology, the class of α_D modulo p^n gives rise to finite étale coverings $V_{n,m} \to U_m$.

For each x_i we choose an affine neighborhood $\mathrm{Spec}\,A_i$ of x_i in X, containing no other x_j. We may assume, by shrinking if necessary, that the ideal of E in A_i

is generated by a single element, say $f_i \in A_i$. Then, locally around x_i, we can describe our spaces as follows:

$$\begin{array}{ccc} X_m & \text{corresponds to the ring} & A_i/t^m A_i \\ U_m & \text{—— // ——} & A_i/t^m A_i[f_i^{-1}] \\ X_{/E} & \text{—— // ——} & (A_i)^\wedge_{(f_i)} \\ X_{/E} \otimes k[t]/(t^m) & \text{—— // ——} & (A_i/t^m A_i)^\wedge_{(f_i)} \end{array}$$

We are going to glue the finite flat coverings $\mathfrak{Z}_n \otimes k[t]/(t^m) \to X_{/E} \otimes k[t]/(t^m)$ to the coverings $V_{n,m} \to U_m$. For this we use the formulation of [6, 4.6]; see [25], [1], [10] for proofs. Thus we need to prove that these coverings agree over the rings

$$\mathcal{R}_{i,m} = (A_i/t^m A_i)^\wedge_{(f_i)}[f_i^{-1}].$$

Both $\mathfrak{Z}_n \otimes k[t]/(t^m)$ and $V_{n,m}$ give rise to a finite étale ring extension of $\mathcal{R}_{i,m}$. We leave it to the reader to see that $\mathcal{R}_{i,m}$ is an Artinian local ring with residue field K_i. Thus we have to see that these finite étale extensions restrict to isomorphic étale ring extensions of K_i. This follows from our choice of α_D; see the end of Subsection 3.7.

By the references given above, we deduce the existence of finite flat coverings $X_{n,m} \to X_m$, agreeing with $\mathfrak{Z}_n \otimes k[t]/(t^m)$ over $X_{/E} \otimes k[t]/(t^m)$ and with $V_{n,m}$ over U_m.

Everything is compatible with varying m. Thus we obtain a formal scheme $\mathfrak{X}_n$ and a finite flat morphism $\mathfrak{X}_n \to \mathfrak{X}$; here $\mathfrak{X}$ denotes the formal completion of X along X_0. By Grothendieck's theorem on algebraization of formal schemes [EGA, III 5.1.4] we conclude that $\mathfrak{X}_n$ is the completion of a scheme Y_n finite flat over X. It is clear from the construction that the morphism $Y_n \to X$ ramifies at most along $E \subset X$.

3.10. Conclusion. For every element α as above there exists an element $\beta \in H^1_{ét}(X_\eta, \mathbb{Z}_p)$ which extends to an étale cohomology class over $X \setminus E = X' \setminus \{P\}$ (see Situation 3.5) and which pulls back to α under the morphism $U \to X' \setminus \{P\}$.

3.11. Let J_η be the Jacobian of the projective curves X_η smooth over $\kappa(\eta)$; we think of J_η as Pic^o. Let J be the Néron model of J_η over $\mathrm{Spec}\, k[[t]]$. Since the curve X_η has a semi-stable model over $k[[t]]$, the group scheme J is semi-abelian over $k[[t]]$. See [9, Theorem 2.4 and Proposition 2.3]. Choose a section $\sigma : \mathrm{Spec}\, k[[t]] \to X$ of the structural morphism $X \to \mathrm{Spec}\, k[[t]]$; the image of σ lies in the smooth locus $\mathrm{Sm}(X/k[[t]])$ of $X \to \mathrm{Spec}\, k[[t]]$. We may and do assume that $\sigma(\mathrm{Spec}\, k)$ lies in $X_0 \setminus E$. We can use the rational point $\sigma\big(\mathrm{Spec}\, k((t))\big)$ to obtain the morphism $X_\eta \to J_\eta$. By the universal property of Néron models this extends to a morphism

$$\mathrm{Sm}(X/k[[t]]) \longrightarrow J$$

which maps σ to the zero section.

3.12. Lemma. *Let C be a smooth projective irreducible curve over a field K. Let $c : \mathrm{Spec}\, K \to C$ be a rational point. Let $C \to J$ be the map of C into its Jacobian, mapping c to zero. Suppose $\beta \in H^1_{ét}(C, \mathbb{Z}_p)$ is a nonzero étale cohomology class such that $c^*\beta$ is zero in $H^1_{ét}(\mathrm{Spec}\, K, \mathbb{Z}_p)$. In this situation, there exists a surjective homomorphism of p-divisible groups $\varphi : J[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p$ such that the following system of finite étale coverings $\{J_n \to J\}$ corresponds to an étale cohomology class* *on J that pulls back to β:*

$$\begin{array}{ccccccccc} 0 & \to & J[p^n] & \to & J & \xrightarrow{p^n} & J & \to & 0 \\ & & \Big\downarrow{\scriptstyle \varphi[p^n]} & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{Z}/p^n\mathbb{Z} & \to & J_n & \longrightarrow & J & \to & 0 \end{array}$$

Proof. This is geometric class field theory. See for example [34, VII §2, Proposition 9]. ◻

We apply this lemma to the situation described just before the lemma. Note that $\sigma^*\beta$ is trivial as $\operatorname{Spec} k[[t]]$ has trivial fundamental group and β extends to $X \setminus E$. We get $\varphi : J_\eta[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p$, and the sequence of abelian varieties $J_{n,\eta}$. Let J be the Néron model of J_η as above, and let J_n be the Néron model of $J_{n,\eta}$. We remark that J_n is a semi-abelian scheme, as $J_{n,\eta}$ is isogenous to J_η. Our goal is to show that the map $J_n \to J$ is étale.

Recall that the p-divisible group $J_\eta[p^\infty]$ has a natural filtration $0 \subset G^f_\eta \subset J_\eta[p^\infty]$. See [24, Chapter IV, Section 2] and compare [8, Section 2]. In fact, G^f_η is the generic fiber of the p-divisible group G^f of the Raynaud extension associated to the semi-abelian scheme J over $k[[t]]$. Further, the quotient $J_\eta[p^\infty]/G^f_\eta$ is a p-divisible group which extends to an étale p-divisible group over $k[[t]]$.

By [8], the restriction of the map φ to G^f_η extends to a homomorphism of p-divisible groups $\psi : G^f \to \mathbb{Q}_p/\mathbb{Z}_p$ over $\operatorname{Spec} k[[t]]$.

There are two cases.

Case 1: ψ is not trivial. Let s be the largest integer such that ψ is divisible by p^s. Set $\psi' = p^{-s}\psi$. Then the restriction of $\psi' : G^f[p] \to \underline{\mathbb{Z}/p\mathbb{Z}}$ is a surjection of finite flat groups schemes over $k[[t]]$. In view of this, the pushout H_n in the diagram

$$\begin{array}{ccccccccc} 0 & \to & G^f[p^n] & \to & G^f & \xrightarrow{p^n} & G^f & \to & 0 \\ & & \Big\downarrow{\scriptstyle \psi'[p^n]} & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{Z}/p^n\mathbb{Z} & \to & H_n & \longrightarrow & G^f & \to & 0 \end{array}$$

is a p-divisible group over $\operatorname{Spec} k[[t]]$. We remark that, for $n \geq s$, there is a homomorphism $H_{n-s,\eta} \to J_{n,\eta}[p^\infty]$. In terms of elements this is given by class of $(a, g) \mapsto$ class of $(p^s a, g')$, where $a \in \mathbb{Z}/p^{n-s}\mathbb{Z}$, $g \in G^f_\eta$ and $g' \in G^f_\eta$ is an element such that $p^s g' = g$. It is clear that $H_{n-s,\eta} \to J_{n,\eta}[p^\infty]$ is an injection; in fact it can easily be seen that $H_{n,\eta}$ is the p-divisible part of the inverse image of G^f_η under the map $J_{n,\eta}[p^\infty] \to J_\eta[p^\infty]$.

On the other hand, let G^f_n denote the p-divisible group of the Raynaud extension associated to the Néron model J_n. Clearly, we also have that $G^f_{n,\eta}$ is the p-divisible part of the inverse image of G^f_η in $J_{n,\eta}[p^\infty]$. We conclude that $G^f_{n,\eta} \cong H_{n,\eta}$, and hence by [8] we obtain $G^f_n \cong H_n$. We conclude that the map $G^f_n \to G^f$ has kernel group scheme $\underline{\mathbb{Z}/p^{n-s}\mathbb{Z}}$ over $k[[t]]$. Thus the induced map on Raynaud extensions is étale and hence the morphism of Néron models $J_n \to J$ is étale.

Case 2: the map ψ is trivial. In this case one sees easily that $G^f_n \to G^f$ is an isomorphism. Thus the same conclusion holds. (In this case we can think of the integer s as being infinite.)

Thus the morphism $J_n \to J$ is étale, and the group $\mathbb{Z}/p^n\mathbb{Z}$ operates on J_n. This means that $J_n \to J$ is finite étale over its image. Let $J^\circ \subset J$ denote the connected Néron model. The image of $J_n \to J$ certainly contains J°. We conclude that there exists a class $\gamma \in H^1_{\acute{e}t}(J^\circ, \mathbb{Z}_p)$ which pulls back to β under the map $X_\eta \to J_\eta$.

There exists a natural number $N \in \mathbb{N}$ such that the composition

$$\mathrm{Sm}(X/k[[t]]) \longrightarrow J \xrightarrow{[N]} J$$

has image contained in J°. The pullback of γ via this map agrees with $N\beta$ on the generic fiber X_η. We conclude that the étale cohomology class α extends to the generic points of E, and hence by Lemma 3.3 we are done.

3.13. *Remark.* In the case that A has mixed characteristic we can argue in exactly the same fashion. Algebraizable now means that S is isomorphic to the formal completion in a point of an algebraic family of projective curves over $W(k)$. One can perform stable reduction for these and one reduces to a situation as in Situation 3.5, with base not $k[[t]]$, but a finite extension $R \supset W(k)$. In Subsections 3.6, 3.7 and 3.9 the reader should think of t as a uniformizer of R; then these go through with only minor modifications. Finally, in the last part of the argument, we use the result of Tate [35] on extensions of homomorphisms of p-divisible groups, instead of that of [8].

However, there is another way to prove the result in the mixed characteristic case, namely by using a result of Miki [23]. This result says that $\mathbb{Z}_p$-extensions of complete discretely valued fields of characteristic $(0, p)$ are "elementary". Let $S \to \operatorname{Spec} W(k)$ be the morphism given by the canonical map $W(k) \to A$; we assume for simplicity that $W(k)$ is algebraically closed in A. For any finite extension of dvr $W(k) \subset R$, we let $\tilde{S}_R$ denote the resolution of singularities of $\operatorname{Spec} A \otimes_{W(k)} R$. The result of Miki implies that the $\mathbb{Z}/p^n\mathbb{Z}$-coverings that define α extend to finite étale coverings of the scheme $\tilde{S}_R$ for some $R = R_n$. We leave it to the reader to see that $\pi_1(\tilde{S}_R) \cong \pi_1(\tilde{S})$ for every R. The desired result follows.

4. Stratification by Newton polygons

Let S be a locally Noetherian scheme of characteristic p, and let $\mathcal{E}$ be a nondegenerate F-crystal of rank r over S. Let $\mathcal{B}$ denote the set of Newton polygons on $[0, r]$ as described in [16, page 122]. By [16] we obtain a stratification $\{U_\beta\}$, indexed by polygons $\beta \in \mathcal{B}$, of S characterized by the following properties: (1) The points $s \in U_\beta$ are the points $s \in S$, where the Newton polygon of $\mathcal{E}$ is β. (2) Each stratum U_β is a reduced locally closed subscheme of S. Of course $S = \coprod U_\beta$. Warning: It is not true in general that the closure $\overline{U}_\beta$ of a stratum is the union of irreducible components of strata U_γ.

Our goal is to show that this stratification has all its "jumps" in codimension 1. This can be formulated in the following way: Suppose that η is a generic point of the scheme $\overline{U}_\beta \setminus U_\beta$. Then

$$\dim \mathcal{O}_{\overline{U}_\beta, \eta} = 1.$$

This follows immediately from the following result.

4.1. Theorem (Purity Theorem)**.** *Let A be a Noetherian local ring of characteristic p. Let $\mathcal{E}$ be an F-crystal over* $\operatorname{Spec} A$. *Assume that the Newton polygon of $\mathcal{E}$ is constant over* $\operatorname{Spec} A \setminus \{\mathfrak{m}_A\}$. *Then either* $\dim A \le 1$ *or the Newton polygon of $\mathcal{E}$ is constant over* $\operatorname{Spec} A$.

Proof. Let us assume that $\dim A > 1$. We will prove that $\mathcal{E}$ has constant Newton polygon over A.

Suppose that $A \to B$ is a local homomorphism of Noetherian local rings such that $\mathfrak{m}_B = \sqrt{\mathfrak{m}_A B}$, and $\dim B \ge 2$. If we prove the result for $\mathcal{E}|_{\operatorname{Spec} B}$, then the result for $\mathcal{E}$ follows. Thus we may perform the following reductions. We may replace A by the completion of A. If $\dim A > 2$, then we choose some nonzero divisor $x \in \mathfrak{m}_A$, and we see that it suffices to prove the theorem for A/xA. Thus we reduce to the case where A is a two-dimensional Noetherian complete local ring. We can find a homomorphism $A \to B$ as above, where B is a Noetherian complete local normal domain of dimension 2 with algebraically closed residue field. (E.g., if $A \cong k[[x_i]]/(f_j)$, then take B to be the normalization of a quotient of $\bar{k}[[x_i]]/(f_j)$.)

Thus we may assume that A is as in Section 3. We write $U = \operatorname{Spec} A \setminus \{0\}$. We remark that U is a 1-dimensional regular scheme.

Let K be the field of fractions of A. Let

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_s \longrightarrow \mathcal{E}|_{\operatorname{Spec} K}$$

be the slope filtration as in Claim 2.8. Let r_i denote the rank of $\mathcal{E}_i$. Then the slope filtration of the F-crystal $\bigwedge^{r_i} \mathcal{E}$ over $\operatorname{Spec} K$ starts with $\bigwedge^{r_i} \mathcal{E}_i \subset \bigwedge^{r_i} \mathcal{E}|_{\operatorname{Spec} K}$. Of course $\bigwedge^{r_i} \mathcal{E}_i$ has rank one, and an integral slope, say $n_i \in \mathbb{Z}_{\geq 0}$. We remark that the Newton polygon of $\mathcal{E}|_U$ is determined by the set of points $\{(r_i, n_i)\}$; these are the break points of the Newton polygon in question. Consider the Galois representation

$$\rho_i : \operatorname{Gal}_K \longrightarrow \operatorname{GL}_1(\mathbb{Q}_p)$$

associated to $\bigwedge^{r_i} \mathcal{E}_i$ over $\operatorname{Spec} K$. See Subsection 2.9. By assumption the Newton slopes of $\bigwedge^{r_i} \mathcal{E}$ are constant over U, and according to Proposition 2.10 this implies that ρ_i is unramified over U. Thus we can write ρ_i as a continuous representation

$$\rho_i : \pi_1(U) \longrightarrow \operatorname{GL}_1(\mathbb{Q}_p) \cong \mathbb{Z}_p \times \mathbb{F}_p^* \times \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z} \text{ if } p = 2).$$

We may replace A by the integral closure of A in a finite extension of K. Thus we may assume that the image of ρ_i lies in the factor $\mathbb{Z}_p$ for all i.

At this point we apply the main result of Section 3. Thus we see that on a resolution of singularities $\widetilde{S} \to \operatorname{Spec} A$, the representations ρ_i extend to representations $\rho_i : \pi_1(\widetilde{S}) \to \operatorname{GL}_1(\mathbb{Q}_p)$. By the equivalence of categories 2.5, this means that $\bigwedge^{r_i} \mathcal{E}_i$ extends to a rank 1 F-crystal $\mathcal{L}_i$ of slope n_i over $\widetilde{S}$. Let us write $\varphi_i : (\mathcal{L}_i)_{\operatorname{Spec} K} \to (\bigwedge^{r_i} \mathcal{E})_{\operatorname{Spec} K}$ for the nontrivial map we have by the above.

Let η denote the generic point of a component of the exceptional fiber of $\widetilde{S} \to \operatorname{Spec} A$. (We remark that if A happens to be regular, then we let $\widetilde{S}$ be the blowing up of the special point of $\operatorname{Spec} A$.) Let $\mathcal{O} = \mathcal{O}^{\wedge}_{\widetilde{S},\eta}$ be the completed local ring of $\widetilde{S}$ at η. Since $\kappa(\eta)$ is the function field of a curve over k, it has a p-basis consisting of 1 element. Therefore $\mathcal{O}$ has a p-basis; see [6, Lemma 1.1.3] for example. Thus we may apply [8, Main Theorem] to the two nondegenerate F-crystals $\bigwedge^{r_i} \mathcal{E}|_{\operatorname{Spec} \mathcal{O}}$ and $\mathcal{L}_i|_{\operatorname{Spec} \mathcal{O}}$ and the map φ_i (which lives over K and by pullback over the fraction field of $\mathcal{O}$). We obtain a nontrivial map $\mathcal{L}_i|_{\operatorname{Spec} \mathcal{O}} \to \bigwedge^{r_i} \mathcal{E}|_{\operatorname{Spec} \mathcal{O}}$. Restricting to η, we see that $(\bigwedge^{r_i} \mathcal{E})|_\eta$ contains a 1-dimensional subcrystal of slope n_i. Thus the Newton polygon of $\mathcal{E}|_\eta$ goes through or below the point (r_i, n_i); see [16, page 122]. On the other hand, by Grothendieck's specialization theorem [16, 2.3.1] it lies above the unique polygon P with break points $\{(r_i, n_i)\}$, since P is the Newton polygon of $\mathcal{E}$ over U. Thus the Newton polygon of $\mathcal{E}|_\eta$ equals P. Since η maps to 0, we derive that $\mathcal{E}|_0$ has Newton polygon P as desired. □

4.2. Question. Let G be the universal deformation of a p-divisible group G_0 over $R = k[[t_1, \ldots, t_{dd^*}]]$. We obtain a stratification $S = \operatorname{Spec} A = \bigcup U_\beta$ as above. Are the closed subschemes $\overline{U}_\beta$ set theoretic complete intersections in S?

5. Irreducibility of a catalogue of Barsotti-Tate groups

Let $\mathcal{P}$ be a moduli problem, for example moduli of elliptic curves or moduli of p-divisible groups, or moduli of finite group schemes. A *catalogue* for $\mathcal{P}$ is a family $\mathcal{X} \to T$ in $\mathcal{P}$ such that every object of $\mathcal{P}$ over an algebraically closed field k appears as the fiber of the family $\mathcal{X}$ over a point $\operatorname{Spec} k \to T$. Of course such a catalogue, if it exists, is not unique. If the moduli problem admits a fine moduli scheme (i.e. if it is representable), the pull back by any surjective morphism gives a catalogue. For elliptic curves over $\mathbb{Q}$ one can take either the universal Weierstrass elliptic curve, or the family defined by $Y^2 = X^3 + AX + B$ over $\mathbb{Q}[A, B, 1/(4A^3 + 27B^2)]$. The advantage of studying catalogues is that they often exist where moduli spaces do not, for example in the case of p-divisible groups. In addition we can ask meaningful questions about them, e.g.: Does there exist an irreducible catalogue for $\mathcal{P}$?

In this section we show that all p-divisible groups in characteristic p isogenous with a given iso-simple p-divisible group fit into an *irreducible* catalogue T. From this we derive the corollaries (4a) and (4b) mentioned in the introduction.

Let $G_{m,n}$ be the iso-simple p-divisible group introduced by Manin [20, page 35] (it is a formal group of dimension m with dual of dimension n). Our choice of T is described in Subsection 5.9. Let X denote some p-divisible group isogenous to $G_{m,n}$. For the proof of the irreducibility we show on the one hand that the Purity Theorem 4.1 implies that any component of the Newton polygon stratum in the local deformation space of X has dimension at least $r := (m-1)(n-1)/2$; see 5.16. By the Isogeny Theorem 2.17 the same holds for every component of the catalogue; see 5.20. On the other hand we can construct a stratification of T by types A of X; see 5.27. The strata with $A \neq \langle 0 \rangle$ have dimension $< r$ and the stratum with $A = \langle 0 \rangle$ is irreducible. Hence T is irreducible.

The types involved are *semi-modules*; we need some elementary combinatorial facts to describe them. Description of the strata is elementary. However, the fact that every stratum is in the Zariski-closure of the stratum with $A = \langle 0 \rangle$ seems a nontrivial fact. (And we have used the results of the previous sections to prove this.)

Acknowledgements. The combinatorial facts needed are recorded in the appendix of this section. We thank Don Zagier and Daan Krammer for helpful discussions on these combinatorial facts; especially for the suggestion of introducing Young diagrams in this situation.

5.1. We fix in this section positive integers $m, n \in \mathbb{Z}_{>0}$ which are coprime. We write

$$r := (m-1)(n-1)/2.$$

The symbol δ will denote the Newton polygon consisting of $m + n$ times the slope $n/(m+n)$.

5.2. The p-divisible group $G_{m,n}$ over $\operatorname{Spec} \mathbb{F}_p$. This is introduced in [20, page 35]. We describe $G_{m,n}$ in terms of its covariant Dieudonné module (in [20] contravariant Dieudonné module theory is used). This module can be given as

$$M_{m,n} = \mathbb{Z}_p[F, V]/\mathbb{Z}_p[F, V](F^m - V^n),$$

where $\mathbb{Z}_p[F, V]$ is the usual ring. We recall that $G_{m,n}$ is a formal group of dimension m and has dual $(G_{m,n})^t = G_{n,m}$ of dimension n. It has height $h = m + n$.

5.3. The p-divisible group $H = H_{m,n}$ over $\operatorname{Spec}\mathbb{F}_p$. In the following we will use more often the p-divisible group H which is isogenous to $G_{m,n}$. Again we describe H by its covariant Dieudonné module $M(H)$. This is a free $\mathbb{Z}_p$-module with basis $e_0, e_1, \dots, e_{m+n-1}$. We will write $e_j = p^a e_i$, for $j \in \mathbb{N}$ such that $j = i + a(m+n)$. The actions of F and V are given by $F(e_i) = e_{i+n}$ and $V(e_i) = e_{i+m}$. Note that $M(H)$ (and hence H) has an endomorphism π given by $\pi(e_i) = e_{i+1}$. Clearly, the isogeny $\pi : H \to H$ has a kernel of order p, which is isomorphic to α_p.

The reader checks immediately that this Dieudonné module is isoclinic of slope $n/(m+n)$. Since $(F^m - V^n)e_0 = 0$, the Dieudonné module generated by e_0 is isomorphic with $M(G_{m,n})$, and we have an inclusion $M(G_{m,n}) \to M(H)$. Thus H is isogenous to $G_{m,n}$ over $\mathbb{F}_p$. Choose $a, b \in \mathbb{Z}$ such that $am + bn = 1$. We observe that $\pi^{m+n} = p$ and that $F^b V^a(e_0) = e_1$.

5.4. Lemma. *For every algebraically closed field k of characteristic p we have*

$$\operatorname{End}(H_k) = W(\mathbb{F}_{p^{m+n}})[\pi],$$

where $\lambda \cdot \pi = \pi \cdot \sigma^{b-a}(\lambda)$ for $\lambda \in W(\mathbb{F}_{p^{m+n}})$, and σ is the Frobenius map. Here $\lambda \in W(\mathbb{F}_{p^{m+n}})$ acts on e_0 via multiplication by λ. The ring $\operatorname{End}(H_k)$ *is a (non-commutative) discrete valuation ring with uniformizer π and valuation given by* $\log_p \deg(-)$. *Consider the filtration $N^\bullet$ of $M(H_k) = M(H) \otimes W(k)$ given by*

$$M(H_k) = N^0 \supset \dots \supset N^j \supset \dots$$

with

$$N^j = \langle e_j, e_{j+1}, \dots \rangle = \sum_{t \geq j} W(k) \cdot e_t = \pi^j(M(H_k)).$$

For every element τ of $\operatorname{End}(H_k)$ *and every j we have $\tau(N^j) = N^{j+v}$, where $v = v(\tau)$ is the valuation of τ.*

The proof of this lemma is left to the reader. Note that the algebra $\operatorname{End}(H_k)[1/p]$ is a central simple algebra over $\mathbb{Q}_p$ of rank $(m+n)^2$. Its invariant can be computed as follows. Write $\mathbb{Q}_{p^{m+n}} := W(\mathbb{F}_{p^{m+n}})[1/p]$. In the notation of [31, page 277] the algebra $\operatorname{End}(H_k)[1/p]$ is $(\mathbb{Q}_{p^{m+n}}, \sigma^{b-a}, p)$. Note that $n(b-a) = 1 \bmod (n+m)$ and hence $(\mathbb{Q}_{p^{m+n}}, \sigma^{b-a}, p) = (\mathbb{Q}_{p^{n+m}}, \sigma, p^n)$ [31, ibid]. Hence by [31, page 338] we see that its invariant equals $n/(m+n)$.

5.5. Lemma. *Let k be an algebraically closed field of characteristic p. Let $\varphi : H_k \to X$ and $\psi : H_k \to X$ be isogenies of p-divisible groups. Then either $\varphi = \psi \circ \tau$ or $\psi = \varphi \circ \tau$ for some $\tau \in$* $\operatorname{End}(H_k)$. *If* $\deg(\varphi) = \deg(\psi)$, *then τ is an automorphism of H_k. A similar result holds for isogenies $X \to H_k$.*

Proof. Let $\beta : X \to H_k$ be any isogeny. Then both $\beta \circ \varphi$ and $\beta \circ \psi$ are endomorphisms of H_k. Since $\operatorname{End}(H_k)$ is a discrete valuation ring, either $\tau = (\beta \circ \psi)^{-1} \circ (\beta \circ \varphi) \in \operatorname{End}(H_k)$ or $\tau = (\beta \circ \varphi)^{-1} \circ (\beta \circ \psi) \in \operatorname{End}(H_k)$. The result follows. □

5.6. Canonical filtrations and the semi-module attached to a formal group. This can be found in [20, page 47].

Let k be an algebraically closed field of characteristic p. Let X be a formal group over k isogenous to $G_{m,n}$. Then there exists an isogeny $\psi : X \to H_k$. This induces an inclusion of Dieudonné modules $M(X) \subset M(H_k)$. The filtration $N^\bullet$ on $M(H_k)$

(see Lemma 5.4) induces a filtration on $M(X)$. This filtration has jumps, and we obtain a set $A_\psi \subset \mathbb{Z}$ such that

$$a \in A_\psi \iff M(X) \cap N^a \neq M(X) \cap N^{a+1}.$$

If $\psi' : X \to H_k$ is isogeny, then either $\psi' = \tau \circ \psi$ or $\psi = \tau \circ \psi'$ for some $\tau \in \mathrm{End}(H_k)$; see Lemma 5.5. Thus, by Lemma 5.4, we see that the filtration of $M(X)$ is well determined up to a shift in numbering, and that $A_\psi \subset \mathbb{Z}$ is well determined up to translation.

Note that $F : M(H_k) \to M(H_k)$ and $V : M(H_k) \to M(H_k)$ are strict for the filtration $N^\bullet$, but with a shift of n, resp. m. See Subsection 5.3. This implies that A_ψ is a semi-module (see Subsection 6.1), i.e., $a \in A_\psi \Rightarrow a+n, a+m \in A_\psi$.

We conclude that the semi-module A_ψ obtained is unique up to equivalence (translation inside $\mathbb{Z}$) once X is given. By Lemma 6.6 we see that there is a unique *admissible* semi-module A attached to this filtration. This semi-module will be called the *type* of X,

$$A = \mathrm{Type}(X).$$

5.7. Example. If $X = G_{m,n}$, we obtain the "maximal" admissible semi-module. This is the semi-module generated by 0, i.e., $A = \langle 0 \rangle := \mathbb{Z}_{\geq 0} \cdot m + \mathbb{Z}_{\geq 0} \cdot n \subset \mathbb{Z}$.

If $X = H_{m,n}$ we obtain the "minimal" admissible semi-module $A = [r, \infty)$.

5.8. We continue the discussion of the previous subsection. The fact that A is admissible means that after renumbering the filtration $N^\bullet$ on $M(H_k)$ we may assume that

$$N^0 \supset M(X) \supset N^{2r},$$

and that both inclusions have colength r over $W(k)$. Note that any N^j is isomorphic (as a Dieudonné module) to the Dieudonné module of H_k. Therefore we conclude that, for any X as above, there exist isogenies $H_k \to X$ and $X \to H_k$ of degree p^r.

5.9. In this subsection we introduce the *catalogue* that we study in the rest of this section. It is the scheme representing the following contravariant functor:

$$\begin{array}{ccc} Sch/\operatorname{Spec}\mathbb{F}_p & \longrightarrow & Sets \\ S & \longmapsto & \{(\varphi, X) \mid \varphi : H_S \to X, \ \deg(\varphi) = p^r\}/\sim . \end{array}$$

More precisely, this functor associates to S the set of isomorphism classes of isogenies φ of degree p^r with source $H_S := H \times_{\operatorname{Spec}\mathbb{F}_p} S$ and target a p-divisible group X over S. In other words, the functor associates to S the set of finite locally free closed subgroup schemes of H_S of rank p^r. (Recall $r = (n-1)(m-1)/2$.) This functor is representable by a scheme $T = T_{m,n} \to \operatorname{Spec}\mathbb{F}_p$ projective over $\operatorname{Spec}\mathbb{F}_p$. For a proof, see [17, Lemma (2.8)].

The universal object over T will be denoted $(\mathcal{G}, \Phi)$, so $\mathcal{G}$ is a p-divisible group over T and $\Phi : H_T \to \mathcal{G}$ is an isogeny of degree p^r. Note that T depends on the choice of the coprime positive integers m and n; having fixed this choice, they will be omitted from notations (otherwise we should have to write $T_{m,n}$, etc.).

5.10. Proposition. *The family $\mathcal{G} \to T$ of p-divisible groups over T is a catalogue for p-divisible groups isogenous to $G_{m,n}$: if X is a p-divisible group over an algebraically closed field k of characteristic p, and X is isogenous to $G_{m,n}$, then there exists a point $t : \operatorname{Spec} k \to T$ such that $X \cong \mathcal{G}_t$.*

Proof. This follows from Subsection 5.8. ∎

The following is the main result of this section.

5.11. Theorem. *The scheme T is geometrically irreducible of dimension r over $\mathbb{F}_p$.*

To state a consequence of the theorem, we recall the following notation. If K is a perfect field of characteristic p, and X is a group scheme over K, then the *a-number* of X is $a(X) := \dim_K \operatorname{Hom}(\alpha_p, X)$.

5.12. Corollary. *Let X be a p-divisible group over an algebraically closed field k isogenous to $G_{m,n}$. Then there exists a deformation $\mathcal{X}/k[[t]]$ of X over $k[[t]]$ such that the Newton polygon is constant, and such that $a(\mathcal{X}_\eta) = 1$.*

To deduce the corollary from the theorem, consider T_k. There is a point $t \in T(k)$ such that $\mathcal{G}_t \cong X$ by Proposition 5.10. To find a point of $T(k)$ with $a = 1$, one simply finds any p-divisible group isogenous to H with a-number 1 (as T_k is a catalogue for p-divisible groups isogenous to H). For example $G_{m,n} \otimes k$ will do. Since T_k is irreducible, we conclude that the $(a = 1)$-locus of the family $\mathcal{G}$ is an open dense subscheme U of T_k. Thus we can find a morphism $\varphi : \operatorname{Spec} k[[t]] \to T_k$ over k such that $\varphi(\eta) \in U$ and $\varphi(\operatorname{Spec} k) = t$. The pullback $\mathcal{X} = \varphi^*\mathcal{G}$ gives the desired deformation.

The following was conjectured by Grothendieck; see [12, page 150].

5.13. Corollary. *Let X be a p-divisible group over an algebraically closed field k isogenous to $G_{m,n}$. Let γ be a Newton polygon with end points $(0,0)$ and $(m+n, n)$. Then there exists a deformation $\mathcal{X}/k[[t]]$ of X over $k[[t]]$ such that the Newton polygon of $\mathcal{X}_\eta$ is γ.*

Proof. Let $\mathcal{Y}/k[[t]]$ be the deformation of X that we produced in the previous corollary. According to [6, 3.2.2, 3.2.3 and 3.2.4] we can find a deformation $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ over $k[[t, t_1, \dots, t_{mn}]]$ which is versal in the directions given by $t_1, \dots, t_{mn}$. Thus the restriction of this family of p-divisible groups to $\overline{k((t))}[[t_1, \dots, t_{mn}]]$ is the universal deformation of a p-divisible group with $a = 1$. According to [30, Corollary 3.8], the Newton polygon γ occurs in this deformation. This proves that the locally closed stratum $U_\gamma \subset \operatorname{Spec} k[[t, t_1, \dots, t_{mn}]]$ (see Section 4) is not empty. Finally, one finds a suitable map $\operatorname{Spec} k[[t']] \to \operatorname{Spec} k[[t, t_1, \dots, t_{mn}]]$ mapping the generic point into U_γ. ∎

5.14. Before we start the proof of the theorem we make some remarks on deformations of p-divisible groups. Let k be an algebraically closed field of characteristic p. Let X be a p-divisible group over k, of dimension d and with Serre dual X^t of dimension c. As is well known, the deformation problem posed by X over k is pro-representable by a formally smooth complete local k-algebra $R \cong k[[t_1, \dots, t_{dc}]]$; see [14] for example. Any p-divisible group over $\operatorname{Spf} R$ comes from a p-divisible group over $\operatorname{Spec} R$; see [6, Lemma 2.4.4] for example. Let $\mathcal{X}$ over $\operatorname{Spec} R$ be the universal deformation of X. We need the following well known result for which we were unable to find a reference.

5.15. Lemma. *The generic fiber of $\mathcal{X}$ has Newton polygon ρ equal to the Newton polygon of $(\mathbb{G}_m)^d \times (\mathbb{Q}_p/\mathbb{Z}_p)^c$ (i.e., d times slope 0 and c times slope 1 on the covariant Dieudonné module).*

Proof. By Grothendieck's specialization theorem we need only find some point of $\operatorname{Spec} R$ where $\mathcal{X}$ has the desired Newton polygon. Write $X = X^\mu \oplus X' \oplus X^{\acute{e}t}$

as the direct sum of its multiplicative part, its local-local part and its étale part. Deforming X' while keeping fixed $X^{ét}$ and X^{μ} constant, and keeping the direct sum decompositon we obtain a closed subscheme of $\operatorname{Spec} R$. Thus, if we prove the lemma for X', then the lemma follows for X.

Hence we may assume that X is local-local. We will use the display of a deformation of X. See [27], [28] and [36]. According to these references, the display of X is given by a matrix

$$(*) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with A of size $d \times d$ over $W = W(k)$ and B of size $d \times c$, etc. The universal deformation over $k[[t_{i,j}]]$ is given by the matrix

$$\begin{pmatrix} A + TC & B + TC \\ C & D \end{pmatrix},$$

where $T = ([t_{i,j}])$, and $[t_{i,j}] \in W(k[[t_{i,j}]])$ is the Teichmüller lift of $t_{i,j}$. See [28, Section 1] and [36, (2.34)]. Let A_0 and C_0 denote the reductions of A and C modulo p. The Hasse-Witt matrix of this deformation is the matrix $A_0 + (t_{i,j})C_0$, seen as a matrix over $k[[t_{i,j}]]$. We want to show that the determinant of this matrix is nonzero in $k[[t_{i,j}]]$.

If not, then this determinant is zero in $k[t_{i,j}]$ as well. This means that every matrix of the form $A_0 + TC_0$ over k has determinant zero. Clearly this can only be the case when $\operatorname{Ker}(A_0) \cap \operatorname{Ker}(C_0)$ is not zero. However, this contradicts the axiom of displays that the matrix $(*)$ is invertible over W. $\square$

5.16. Assume k, X and other notations are as in Subsection 5.14, but now let X be isogenous to $G_{m,n} \otimes k$. The Newton polygon of the covariant Dieudonné module of X is δ; see 5.1. According to Grothendieck's specialization theorem (see [16, Theorem 2.3.1]), the locus of points of $\operatorname{Spec} R$ where the Newton polygon of $\mathcal{X}$ lies above δ is a closed subset $W_\delta \subset \operatorname{Spec} R$. Remark that, since the Newton polygon in the unique closed point of $\operatorname{Spec} R$ is δ, the Newton polygon of $\mathcal{X}|_{W_\delta}$ is constant.

5.17. Proposition. *Use the notations above. The dimension of any irreducible component of W_δ is at least r.*

Proof. The fiber of $\mathcal{X}$ at the generic point of $\operatorname{Spec} R$ gives a p-divisible group with Newton polygon equal to $\rho := m \times (\text{slope } 0) + n \times (\text{slope } 1)$. See Lemma 5.15. Consider Newton polygons of height $m+n$ and dimension m, i.e., those which start at the point $(0,0) \in \mathbb{R}^2$ and end at the point $(m+n, n)$. The number of points with integral coordinates in $\mathbb{R}^2$ strictly below δ and on or above ρ equals $mn - r$, as is easy to see. Hence any chain of comparable, mutually different Newton polygons starting at δ and ending at ρ has length at most $mn - r$. Using the Purity Theorem 4.1, the result of the proposition follows. $\square$

5.18. Suppose $\mathcal{G} \to T$ and $\Phi : H_T \to \mathcal{G}$ are the catalogue as in Subsection 5.9. Let $t : \operatorname{Spec} k \to T$ be a geometric point, and let (X, φ) be the fiber at this point: $X = \mathcal{G}_t$ and $\varphi = \Phi_t : H_k \to X$. We denote by $\mathcal{O}^\wedge$ the complete local ring of T_k at t. The pullback of $\mathcal{G}$ to $\mathcal{O}^\wedge$ is a deformation of X to $\operatorname{Spec} \mathcal{O}^\wedge$. By the universal property of the deformation ring R (see 5.14), we obtain a k-algebra homomorphism $R \to \mathcal{O}^\wedge$. Of course, the (reduced) image of the associated morphism $\operatorname{Spec} \mathcal{O}^\wedge \to \operatorname{Spec} R$ is contained in W_δ. (We do not know whether T and hence $\mathcal{O}^\wedge$ is reduced.)

5.19. Proposition. *The morphism* $\operatorname{Spec}\mathcal{O}^\wedge \to \operatorname{Spec} R$ *is a closed immersion. The (reduced) image of this morphism equals* W_δ*.*

Proof. To prove that it is a closed immersion we need only show that it is unramified at the special point. This is clear: if not then we would obtain a nontrivial deformation $\varphi_A : H_A \to X_A$ of the initial map $\varphi : H_k \to X$ over an Artinian k-algebra A. By rigidity of homomorphisms of p-divisible groups this is impossible.

Let S be an irreducible component of W_δ. Note that $S = \operatorname{Spec} A'$, and that A' is an excellent domain. Thus the normalization A of A' is another complete local Noetherian k-algebra with residue field k. Let $\mathcal{Y}$ be the restriction of the p-divisible group $\mathcal{X}$ over $\operatorname{Spec} R$ to $\operatorname{Spec} A$. We may apply the Isogeny Theorem 2.17 to $\mathcal{Y}$ over $\operatorname{Spec} A$: we obtain an isogeny $\psi'_A : H_A \to \mathcal{Y}$. We may assume that the degree $p^t = \deg(\psi'_A)$ is $\geq p^r$.

Let K be the field of fractions of A, and let $\bar{K}$ be an algebraic closure of K. According to Subsection 5.8, we may factor $\psi'_{\bar{K}}$ as $\psi_{\bar{K}} \circ \pi^{t-r}$, for some $\psi_{\bar{K}} : H_{\bar{K}} \to X$ of degree p^r. This means that $\operatorname{Ker}(\pi^{t-r})_{\bar{K}} \subset \operatorname{Ker}(\psi'_{\bar{K}})$. We conclude that $\operatorname{Ker}(\pi^{t-r}) \times \operatorname{Spec} A$ is contained in $\operatorname{Ker}(\psi'_A)$, as $A \to \bar{K}$ is injective. In other words, there exists an isogeny $\psi'_A : H_A \to \mathcal{Y}$ of degree p^r, such that $\psi_A \circ \pi^{t-r} = \psi'_A$.

Let $\psi = \psi_A \otimes k$ be the special fiber of ψ_A. It may not be true that $\psi = \varphi$ (see 5.18 for the meaning of φ). However, according to Lemma 5.5, there exists an automorphism τ of H_k such that $\varphi = \psi \circ \tau$. Thus the pair $(\mathcal{Y}, \psi_A \circ \tau)$ is a deformation of (X, φ) over A. Hence, by the defining property of the moduli scheme T, we get a morphism $\operatorname{Spec} A \to T$, whose special point is t. By definition of $\mathcal{O}^\wedge$, this induces a map $\mathcal{O}^\wedge \to A$. We leave it to the reader to see that the diagram

$$\begin{array}{ccccc} \operatorname{Spec} A & \longrightarrow & \operatorname{Spec}\mathcal{O}^\wedge & \hookrightarrow & \operatorname{Spec} R \\ \downarrow & & & & \| \\ S = \operatorname{Spec} A' & \longrightarrow & W_\delta & \subset & \operatorname{Spec} R \end{array}$$

is commutative. Since S was any irreducible component of W_δ, the last statement of the proposition follows. □

5.20. Corollary. *Every component of* T *has dimension at least* r*.*

Proof. This follows trivially from the previous two propositions. □

5.21. Let K be a *perfect* field of characteristic p. Suppose $\varphi : H_K \to X$ defines a point of T over K. We are going to find standard elements in $M(X)$. Note that since K is perfect, we can use (covariant) Dieudonné module theory over K; the reader will have to check in what follows that some of the results obtained above for algebraically closed fields k remain true for perfect fields.

First, we recall from 5.8 that there exists an isogeny $\beta : X \to H_K$ of degree p^r. After changing β with an automorphism of H_K, we may assume that $\beta \circ \varphi = \pi^{2r}$; compare 5.5. Note that β is uniquely determined by this condition. Thus we obtain a canonical injection $M(\beta) : M(X) \hookrightarrow M(H_K)$ determined by (X, φ). From now on we think of $M(X)$ as a submodule of $M(H_K)$.

Any element z of $M(H_K)$ can be written uniquely in the form

$$z = [c_0]e_0 + [c_1]e_1 + [c_2]e_2 + \dots$$

with $c_j \in K$. Here $[c] = (c, 0, 0, 0, \dots)$ is the Teichmüller lift of $c \in K$. (Recall that e_i are defined for every $i \geq 0$.) Suppose we have a second element $z' \in M(H_K)$

which has the form

$$z' = [c_t]e_t + \sum_{j>t}[c'_j]e_j.$$

Then the difference $z - z'$ has the expression

$$z - z' = \sum_{0\geq j<t}[c_j]e_j + \sum_{j>t}[p_j]e_j,$$

where the $p_j \in K$ have the following shape: there exist integers $n(j)$ and certain universal polynomial expressions P_j in the elements c_i and c'_i with $t < i \leq j$ such that

$$(p_j)^{p^{n(j)}} = P_j(c_{t+1}, c'_{t+1}, \dots , c_j, c'_j).$$

We will call such an expression p_j a *quasi-polynomial expression in* $c_{t+1}, c'_{t+1}, \dots ,$ c_j, c'_j. This statement follows from the following two assertions: (1) the addition of Witt vectors is given by certain universal polynomial expressions, and (2) when we write a Witt vector $\bar{c} = (c_0, c_1, c_2, \dots)$ in the form $\bar{c} = [c_0] + p(\gamma_0, \gamma_1, \gamma_2, \dots)$ the elements $(\gamma_i)^p$ are certain universal polynomial combinations of $c_0, \dots , c_i$. In addition, one easily sees that polynomial P_j has the following shape:

$$P_j = (c_j)^{p^{n(j)}} - (c'_j)^{p^{n(j)}} + Q(c_{t+1}, c'_{t+1}, \dots , c_{j-1}, c'_{j-1}).$$

Let A be the type of X, defined as in Subsection 5.6. We say that $x \in M(X)$ is a *standard element of* $M(X)$ if for some $a \in A$

$$x = e_a + \sum_{j>a}[c_j]e_j, \tag{$*$}$$

with $c_j \in K$, and $c_j = 0$ if $j \in A$.

5.22. Lemma (See [20, Lemma 3.9]). *Let X be as above. For every $a \in A$, there exists a standard element $x_a \in M(X)$ as in $(*)$, and it is unique.*

Proof. By definition of A there exists an element $y \in M \cap N^a$, which is nonzero modulo $M \cap N^{a+1}$. Scaling gives y the form $y = e_a + \sum_{j>a}[c_j]e_j$, with $c_j \in K$.

Suppose that for some t, $a < t$, we have already that $c_{a'} = 0$ for $a < a' < t$ with $a' \in A$. Consider t and c_t. If $t \notin A$, then we may replace t by $t + 1$. If $t \in A$, then we find an element $y' \in M(X)$ such that $y' \in M(X) \cap N^t$, but $y' \notin M(X) \cap N^{t+1}$. After subtracting a suitable multiple of y' from y, we obtain a new y as above with in addition $c_t = 0$. We can continue like this until $t \geq 2r$. Then since $N^{2r} \subset M(X)$, we can subtract the tail end $\sum_{j\geq 2r}[c_j]e_j$ from y to get the desired result. $\square$

5.23. Let B be the cycle associated to the type A of X. We freely use the definitions introduced in the appendix, in particular Subsection 6.7. We are going to consider the standard elements corresponding to $t(i) \in A$; we write them as

$$x_{t(i)} = e_{t(i)} + \sum_{j>t(i)}[X_{t(i),j}]e_j.$$

Here $i \in \{1, \dots , a\}$. Similarly, we also consider the standard elements

$$x_{c(i)} = e_{c(i)} + \sum_{j>t(i)}[Y_{c(i),j}]e_j.$$

Since $\top(B) \cap \bot(B) = \emptyset$, no confusion can result from these formulae. These formulae define elements $X_{t(i),j}$ and $Y_{c(i),j}$ of K. Note that since the elements above are standard, only the pairs $(t(i), j) \in \mathcal{P}(A)$ occur, and similarly only $(c(i), j)$ belonging to $\mathcal{P}(A)$ occur.

5.24. Lemma. *Given the elements $X_{t(i),j}$ and $Y_{c(i),j}$ of K we can reconstruct the submodule $M(X) \subset M(H_K)$.*

Proof. This follows, since the elements $x_{t(i)}$ generate the Dieudonné module $M(X)$. This is proved in [20, Lemma 3.9, part b]. $\square$

In particular, the isogeny $H_K \to X$ can be defined over the perfect closure in K of the subfield of K generated by the elements $X_{t(i),j}$ and $Y_{c(i),j}$. Hence the morphism $\operatorname{Spec} K \to T$ can be defined over this perfect closure. We will assume this is the case in the following discussions.

The strategy of the proof of Theorem 5.11 is as follows. We will show that, given A, there are certain algebraic relations between the elements $X_{t(i),j}$ and $Y_{c(i),j}$ of K. In fact, we will show that the maximal number of algebraically independent elements in the set $\Gamma := \{X_{t(i),j}, Y_{c(i),j}\}$ is at most the volume of the Young diagram $\mathcal{Y}(A)$ associated to A. Thus the morphism $\operatorname{Spec} K \to T$ does not map to a generic point of T, unless $A = \langle 0 \rangle$, by Corollary 5.20. Finally, we will show that T has only one generic point by considering the case $A = \langle 0 \rangle$ in more detail.

5.25. We produce relations among the elements $X_{t(i),j}$ and $Y_{c(i),j}$ as follows. This will be an increasing induction with respect to a variable $t \in \mathbb{Z}_{>0}$.

Consider the subset $\Gamma^t \subset \Gamma$ defined by

$$\Gamma^t = \{X_{t(i),j} \mid j - t(i) < t\} \cup \{Y_{c(i),j} \mid j - c(i) < t\}.$$

Obviously we have $\emptyset = \Gamma^1 \subset \Gamma^2 \subset \ldots \subset \Gamma^{2r} = \Gamma$, as only pairs in $\mathcal{P}(A)$ occur and $A \supset [2r, \infty)$.

Pick an equivalence class $C \subset \mathcal{P}(A)$, and let i_0, i_1 be the integers associated to C by Lemma 6.10. Let t denote the integer $t = j - a$ which is fixed for elements $(a, j) \in C$.

5.26. Lemma. *The elements $X_{t(i),t(i)+t}$, $i_0 \leq i \leq i_1$, and $Y_{c(i),c(i)+t}$, $i_0 < i \leq i_1$, generate a field extension of transcendence degree at most one over the subfield of K generated by Γ^t.*

Let us assume the lemma for the moment and finish the proof of Theorem 5.11. We bound the transcendence degree of the field $\mathbb{F}_p(\Gamma) \subset K$ as follows. By induction on t we show that the transcendence degree of $\mathbb{F}_p(\Gamma^t)$ is at most equal to the number of equivalence classes $C \subset \mathcal{P}(A)$ such that $(a, j) \in C$ implies $j - a < t$. The lemma above, together with Lemma 6.10, furnishes the induction step. Hence the transcendence degree of $\mathbb{F}_p(\Gamma)$ is at most the number of equivalence classes of $\mathcal{P}(A)$. By Lemma 6.12 this is the number of elements of $\mathcal{V}(A)$; by definition of the Young diagram of A (see 6.13) this equals the volume of $\mathcal{Y}(A)$ and by Proposition 6.15, this is less than r, unless $A = \langle 0 \rangle$. We conclude that if $t : \operatorname{Spec} K \to T$ maps to a generic point, then $A = \langle 0 \rangle$.

The case $A = \langle 0 \rangle$. Here we note that there are only $X_{t(i),j}$ and no $Y_{c(i),j}$. Furthermore, every class $C \subset \mathcal{P}(A)$ has one element in this case, hence the number of $X_{t(i),j}$ equals r; see Example 6.14. Thus, if $\operatorname{Spec} K \to T$ maps to a generic point of T, then the elements $X_{t(i),j}$ are algebraically independent. We may (see above) also assume that K is the perfect closure of $\mathbb{F}_p(X_{t(i),j})$. Clearly, there is only one such K (up to isomorphism), and only one $M(X) \subset M(H_K)$ with the correct parameters $X_{t(i),j}$. This means that T has a unique generic point, hence T is irreducible. This finishes the proof of Theorem 5.11.

Proof of Lemma 5.26. We first introduce specific elements x_a for all $a \in A$. When $a \in \top(B)$, or $a \in \bot(B)$, we already have defined x_a above. If $b \in B$ and $a \notin \bot(B) \cup \top(B)$, then we can write $b = t(i) + s \cdot m$ for some i and s, $0 < s < n_i$, or we can write $b = t(i) + s \cdot n$ for some i and s, $0 < s < m_i$; see Subsection 6.7. In the first case we put $x_b = V^s(x_{t(i)})$; in the second case we put $x_b = F^s(x_{t(i)})$. If $a \in A$ and $a \notin B$, then we can write $a = b + s \cdot h$ for a unique $b \in B$ and $s > 0$. In this case we put $x_a = p^s x_b$ (since $pe_j = e_{j+h}$ this just means shifting the coefficients). The elements x_a so defined are in general *not* standard elements. However, they all have the form

$$x_a = e_a + \sum_{j>a} [Z_{a,j}] e_j,$$

where each $Z_{a,j}$ is either 0 or a p-power (negative or positive) of some $X_{t(i),j'}$, or some $Y_{c(i),j'}$ with the property that $t(i) - j' = j - a$ or $c(i) - j' = a - j$. In particular, $Z_{a,j}$ is a quasi-polynomial expression in the elements of Γ^{j-a+1}. (See Subsection 5.21 for an explanation of the term "quasi-polynomial".)

We show how to produce an algebraic relation between $X_{t(i),t(i)+r}$ and $Y_{c(i),c(i)+r}$ over $\mathbb{F}_p(\Gamma^r)$. We will leave it to the reader to establish the analogous relation between $X_{t(i),t(i)+r}$ and $Y_{c(i+1),c(i+1)+r}$. Recall from 6.7 that $c(i) = t(i) + n_i \cdot m$. Thus if we apply V^{n_i} to $x_{t(i)}$ we obtain

$$V^{n_i}(x_{t(i)}) = e_{c(i)} + \sum_{j>t(i)} [X_{t(i),j}^{p^{-n_i}}] e_{j+n_i \cdot m}.$$

If this happens to be a standard element, then by uniqueness of standard elements we deduce that this equals $x_{c(i)}$ and hence

$$Y_{c(i),c(i)+r} = X_{t(i),t(i)+r}^{p^{n_i}}.$$

In general the element $V^{n_i}(x_{t(i)})$ need not be standard. As explained in the proof of Lemma 5.22 we can modify the expression until we obtain a standard element. We claim that during this process the coefficient $X_{t(i),t(i)+t}^{p^{n_i}}$ will only be modified by adding onto it a quasi-polynomial expression in the variables in Γ^t. This will prove that

$$Y_{c(i),c(i)+t} = X_{t(i),t(i)+t}^{p^{n_i}} + \text{a quasi-polynomial expression in } \Gamma^t,$$

which is a precise version of what we wanted to prove.

Let us consider any element

$$z = e_a + \sum_{j>a} [W_j] e_j$$

of $M(X)$ such that $W_j \in \mathbb{F}_p(\Gamma^{j-a+1})$. We want to bring it into standard form. Thus, let $a' \in A$, $a < a'$, be the smallest element such that $W_{a'} \neq 0$. We form the difference

$$z' = z - [W_{a'}] x_{a'} = \left(e_a + \sum_{j>a} [W_j] e_j \right) - \left([W_{a'}] e_{a'} + \sum_{j'>a'} [W_{a'} Z_{a',j'}] e_{j'} \right).$$

Next, we write z' in the form $e_a + \sum_{j>a}[W'_j]e_j$. The reader easily checks, using the discussion in 5.21, that (1) $W'_j = W_j$ for $j < a'$, (2) $W'_{a'} = 0$, and (3) for $j > a'$, W'_j equals W_j+ a quasi-polynomial expression in $W_{a'}$ and $Z_{a',j'}$, with $a' < j' \leq j$. Note that in case (3) the elements $W_{a'}$ and $Z_{a',j'}$ have "lower level" than W_j (more precisely, they lie in the perfect closure of $\mathbb{F}_p(\Gamma^{j-a})$). By repeatedly applying this procedure, starting with $V^{n_i}(x_{t(i)})$ we arrive at the claim that was stated in the middle of the preceding paragraph. This ends the proof of Lemma 5.26. $\square$

5.27. *Remark.* Possibly the proof of the theorem becomes clearer by the following explanation. For every admissible semi-module A we define the set $U_A \subset T$ consisting of all points corresponding with a p-divisible group, isogenous with H, with semi-module equal to A. We can show that $U_A \subset T$ is locally closed in T, irreducible and of dimension equal to the volume of $\mathcal{Y}(A)$. In this way we have a stratification $T = \bigsqcup U_A$, which is implicitly used above. The group H admits a unique $H \to H$ of degree p^r; this gives the stratum defined by $A = [r, \infty)$. It is the unique zero-dimensional stratum. It is easy to see that every stratum contains this in its closure. One of the nonelementary facts seems to be to determine in which way these strata fit together. The central idea of this section is the fact that the largest stratum contains all others in its closure, plus a proof of this via the Purity Theorem.

6. Appendix, combinatorial facts on semi-modules

In this appendix we give the combinatorial definitions and facts used above. Some of these can already be found in [20, pp. 45–46]. As above, we fix coprime positive integers m and n; we write $h := n + m$ and $r := (m-1)(n-1)/2$.

For integers $a < b$ we write $[a, b] := \{a, a+1, \dots, b\}$. We write $[a, \infty) = \mathbb{Z}_{\geq a}$. For subsets $V_1, V_2 \subset V$ we write $V_1 \setminus V_2$ for the set of elements in V_1 not in V_2. For a subset $A \subset \mathbb{Z}$ and $t \in \mathbb{Z}$ we write $A + t = \{a + t \mid a \in A\}$.

6.1. A *semi-module* is a subset $A \subset \mathbb{Z}$ bounded from below such that $x \in A \Rightarrow x + m,\ x + n \in A$. We say that semi-modules A, A' are *equivalent* if one is obtained from the other by translation by an element of $\mathbb{Z}$.

A *numerical type* is a map $\delta : \{1, \dots, m+n\} \to \{+, -\}$ such that $\#\delta^{-1}(+) = m$ and $\#\delta^{-1}(-) = n$. Two numerical types are called *equivalent* if they can be obtained from each other by a cyclic permutation.

A *cycle* is a sequence $B = (b_0, b_1, \dots, b_{m+n-1})$ of integers with a partition in disjoint subsets: $B = B^+ \coprod B^-$ with $\#B^+ = m$ and $\#B^- = n$ such that for every i one of the following two conditions holds (here we set $b_{m+n} := b_0$ for convenience): (1) $b_i + n \in B$, $b_i \in B^+$, and $b_{i+1} = b_i + n$, or (2) $b_i - m \in B$, $b_i \in B^-$, and $b_{i+1} = b_i - m$. Thus the subsets B^+ and B^- are well defined by B. Cycles are called *equivalent* if they can be obtained from each other by a combination of a translation and a cyclic permutation.

6.2. We will show the three types of objects are in one-to-one correspondence up to equivalence.

The correspondence between cycles and numerical types is rather obvious. If δ is a numerical type, then we choose some $b_0 \in \mathbb{Z}$ and we inductively define $b_{i+1} = b_i + n$ if $\delta(i) = +$ and $b_{i+1} = b_i - n$ if $\delta(i) = -$. (With this definition we indeed have $b_{n+m} = b_0$.) We leave the inverse construction to the reader.

We define the *cycle* B of a semi-module A by setting $B^+ = A \setminus (m + A)$, $B^- = (m+A) \setminus (h+A)$ and $B = B^+ \cup B^-$ as sets. It is easy to show that $\#B^+ = m$ and $\#B^- = n$. The reader shows that $+n$ induces a map $B^+ \to B$ and $-m$ induces a map $B^- \to B$. Starting at any $b_0 \in B$ these operations define a walk through the set B. This walk has to close; an easy number theoretic lemma shows that such a loop has length at least $h = n + m$. Therefore, all members of B are visited in this manner and B is a cycle. Conversely, a cycle B defines a semi-module by considering the semi-module generated by it: $A = B + m\mathbb{Z}_{\geq 0} + n\mathbb{Z}_{\geq 0}$.

6.3. *Remark.* The number of equivalence classes equals $\binom{m+n}{m}/(m+n)$. This follows from the description in terms of numerical types.

6.4. A semi-module A is called *admissible* if: (1) $A \subset \mathbb{Z}_{\geq 0}$, (2) the first r elements $a_1, \ldots, a_r$ of A are all smaller than $2r$, and (3) $A = \{a_1, \ldots, a_r\} \cup [2r, \infty)$.

For any semi-module A we define the *dual* semi-module to be the set $A^t = \mathbb{Z} \setminus (2r - 1 - A) = \{b \mid b \in \mathbb{Z}, \quad 2r - 1 - b \notin A\}$. Trivially $A^{tt} = A$. Also, if A is an admissible semi-module, then A^t is an admissible semi-module (proof left to the reader).

Finally, we define the symbol $\langle 0 \rangle$ to indicate the semi-module generated by 0. So $\langle 0 \rangle = \mathbb{Z}_{\geq 0}n + \mathbb{Z}_{\geq 0}m$.

6.5. Lemma. *(i) The semi-module $\langle 0 \rangle$ is admissible. We have $2r - 1 \notin \langle 0 \rangle$.*

(ii) For an admissible semi-module A the following are equivalent: (1) $A = \langle 0 \rangle$, (2) $0 \in A$, and (3) $2r - 1 \notin A$.

Proof. The first statement is a combination of [20, Lemma 3.8] and its two corollaries. For the second: Clearly (1) implies (2). We have (2) $\Rightarrow$ (1) as $0 \in A$ implies $\langle 0 \rangle \subset A$, and the equality follows by counting. In particular, by (i), we see that $\langle 0 \rangle^t = \langle 0 \rangle$. By considering the duality $A \mapsto A^t$, we see that (3) is dual to (2). □

6.6. Lemma. *For every semi-module there is a unique equivalent semi-module which is admissible.*

Proof. Let A be a semi-module. If necessary, we translate A such that A has exactly r elements smaller than $2r$, and such that $2r \in A$. This translation is possible and it is unique. If $a \in A$ with $a < 0$, then $a + \langle 0 \rangle \subset A$ and we obtain a contradiction, using $\#\left(a + \langle 0 \rangle \cap (-\infty, 2r-1]\right) = r - a > r$. Hence $A \subset \mathbb{Z}_{\geq 0}$.

Next, we consider A^t. There are exactly r elements of A^t which are smaller than $2r$ by what we just proved. Also, $2r \in A^t$, as $-1 \notin A$. Thus we can apply the argument above to see that A^t does not contain any negative integers. These two facts imply that A is admissible. □

6.7. For a given semi-module A we define the *a-number* to be the number of generators of A, i.e., $a(A) := \#\left(A \setminus (n + A) \cup (m + A)\right)$. In the language of cycles this is the number of elements $b \in B$ such that $b + n \in B$ and $b + m \in B$. We leave the proof of this to the reader.

More definitions. Let $B = (b_0, b_1, \ldots, b_{n+m-1})$ be a cycle. Let us write

$$\top(B) = \{b \in B | b + m \in B \quad \text{and} \quad b + n \in B\}.$$

Similarly,

$$\perp(B) = \{b \in B | b - n \in B \quad \text{and} \quad b - m \in B\}.$$

In terms of the numerical type δ associated to B, we have $b_i \in \top(B)$ if $\delta(i) = +$ and $\delta(i-1) = -$, and analogously for elements of $\bot(B)$. This implies that $\#\top(B) = \#\bot(B)$. This number is the a-number a of the semi-module associated to B; see above.

We can use $\top(B)$ and $\bot(B)$ to write B in a standard form. After a cyclic renumbering, we may assume that $b_0 \in B$ is the largest element of B, so $b_0 \in \bot(B)$. This uniquely determines bijections

$$t : \{1, \cdots, a\} \to \top(B)$$

and

$$c : \{1, \cdots, a\} \to \bot(B)$$

which preserve the order in which the elements are numbered in B. (In other words, if $t(i) = b_j$ and $t(i+1) = b_{j'}$, then $j < j'$. Similarly for c; in particular $b_0 = c(1)$.) The definitions above imply that there are positive integers $n_1, \dots, n_a$ and $m_1, \dots, m_a$ such that

$$c(1) = n_1 \cdot m + t(1),\ t(1) + m_1 \cdot n = c(2),\ c(2) = n_2 \cdot m + t(2), \dots,$$

and $t(a) + m_a \cdot n = c(1)$. We remark that $n_1 + \ldots + n_a = n$, and $m_1 + \ldots + m_a = m$. Such sequences of integers give us yet another way of finding cycles. Finally, any element b of B can be written uniquely as $t(i) + s \cdot m$ for some i and some s, with $0 \le s \le n_i$ or as $t(i) + s \cdot n$ for some i and some s, $0 \le s \le m_1$.

Here is the way we picture our cycle:

$$\begin{array}{ccccccccccc} & t(1) & & & t(2) & & \cdots & & t(a) & & \\ \swarrow & & \searrow & \swarrow & & \searrow & \cdots & \swarrow & & \searrow & \\ c(1) & & & c(2) & & & \cdots & c(a) & & & c(1) \end{array}$$

The south-west arrows indicate adding various multiples of m, and the south-east arrows indicate adding multiples of n. We think of the rest of A as lying "below" this picture.

6.8. Lemma. *Let $b_0 \in B$ be the largest element of B as above. Then $[b_0, \infty) \subset A$.*

Proof. Let x be an integer, $x \ge b_0$. By definition of B^+ there is an element $b \in B^+$ such that $x - b$ is divisible by m. By our choice of b_0, we see that $s = (x-b)/m$ is a nonnegative integer. Thus $x = b + s \cdot m \in A$. $\square$

6.9. Suppose given a semi-module A. We denote

$$\mathcal{P}(A) := \{(a, j) \mid a \in A,\ \ j \in \mathbb{Z} \setminus A,\ \ a < j\}.$$

On $\mathcal{P}(A)$ we introduce an equivalence relation $\sim$. It is the smallest generated by the following two implications: $(a, j),\ (a+n, j+n) \in \mathcal{P}(A) \Rightarrow (a, j) \sim (a+n, j+n)$, and $(a, j),\ (a+m, j+m) \in \mathcal{P}(A) \Rightarrow (a, j) \sim (a+m, j+m)$. We write $C(a, j) \subset \mathcal{P}(A)$ for the equivalence class containing (a, j).

An equivalence class $C \subset \mathcal{P}(A)$ is mapped to A by the map $q : C \to A$ given by $q(a, j) = a$. Note that q is injective as the difference $k = j - a$ is fixed for (a, j) in C. Hence $C = \{(a, a+k) | a \in q(C)\}$. If $a \in A$ and $a + n \in q(C)$, then $a \in q(C)$. Similarly, if $a \in A$ and $a + m \in q(C)$, then $a \in q(C)$.

6.10. Lemma. *Let C be an equivalence class as above. Then there exist $1 \le i_0 \le i_1 \le a$ such that $t(i) \in q(C)$ if and only if $i_0 \le i \le i_1$, and $c(i) \in q(C)$ if and only if $i_0 < i \le i_1$. (Notations used are as in Subsection 6.7.)*

Proof. According to the remarks made before the lemma, for any element $a \in q(C)$, if $a - n$ (or $a - m$) is in A, then it lies in $q(C)$. Thus we see that $q(C) \cap \top(B)$ is not empty. Let i_0, resp. i_1, be the smallest, resp. the largest, index in $\{1, \dots, a\}$ such that $t(i_0)$, resp. $t(i_1)$, lies in $q(C)$.

Suppose that $c(i) \in q(C)$. By Lemma 6.8 $c(1) \notin C$, so $i \neq 1$. We have $t(i-1) = c(i) - m_{i-1} \cdot n$ and $t(i) = c(i) - n_i \cdot m$. This implies $t(i-1), t(i) \in q(C)$. Hence $i_0 < i \leq i_1$. We have shown the "only if" clause.

By definition of the equivalence relation on $\mathcal{P}(A)$, there exists a sequence $t(i_0) = a_0, a_1, a_2, \dots, a_{s-1}, a_s = t(i_1)$ of elements of C such that for each index $0 \leq j < s$ we have one of the following four possibilities $a_j = a_{j+1} + n$, $a_j = a_{j+1} + m$, $a_j = a_{j+1} - n$, or $a_j = a_{j+1} - m$. We may suppose (by "abbreviating") that this sequence has no repetitions. Suppose that for some $s > j \geq 1$ we have $a_{j-1} = a_j - n$ and $a_{j+1} = a_j - m$. In this case we have $a_j - m - n \in A$ unless $a_{j+1} \in \perp(B)$ (left to the reader). If $a_j - n - m \in A$, then we replace a_j by $a_j - n - m$ which is also in $q(C)$. A similar replacement can be performed if $a_{j-1} = a_j - n$, $a_{j+1} = a_j - m$ and $a_j - n - m \in A$. Continuing this process, we see that after a finite number of steps, all the local maxima in the sequence $a_0, \dots, a_q$ are elements of $\perp(B)$. This implies that the sequence is part of the picture in 6.7: it is contained in B, and the sequence of local maxima of $a_0, \dots, a_q$ is either $c(i_0+1), c(i_0+2), \dots, c(i_1)$ or $c(i_0), c(i_0-1), \dots, c(1), c(a), \dots, c(i_1+1)$. The second possibility is excluded by what was said above. This proves the "if" clause. $\square$

6.11. We write

$$\mathcal{V}(A) := \{(v, f) \mid f \in B^+, \quad v \in B^-, \quad v < f\}.$$

6.12. Lemma. *The natural map*

$$\mathcal{V}(A) \to \mathcal{P}(A)/\!\sim$$

given by $(v, f) \mapsto C(v - m, f - m)$ *is bijective.*

Proof. The map is well defined: $f \in B^+$ implies that $f - m \notin A$, and $v \in B^-$ implies that $v - m \in A$. We will construct the inverse to this map. Let $C \subset \mathcal{P}(A)$ be an equivalence class, and let $i_0 \leq i_1$ be the integers of Lemma 6.10. Let $s \geq 0$ be the largest integer such that $t(i_0) + s \cdot m \in q(C)$. Since $c(i_0) \notin q(C)$, we have $s < n_{i_0}$. Thus $v = t(i_0) + (s+1) \cdot m$ is an element of B. It is trivial to check that $v \in B^-$. We take $f = v + k$, where k is the constant difference $k = j - a$, for $(a, j) \in C$. We leave it to the reader to check that $f \in B^+$. (We remark that in the picture of 6.7, the point v is the left-most-point of $q(C) \cap B$.) $\square$

6.13. We think of a *Young diagram* as a finite subset $Y \subset \mathbb{N} \times \mathbb{N}$ such that $(x, y) \in Y$ and $1 \leq i \leq x$, $1 \leq j \leq y$ imply $(i, j) \in Y$. The volume of Y is the number of elements of Y.

Given a semi-module A, we may write

$$B^+ = \{f_m, \dots, f_1\}, \quad f_m < \dots < f_2 < f_1.$$

The *Young diagram* $\mathcal{Y}(A)$ *associated with* A is the Young diagram with columns for $x = 1, \dots, m$ whose heights are given by $y = \#\{v \in B^- | v \leq f_x\}$. By definition the volume of $\mathcal{Y}(A)$ is equal to the number of elements of $\mathcal{V}(A)$.

6.14. Example. In the notions defined above there is a "maximal" Young diagram (why this is maximal will be specified). This is the case when $A = \langle 0 \rangle$. In this case

$B^+ = \{0, n, 2n, \dots, (m-1)n\}$, and $B^- = \{mn, mn-m, mn-2m, \dots, m\}$. The Young diagram is given as follows: the height of the column x is

$$[(m-x)\cdot n/m]$$

(where [ ] denotes the integral part). We leave it to the reader to see that the volume equals $r = (n-1)(m-1)/2$ in this case.

(min) Consider the admissible semi-module $A = [r, \infty) = \mathbb{Z}_{\geq r}$. In this case $B^+ = [r, r+n-1]$, $B^- = [r+n, r+n+m-1]$. The Young diagram given by this is empty and has volume equal to zero.

6.15. Proposition. *Let A be a semi-module. Then its Young diagram is contained in the maximal one: $\mathcal{Y}(A) \subset \mathcal{Y}(\langle 0 \rangle)$. Moreover, $\mathcal{Y}(A) = \mathcal{Y}(\langle 0 \rangle) \iff A \sim \langle 0 \rangle$.*

Proof. We define a function $g : [0, n+m] \to \mathbb{R}$, where $[0, n+m] \subset \mathbb{R}$ denotes the usual interval. Let $B = \{b_0, \dots, b_{n+m-1}\}$ as in Subsection 6.7. We require that g is piecewise linear with breakpoints only at integers, and such that $g(i) = b_i$ and $g(n+m) = b_0$. Thus the graph of g has only two slopes, namely $+n$ and $-m$: if $b_i \in B^+$, then the slope is $+n$ on the interval $[i, i+1]$, and if $b_i \in B^-$, then the slope is $-m$ on the interval $[i, i+1]$. We may also assume that b_0 is the largest element of B (as in 6.7). If $g(i) = g(j)$ for some integers $0 \leq i < j \leq n+m$, then $i = 0$ and $j = n+m$. (The reader will be able to follow the arguments much more easily after visualizing the graph of g by making some pictures.)

We choose an integer t, $1 \leq t \leq m$. Let $I_t \subset [0, n+m]$ be the part of the interval where the graph of g lies below f_t, so $I_t = \{x \mid g(x) \leq f_t\}$. Let I_t^+ be the part of I_t where the slope of g is positive $(+n)$, and let I_t^- be the part where the slope is negative $(-m)$. By the description of the slopes above, we see that the length of I_t^+ (i.e., its measure) is at most the number of elements of $b_i \in B^+$ with $b_i < f_t$, i.e., $m(I_t^+) \leq (m-t)$. We leave it to the reader to show the trivial equality:

$$m(I_t^-) \cdot m = m(I_t^+) \cdot n.$$

On the other hand, by the description of the slopes of g, we see that the number of $i \in I_t$ with $b_i \in B^-$ is $\leq m(I_t^-)$. Putting all of this together we obtain:

$$\#\{v \in B^- | v \leq f_t\} \leq \left[\frac{(m-t)n}{m}\right].$$

This proves the first assertion.

The second assertion means the following. If we have equality in the last displayed equation for all $1 \leq t \leq m$, then the graph of g is an inverted triangle. Assume that equality holds for all t. Suppose that for some i, $2 \leq i \leq n+m-2$, we have $b_{i-1} + n = b_i$, and $b_i = b_{i+1} + m$. We claim that there exists a t such that $b_{i-1} < f_t < b_i$. To prove this look at the elements $\{b_{i+1}, b_{i+2}, \dots, b_{n+m-1}\}$. Let j be the first index in $\{i+1, i+2, \dots, n+m-1\}$ such that $b_j \in B^+$; note that such an index exists and that $b_j \leq b_{i+1} < b_i$. There are two possibilities: either $b_{i-1} < b_j$, in which case $f_t = b_j$ works, or $b_{i-1} > b_j$. In the last case, the series of elements $b_j, b_{j+1}, \dots, b_{n+m-1}, b_{n+m} = b_0$ starts below b_{i-1} and ends in b_0 which is above $b_i = b_{i-1} + n$. The only upward steps in the series are steps of size $+n$ and the interval $[b_{i-1}, b_i]$ has size n also. Then the pigeon hole principle says that some j', $j < j' < n+m$, exists, with $b_{i-1} < b_{j'} < b_i$, and $b_{j'+1} = b_{j'} + n$. Thus $b_{j'} \in B^+$ and we have proved the claim.

Let $f_t = b_j$ be the element of B^+ that exists by the claim above. There is another cycle B' which is given by $B' = \{b'_0 = b_0, \dots, b'_{i-1} = b_{i-1}, b'_i = b_i - n - m, b'_{i+1} =$

$b_{i+1}, \ldots, b'_{n+m-1} = b_{n+m-1}\}$. The only elements that changed signs are b'_{i-1}, which is now a $-$, and b'_i, which is now a $+$. Thus the element b_j will still be listed as f_t in the enumeration scheme of Subsection 6.13 (because the number of elements of B^+ above it has not changed). However, the number of elements of $(B')^-$ below $f_t = b_j$ has increased by 1, as the element b'_i has been added. However, this is impossible as by assumption the number of elements of B^- below f_t was already maximal. This means that an i as in the third sentence of the previous paragraph doesn't exist, and hence the graph of g is an inverted triangle. □

6.16. *Remark.* We have shown that the set of cycles defined by the numbers m, n maps to the set of Young diagrams contained in $\mathcal{Y}(\langle 0 \rangle)$. Both sets have the same cardinality. It seems plausible that this map is bijective. A "better" proof of the previous proposition would be to construct from every such Young diagram Y a cycle B such that this defines an inverse to $B \mapsto \mathcal{Y}(A)$; we do not know whether this is possible.

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