Putting Risk in its Proper Place

Louis Eeckhoudt and Harris Schlesinger*

January 7, 2005

Abstract

This paper examines preferences towards particular classes of lottery pairs. We show how concepts such as prudence and temperance can be fully characterized by a preference relation over these lotteries. If preferences are defined in an expected-utility framework with differentiable utility, the direction of preference for a particular class of lottery pairs is equivalent to signing the n^{th} derivative of the utility function. What makes our characterization appealing is its simplicity, which seems particularly amenable to experimentation. (*JEL* D81)

Keywords: Properness, prudence, risk apportionment, risk aversion, stochastic dominance, temperance, utility premium

*Eeckhoudt: Facultes Universitaires Catholiques de Mons (FUCAM), Chaussee de Binche 151, Mons 7000, Belgium, and CORE (e-mail: louis.eeckhoudt@fucam.ac.be). Schlesinger: Department of Finance, University of Alabama, Tuscaloosa AL 35487-0224 and CESifo (e-mail: hschlesi@cba.ua.edu). This paper was partly written while both authors were visiting at the IDEI, University of Toulouse. Financial support from the French Federation of Insurance Companies (FFSA) is gratefully acknowledged. We thank seminar participants at the EGRIE meeting in Zurich, CORE, DELTA, the University of Bologna, the University of Pennsylvania and Rice University for many useful comments on a draft version of this paper. Detailed comments from Georges Dionne, Neil Doherty, Christian Gollier, Miles Kimball, Achim Wambach, Claudio Zoli and two anonymous referees were especially helpful.

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The concept of risk aversion has long been a cornerstone for modern research on the economics of risk. Ask several economists to define what it means for an individual to be risk averse and you are likely to get several different answers. Some, assuming an expected-utility framework, will say that the second derivative of the von Neumann-Morgenstern utility function u is concave or, assuming differentiability, that $u'' < 0$. Others might define risk aversion in a more general setting, equating it to an aversion to mean-preserving spreads, as defined by Michael Rothschild and Joseph E. Stiglitz (1970). It is not likely that one would define risk aversion via some behavioral consequence, such as the propensity to purchase full insurance at an actuarially-fair price.

Although somewhat newer, the concept of "prudence" and its relationship to precautionary savings also has become a common and accepted assumption.[1] Ask someone to define what it means for the individual to be "prudent" and they might say that marginal utility is convex, $u''' > 0$, but they also might define prudence via behavioral characteristics. For example, Christian Gollier (2001 p. 236), defines an agent as prudent "if adding an uninsurable zero-mean risk to his future wealth raises his optimal saving." In other words, unlike the case with risk aversion, prudence is often defined via an optimizing type of behavior, rather than some type of more primitive trait.[2]

More recently, some new concepts have entered the literature such as "temperance" ($u^{iv} < 0$) and "edginess" ($u^{v} > 0$), which arise as necessary and/or sufficient conditions for various behavioral results.[3] But what exactly are these concepts and what do they imply about one's preference towards risk?

Within an expected-utility framework, in contrast to ordinal utility, the sign of every derivative of the von Neumann-Morgenstern utility function u has some economic meaning. In this paper, we derive a class of lottery pairs such that the direction of preference between these lotteries is equivalent to signing the n^{th} derivative of utility. The lotteries themselves are particularly simple, involving equal likelihoods for all outcomes, which would seem particularly amenable to experimentation. Moreover, since the signs of the first n derivatives of utility are well-known to coincide with a preference for n^{th}-degree stochastic dominance, our lottery preferences also coincide with stochastic-dominance preference.

Although our results are interpreted in this paper in a context of preferences towards risk, it turns out that they can be given other economic interpretations. The most direct application is likely in the area of income distribution, where concepts such as "inequality aversion" and "aversion to downside inequality" have been employed for some time. See for example the papers by Anthony B. Atkinson (1970) and by Anthony F. Shorrocks and James E. Foster (1987). Our results are also relevant to the literatures on the competitive firm under price uncertainty, labor supply, auctions and portfolio choice.[4]

Justifying the sign of higher order derivatives can often meet with skepticism. For example, Miles Kimball's (1993) "standard risk aversion," which has been shown to have many implications, is becoming a more common assumption in the literature. This condition requires $u^{iv} \leq (u''')^2/u'' < 0$, yet the weaker condition of temperance, $u^{iv} < 0$, typically is met with skepticism.

Our goal in this paper is to provide a set of natural conditions regarding behavior towards risk, in the form of a preference relation between pairs of simple lotteries. In particular, we start out by assuming that an individual dislikes two things: a certain reduction in wealth and adding a zero-mean independent noise random variable to the distribution of wealth. We define "prudence," for example, as a type of preference for disaggregation of these two untoward events. We define "temperance" in a similar manner, except we replace the certain reduction in wealth with a second independent zero-mean risk. Temperance is defined as preference for disaggregating these two independent risks. We then extend and generalize these concepts by nesting the above types of lotteries. By defining our set of preferences over lotteries, we provide relatively simple behavioral characterizations of the mathematical assumption that the derivatives of the utility function are alternating in sign: $sgn\ u^{(n)} = sgn\ (-1)^{n+1}$ for all positive integers n. This describes the class of so-called "mixed risk averse" utility functions, as defined by Jordi Caballé and Alexey Pomansky (1996), a class which includes most all of the commonly used von Neumann-Morgenstern utility functions.[5]

Our "tool" in deriving these results is the utility premium, measuring the degree of "pain" involved in adding risk. Although this measure actually predates more formal analyses of behavior under risk, as pioneered by Kenneth J. Arrow (1965) and John W. Pratt (1964), it has been largely ignored in the literature.[6]

The following section defines preferences over lotteries that correspond to prudence and temperance. We then show how these behaviors can be "nested" into compound lotteries to yield particular types of rational behavior that we term "risk apportionment." Next, we prove how these concepts coincide with those of mixed risk aversion within an expected-utility framework. Finally, we discuss how our results fit in with several results in the literature.

1 Prudence and Temperance

We consider two basic "building blocks" for our analysis. The first is a sure reduction in wealth of arbitrary size k, $k > 0$. The second is the addition of a zero-mean random variable $\widetilde{\varepsilon}$, where $\widetilde{\varepsilon}$ is assumed to be non-degenerate and to be independent of any other random variables that may be present in an individual's initial wealth allocation. We let x denote the individual's initial wealth, where x is arbitrary in size, $x > 0$. We assume x is non-random for simplification, although initial wealth may be random so long as a random $\widetilde{x}$ is statistically independent of $\widetilde{\varepsilon}$. We also assume that random wealth is constructed in such a way as to have its support contained within a range of well-defined preferences.[7]

In order to avoid mathematical nuances, we only consider weak preference relations in this paper.[8] For any two lotteries A and B, we use the notation $B \succcurlyeq A$ to denote the individual's preference relation "lottery B is at least as good as lottery A."

We define preferences as monotonic if $x \succcurlyeq x - k \ \ \forall x$ and $\forall k$. We define preferences to be risk averse if $x \succcurlyeq x + \widetilde{\varepsilon} \ \ \forall x$ and $\forall \widetilde{\varepsilon}$. While not necessary for our definition of risk aversion, one usually thinks of monotonicity as jointly holding. However, it is certainly possible to desire as little wealth as possible and still be risk averse.

To keep the notation consistent, define the "lottery" B_1 as $B_1 = [0, 0]$ and the "lottery" A_1 as $A_1 = [-k, -k]$, where all simple lotteries are assumed to have an equal probability for each outcome. Similarly, define the "lotteries" B_2 and A_2 as $B_2 = [0, 0]$ and $A_2 = [\widetilde{\varepsilon}, \widetilde{\varepsilon}]$. Thus, we can define preferences as being monotone if $B_1 \succcurlyeq A_1$ and as being risk averse if $B_2 \succcurlyeq A_2$ for all initial wealth levels x and for all k and all $\widetilde{\varepsilon}$.

1.1 Prudence

Prudence is defined within expected-utility confines by Kimball (1990), who shows it is analogous to a precautionary-savings motive in a particular type of consumption/savings model. We define prudence in this paper as a type of natural preference over simple lotteries. Later, we will show how this definition coincides with Kimball's characterization.[9]

Definition 1: *An individual is said to be prudent if the lottery* $B_3 = [-k;\ \widetilde{\varepsilon}]$ *is preferred* *to the lottery* $A_3 = [0,\ \widetilde{\varepsilon} - k]$*, where all outcomes of the lotteries have equal probability, for all initial wealth levels* x *and for all* k *and all* $\widetilde{\varepsilon}$*.*

Thus, prudence shows a type of preference for disaggregation of a sure loss of size k and the addition of a zero-mean random variable $\widetilde{\varepsilon}$. If preferences are also monotonic and risk averse, the individual prefers to receive one of the two "harms" for certain, with the only uncertainty being about which one is received, as opposed to a 50-50 chance of receiving both "harms" simultaneously or receiving neither. Borrowing terminology from Kimball (1993), the above property implies that $-k$ and $\widetilde{\varepsilon}$ are "mutually aggravating" for all initial wealth levels x and for all k and all $\widetilde{\varepsilon}$.

We also can interpret prudence as type of "location preference" for one of the harms within a lottery. In particular, consider the lottery $[0;\ -k]$. Now suppose the individual is told that she must accept a zero-mean random variable $\widetilde{\varepsilon}$, but she only must receive it in tandem with one of the two lottery outcomes. The prudent individual will always prefer to attach the risk $\widetilde{\varepsilon}$ to the better outcome 0, rather than to the outcome $-k$. This characterization already has been noted by Louis Eeckhoudt, et al. (1995) and essentially follows from the earlier work of Hanson and Menezes (1971). In a sense, we are more willing to accept an extra risk when wealth is higher, rather than when wealth is lower. Indeed, this logic helps to explain why someone opts for a higher savings when second period income is risky in a two-period model. The resulting higher wealth in the second period helps one to cope with the additional risk, exactly as in Kimball (1990), who uses prudence as equivalent to a precautionary demand for savings.

Equivalently, we can start from the lottery $[0; \widetilde{\varepsilon}]$ and define prudence as a preference for attaching the harm $-k$ to the outcome 0, rather than to the outcome $\widetilde{\varepsilon}$. Again, we prefer to attach the new harm to the better of the two lottery outcomes, where the interpretation of "better" here assumes risk aversion.

1.2 Temperance

We now add a second zero-mean random variable. Let $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$ denote these two zero-mean random variables. We assume that $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$ are statistically independent of each other as well as independent from other random variables that might be owned by the individual.

Definition 2: *An individual is said to be temperate if the lottery* $B_4 = [\widetilde{\varepsilon}_1;\ \widetilde{\varepsilon}_2]$ *is preferred to the lottery* $A_4 = [0;\ \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2]$, *where all outcomes of the lotteries have equal probability, for all initial wealth levels* x *and for all* $\widetilde{\varepsilon}_1$ *and* $\widetilde{\varepsilon}_2$.

Thus, temperance shows a type of preference for disaggregation of the two independent zero-mean random variables. Temperance, as defined above, also can be interpreted as a type of location preference for adding a second independent zero-mean risk to the lottery $[0;\ \widetilde{\varepsilon}_2]$. Suppose the individual must accept a second zero-mean random variable $\widetilde{\varepsilon}_1$, but she only must receive it in tandem with one of the two lottery outcomes. The temperate individual will always prefer to attach the second risk $\widetilde{\varepsilon}_1$ to the better outcome 0, rather than to the worse outcome $\widetilde{\varepsilon}_2$. This means that we must dislike the risk $\widetilde{\varepsilon}_1$ more in the presence of $\widetilde{\varepsilon}_2$. The risks $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$ are "mutually aggravating" in the terminology of Kimball (1993).

2 Generalizing Prudence and Temperance

Let $\{\widetilde{\varepsilon}_i\}$ denote an indexed set of zero-mean non-degenerate random variables, $i = 1, 2, 3, ...$, where we assume that the $\widetilde{\varepsilon}_i$ are all mutually independent and that the $\widetilde{\varepsilon}_i$ are also independent of any existing risks in an individual's wealth. We assume throughout this paper that all lotteries have equally-likely outcomes. We now generalize the concepts of prudence and of temperance as a type of preference for disaggregation of the "harms" $-k$ and $\widetilde{\varepsilon}_i$.

2.1 Risk Apportionment

If C denotes a lottery, we can think of this lottery as essentially defining a random variable. In particular, the lottery C generates a probability distribution over wealth outcomes. As a matter of notation, if $\widetilde{y}$ denotes a random variable that is independent of C , we let $\widetilde{y}+C$ denote the sum of the random variables.[10]

As a matter of terminology, we will say that preferences satisfy risk apportionment of order 1 if they are monotonic, i.e. if $B_1 \succcurlyeq A_1$. If preferences are risk averse, so that $B_2 \succcurlyeq A_2$, we say that preferences satisfy risk apportionment of order 2. In a similar manner we define risk apportionment of order 3 as the equivalence of prudence, $B_3 \succcurlyeq A_3$, and risk apportionment of order 4 as the equivalent of temperance, $B_4 \succcurlyeq A_4$. To define risk apportionment of higher orders, we proceed iteratively.[11]

2.1.1 Risk Apportionment of orders 5 and 6

We define risk apportionment of order 5, RA-5, as follows:

Definition 3: *Assume that outcomes of the lotteries below all have equal probability. Preferences are said to satisfy risk apportionment of order 5 if, for all initial wealth levels* x *and for all* k, $\widetilde{\varepsilon}_1$, $\widetilde{\varepsilon}_2$ *and* $\widetilde{\varepsilon}_3$, *the lottery* $B_5 = [0 + A_3;\ \widetilde{\varepsilon}_2 + B_3]$ *is preferred to the lottery* $A_5 = [0 + B_3;\ \widetilde{\varepsilon}_2 + A_3]$. *Preferences satisfy risk apportionment of order 6 if the lottery* $B_6 = [0 + A_4;\ \widetilde{\varepsilon}_3 + B_4]$ *is preferred to the lottery* $A_6 = [0 + B_4;\ \widetilde{\varepsilon}_3 + A_4]$.

This definition does not require risk apportionment of lower orders. But if we have risk aversion, then we know that $0 \succcurlyeq \widetilde{\varepsilon}_2$, and if we have prudence, then we know that $B_3 \succcurlyeq A_3$. We can thus interpret risk apportionment of order 5 as a preference location for adding the risk $\widetilde{\varepsilon}_2$: Given that we must add $\widetilde{\varepsilon}_2$ to one of the outcomes in the lottery $[B_3;\ A_3]$, we would prefer to add it to the better outcome B_3. Similarly, if we have risk aversion, then we know that $0 \succcurlyeq \widetilde{\varepsilon}_3$, and if we have temperance, then we know that $B_4 \succcurlyeq A_4$. We also can interpret risk apportionment of order 6 as a preference location for adding the risk $\widetilde{\varepsilon}_3$: Given that we must add $\widetilde{\varepsilon}_3$ to one of the outcomes in the simple lottery $[B_4;\ A_4]$, we would prefer to add it to the better outcome B_4. We illustrate A_6 and B_6 and how they relate to risk apportionment of order 6 in Figure 1. Risk apportionment of order 5 is easily illustrated in a similar manner.

INSERT FIG 1 ABOUT HERE

Assuming reduction of compound lotteries, it is trivial to verify that k, $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$ are all interchangeable wherever they appear in lotteries A_5 and B_5. Likewise, we can replace $-k$ with $\widetilde{\varepsilon}_3$ in any formulation of A_5 and B_5 to obtain A_6 and B_6 respectively.

2.2 Risk apportionment of order n

Given the definitions $B_1 = B_2 = [0]$, $A_1 = [-k]$ and $A_2 = [\widetilde{\varepsilon}_1]$ we can iterate on the definitions above to define risk apportionment of order n. First, we define the appropriate lotteries:

Definition 4: *Assume that the outcomes of all lotteries A_i and B_i as listed here have equal probabilities. Further assume that $k > 0$ and that all $\widetilde{\varepsilon}_i$ are mutually independent with a zero mean. Let $Int(y)$ denote the greatest-integer function, i.e. the greatest integer not exceeding the real number y. Then for each $n \geq 3$ we define the following lotteries*:

$$A_n = [0 + B_{n-2};\ \widetilde{\varepsilon}_{Int(n/2)} + A_{n-2}].$$

$$B_n = [0 + A_{n-2};\ \widetilde{\varepsilon}_{Int(n/2)} + B_{n-2}].$$

We now can define risk apportionment for the general case:

Definition 5: *Preferences are said to satisfy risk apportionment order n if, for the* *lotteries A_n and B_n as defined above, the individual always prefers B_n: $B_n \succcurlyeq A_n$.*

For example, consider risk apportionment of order 8. This is a preference of $B_8 \succcurlyeq A_8$, where $B_8 = [0 + A_6;\ \widetilde{\varepsilon}_4 + B_6]$ and $A_8 = [0 + B_6;\ \widetilde{\varepsilon}_4 + A_6]$. Suppose we start from the lottery $[0,\ \widetilde{\varepsilon}_4]$ and we are told that we must add A_6 to one outcome and add B_6 to the other outcome, where A_6 and B_6 are as illustrated in Figure 1. Risk apportionment of order 8 would indicate a preference for attaching the more preferred lottery B_6 to the less preferred outcome $\widetilde{\varepsilon}_4$.

3 Utility Equivalence

In this section, we show how risk apportionment coincides with very particular conditions on the utility function, u, within an expected-utility framework. We assume that u is continuously differentiable over the domain of wealth. The approach we use here is quite a direct use of the utility premium. Lottery B is preferred to lottery A if and only if it causes less pain when added to any initial wealth level x. Since all of our risks are assumed to be mutually independent as well as independent of any risks inherent in initial wealth, it would not matter if we allowed $\tilde{x}$ to be random. For the sake of simplicity, we only consider nonrandom x values below.[12]

3.1 Some Properties of the Utility Premium

Let $\{\tilde{\varepsilon}_i\}$ denote an indexed set of mutually-independent zero-mean random variables. We assume that each $\tilde{\varepsilon}_i$ is a non-degenerate random variable, i.e. $\tilde{\varepsilon}_i$ has a non-zero variance. We define the utility premium for the risk $\tilde{\varepsilon}_1$ at wealth level x as

$$w_1(x) \equiv Eu(x + \tilde{\varepsilon}_1) - u(x). \tag{1}$$

Note that we define the utility premium as the *gain* in expected utility from adding the zero-mean risk $\tilde{\varepsilon}_1$ to wealth x.[13]

By our definition, the utility premium is negative if and only if preferences are risk averse,

$$w_1(x) \equiv Eu(x+\tilde{\varepsilon}_1) - u(x) \leq 0 \;\; \forall x \;\text{ if and only if }\; u'' \leq 0. \tag{2}$$

Similarly, it follows trivially from Jensen's inequality that

$$w_1'(x) \equiv Eu'(x+\tilde{\varepsilon}_1) - u'(x) \geq 0 \;\; \forall x \;\text{ if and only if }\; u''' \geq 0 \tag{3}$$

and

$$w_1''(x) \equiv Eu''(x+\tilde{\varepsilon}_1) - u''(x) \leq 0 \;\; \forall x \;\text{ if and only if }\; u^{iv} \leq 0. \tag{4}$$

Thus we see that w_1 as defined here is increasing and concave whenever $u''' \geq 0$ and $u^{iv} \leq 0$. In other words, w_1 exhibits the properties of a risk-averse utility function on its own. Of course these properties coincide with prudence and temperance in the expected-utility literature. We next show that they are equivalent to our definitions of prudence and temperance from the previous section.

3.2 Prudence and Utility

Condition (3) is equivalent to our definition of prudence, since we can allow our sure reduction in wealth, $-k$, to be arbitrarily small. Note that from (1) - (3) above, it follows that prudence, $u''' \geq 0$, is equivalent to each of the following:

(i) Adding $\tilde{\varepsilon}_1$ to a higher wealth level is "less painful" (i.e. the absolute size of the utility premium is decreasing in x).

(ii) Adding $\tilde{\varepsilon}_1$ to wealth increases the expected marginal utility.

Kimball (1990) noted both of these properties and used them to model precautionary savings. In his set-up, an income risk is added in the second of two periods. This induces the individual to shift some nonrandom wealth to the second period (via more savings in the first period) in order to help mitigate the pain.

From (i) above and inequality (4), if we also have prudence, we can interpret $u^{iv} \leq 0$ as implying that the pain from adding $\tilde{\varepsilon}_1$ to wealth decreases as one gets wealthier, but it decreases at a decreasing rate. We next show that $u^{iv} \leq 0$ is equivalent to our definition of temperance.

3.3 Temperance and Utility

Let $\tilde{\varepsilon}_2$ be a zero-mean risk that is independent of $\tilde{\varepsilon}_1$. We iterate on the above procedure for defining the utility premium, and define w_2 as the utility premium for w_1 (regardless of whether or not w_1 is increasing or concave):

$$w_2(x) \equiv Ew_1(x + \tilde{\varepsilon}_2) - w_1(x). \tag{5}$$

If w_1 is concave, then w_2 will be everywhere negative. From (4), this implies that

$$w_2(x) \equiv Ew_1(x + \tilde{\varepsilon}_2) - w_1(x) \leq 0 \ \ \forall x \ \text{ if and only if } \ u^{iv} \leq 0. \tag{6}$$

Using only Jensen's inequality, in a manner similar to w_1, we can continue to find

$$w_2'(x) \equiv Ew_1'(x + \tilde{\varepsilon}_2) - w_1'(x) \geq 0 \ \ \forall x \ \text{ if and only if } \ u^{v} \geq 0 \tag{7}$$

and

$$w_2''(x) \equiv Ew_1''(x+\tilde{\varepsilon}_2) - w_1''(x) \leq 0 \ \ \forall x \ \text{ if and only if } \ u^{vi} \leq 0. \tag{8}$$

To see that $u^{iv} \leq 0$ is equivalent to temperance, use (1) to expand (6). It follows that $u^{iv} \leq 0$ is equivalent to

$$[Eu(x+\tilde{\varepsilon}_1+\tilde{\varepsilon}_2) - Eu(x+\tilde{\varepsilon}_2)] - [Eu(x+\tilde{\varepsilon}_1) - u(x)] \leq 0 \tag{9}$$

or equivalently

$$\tfrac{1}{2}[Eu(x+\tilde{\varepsilon}_1) + Eu(x+\tilde{\varepsilon}_2)] \geq \tfrac{1}{2}[u(x) + Eu(x+\tilde{\varepsilon}_1+\tilde{\varepsilon}_2)]. \tag{10}$$

Inequality (10) is clearly an expected-utility equivalent to our lottery-preference definition of temperance (Definition 2).

3.4 Risk Apportionment of Orders 5 and 6

We can use w_2 to show that risk apportionment of order 5 (RA-5) is equivalent to $u^{v} \geq 0$ by once again noting that our Definition 3 allows for the sure reduction in wealth $-k$ to be arbitrarily small. Equivalently, we can write (7) as

$$[Ew_1(x+\tilde{\varepsilon}_2) - w_1(x)] - [Ew_1(x-k+\tilde{\varepsilon}_2) - w_1(x-k)] \geq 0. \tag{11}$$

Expanding w_1 in (**??**) and rearranging shows that it is equivalent to the lottery-preference definition for RA-5 (Definition 3).

To show that risk apportionment of order 6 is equivalent to $u^{vi} \leq 0$, we need to iterate once again on the utility premium and define

$$w_3(x) \equiv E w_2(x + \tilde{\varepsilon}_3) - w_2(x), \tag{12}$$

where $\tilde{\varepsilon}_3$ is a zero-mean risk independent of $\tilde{\varepsilon}_2$ and $\tilde{\varepsilon}_2$. Similar to our analysis above, it follows from Jensen's inequality that $w_3 \leq 0$ if and only if w_2 is concave, which we have already proven is equivalent to $u^{vi} \leq 0$. Expanding the inequality $w_3 \leq 0$ by using (1) and (5), it is straightforward to show that $u^{vi} \leq 0$ is equivalent to our lottery-preference characterization of RA-6 in Definition 4.

3.5 Risk Apportionment of Order n

One can continue on in this manner by demonstrating that $w_3' \geq 0$ is equivalent to $u^{vii} \geq 0$, as well as equivalent to our definition of RA-7. To obtain the equivalence of $u^{viii} \leq 0$ and RA-8, we need to define w_4 as the utility premium of w_3. We can iterate in this manner for any $n \geq 3$:

(*i*) For n even, we define $w_{n/2}(x) \equiv E w_{(n/2)-1}(x + \tilde{\varepsilon}_{(n/2)-1}) - w_{(n/2)-1}(x)$. Expanding this expression we can show that $u^{(n)} \leq 0$ iff $w_{n/2}(x) \leq 0$ iff RA-n holds.

(*ii*) For n odd, we use the equivalence of $u^{(n)} \geq 0$ and $w'_{(n-1)/2}(x) \geq 0$ and demonstrate how this non-negative derivative is equivalent to the lottery preference for RA-n.

This leads to the following main result, showing how risk apportionment relates to derivatives of the utility function.

Theorem: *In an expected-utility framework with differentiable u, risk apportionment of order n is equivalent to the condition sgn $u^{(n)} =$ sgn $(-1)^{n+1}$.*

4 Related Concepts

Many papers have looked at the implications of signing higher order derivatives of utility in an expected-utility framework, but very few have pinned down the meaning of these signs in and of themselves. The advantage of risk apportionment lies mainly in its simplicity. The fact that it is defined over lottery preferences also makes it applicable outside of an expected-utility framework. Thus, concepts like "prudence" and "temperance" can be generalized and embedded into other frameworks for choice under risk. In this section, we examine how our results in this paper relate to some of the extant literature.

4.1 Higher Order Effects

Within expected-utility models, growth rates and elasticities are typically second-order effects because they relate the effect of changes in an exogenous variable on a first-order condition.[14] Decreasing absolute risk aversion (DARA) is a third-order property because it has to do with changes in risk aversion (a second-order property). Prudence is also a third-order property, since it relates the effect of risk on a first-order condition. However, DARA is a stronger condition than simply assuming prudence, in particular, requiring that $u''' \geq (u'')^2/u'$.

In a sense, we can think of prudence itself, $u''' > 0$, as a pure third-order effect. A straightforward interpretation of inequality (3) is that the "pain" of adding a risk $\widetilde{\varepsilon}$ decreases as one gets wealthier. On the other hand, decreasing risk aversion implies that one's willingness to pay to remove a risk is decreasing as one gets wealthier. But this "willingness to pay" in a sense contains too much information, since it must relate the changing level of "pain" to the marginal valuation of paying a dollar to remove this "pain."[15]

We can, take this argument to higher order. Consider the interaction of two risks, $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$, which is a fourth-order effect. Many authors have formulations similar to our lottery defining prudence, in Definition 2. For example, Pratt and Richard Zeckhauser (1987) define preferences as being "proper" if $[\widetilde{\varepsilon}_1;\ \widetilde{\varepsilon}_2] \succeq [0;\ \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2]$ not for all zero-mean risks $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$, but rather risks that are undesirable to the individual: each reduces expected utility of the individual when added to wealth.[16] Gollier and Pratt (1996), in defining the very useful concept of risk vulnerability, essentially look at this same lottery preference, but where one of the risks, say $\widetilde{\varepsilon}_2$, is restricted to the set of risks that are undesirable for *all* risk-averse individuals, which implies $\widetilde{\varepsilon}_2$ has a non-positive mean. Kimball, defines standard risk aversion in much the same manner, but where $\widetilde{\varepsilon}_2$, is restricted to the set of risks that increase marginal utility. Naturally, temperance is a necessary condition for both of these formulations, since they both include zero-mean risks $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$ as a special case. By allowing for non-zero means, all of these formulations include effects of other orders, and do not isolate the pure fourth-order effect of temperance.[17]

4.2 Stochastic Dominance

One obvious related area is that of stochastic dominance. Stochastic dominance establishes a partial ordering of probability distributions for which it is well known that wealth distribution F dominates wealth distribution G in the sense of n^{th}-order stochastic dominance if and only if everyone with a utility function u for which $sgn\ u^{(j)} = sgn\ (-1)^{j+1}$ for $j = 1,\ 2,\ ...,n$ prefers F to G. [18] Such a utility function is said to satisfy stochastic-dominance preference of order n. Hence, from our Theorem it follows that preferences satisfy stochastic-dominance preference of order n if and only if they satisfy risk apportionment of order j for all $j = 1,\ 2,\ ...,n$.

Steinar Ekern (1981) limits the distributions F and G to those for which F dominates G by stochastic dominance of order n, but not for any orders less than n. In this case, he says that G has more n^{th} degree risk than F. He then shows how this condition is equivalent to saying that every individual with $sgn\ u^{(n)} = sgn\ (-1)^{n+1}$ would prefer F to G. He labels such an individual as "n^{th} degree risk averse." Obviously then, it follows from our Theorem that Ekern's n^{th} degree risk aversion is equivalent to preferences satisfying risk apportionment of order n.

Given the comments above, it is clear that others have already characterized the signs of the derivatives of the utility function. What makes risk apportionment so appealing is its simplicity. For instance, consider RA-4 (temperance, or equivalently $u^{iv} \leq 0$). For those readers familiar with stochastic dominance, think of describing distributions where there is stochastic dominance of order 4, but not of orders 1,2 or 3. Of course this is possible, but it is hardly simple. Compare this to the simplicity of assuming the lottery $[\widetilde{\varepsilon}_1,\ \widetilde{\varepsilon}_2]$ is preferred to $[0,\ \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2]$.

This simplicity of our lottery design with equal probabilities, also lends itself well to experimental design. While framing contexts and situationalism will surely still play a role, the complexity of understanding the lottery itself is not an issue, especially for RA-n where n is not too large. Thus, a concept like temperance seems quite plausible. On the other hand, our definition of temperance (n =4) requires that $[\widetilde{\varepsilon}_1,\ \widetilde{\varepsilon}_2]$ be preferred to $[0,\ \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2]$ for *all* independent $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$. This must hold not only if $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$ are identically distributed, but even if, say, $\widetilde{\varepsilon}_1$ has a very large variance and the variance of $\widetilde{\varepsilon}_2$ is extremely small. In such a setting, behaviorists might have us believe that many individuals will be lured by the "certainty" of the first outcome in the lottery $[0,\ \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2]$, and thus prefer it to $[\widetilde{\varepsilon}_1,\ \widetilde{\varepsilon}_2]$.

4.3 Aversion to Outer Risk

Perhaps the closest approach to our own is that of Menezes and X. Henry Wang (2005), who relate the property of temperance to the notion of outer risk. In their model, they formally show how $[\widetilde{\varepsilon}_1;\ \widetilde{\varepsilon}_2] \succeq [0;\ \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2]$ implies fourth-order stochastic dominance of the corresponding lottery distribution functions, thus equating this lottery preference to $u^{iv} \leq 0$.[19] We can generalize their notion of outer risk as follows.

In general, we cannot order $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$, with respect to preferences. But we can construct the chain $0 \succeq \widetilde{\varepsilon}_i \succeq \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2$, where $i = 1$ or $i = 2$. To this end, consider $\{\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2\}$ as the "inner risks" and $\{0, \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2\}$ as the "outer risks." Our definition of temperance (Definition 2) thus states that a 50-50 gamble between the inner risks is preferred to one between the outer risks.

We can also use Menezes' and Wang's concept of inner and outer risks to describe higher-order risk apportionment. For example, consider the simple lottery $[0, \widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2, \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2]$, where all four outcomes have equal probability. If we must attach a sure loss of $k > 0$ to either the two inner risks or to the two outer risks, RA-5 is equivalent to always preferring to attach $-k$ to the two inner risks. RA-6 can be defined in a similar manner, where we replace the sure loss $-k$ with an independent third risk $\widetilde{\varepsilon}_3$. We can achieve all higher orders of risk apportionment by simple iteration on these results.

5 Concluding Remarks

For a long time, risk aversion has played a key role in the theory of choice under uncertainty; not only within expected-utility (EU) models, but also within other decision-theoretic frameworks. It was recognized quite early on, that the sign of u''' played a key role within EU, but it was not until Kimball (1990) that this role was formalized into the concept of "prudence." Since this formalization, models of consumption and savings decisions have received a new focus and made many advancements. Outside of EU, these advances have come mostly from trying to mimic either the consequences that follow within EU, or to mimic some of the parametric nuances of properties such as DARA and prudence. The role of signing higher order derivatives, such as assuming "temperance" or "edginess," is only recently receiving more interest in the literature.

By considering simple lottery preferences, we are able to provide a characterization of these properties based only on underlying preferences. In particular, we *define* such properties by our lottery preference, and then we show how these definitions are equivalent to signing the n^{th} derivative within EU models. Since our definitions are not confined to EU, they are applicable within other choice-theoretic frameworks as well. The types of lotteries we look at are rather simple, especially for fairly low values of n. Indeed the simplicity of our lotteries seems quite amenable to experiments about individual behavior towards risk.

Notes

[1] The term "prudence" was coined by Miles Kimball (1990), although the importance of the third derivative of utility in determining a precautionary savings demand was noted much earlier by Hayne E. Leland (1968) and Agnar Sandmo (1970).

[2] One notable exception is the paper by Carmen F. Menezes, C. Geiss and John Tressler (1980), who describe "aversion to downside risk" and relate it to the sign of u'''.

[3] We use the notations $u^{(4)}(x)$ and $u^{iv}(x)$ interchangeably to denote the fourth derivative of u, $\frac{d^4u(x)}{dx^4}$. Similarly, we denote the n^{th} derivative by $u^{(n)}$ as well as by a Roman-numeral superscript.

[4] A summary of results relating stochastic dominance, and hence our lottery preference, to income distribution can be found in Patrick Moyes (1999). The other economic applications mentioned above are scattered throughout the literature, but a good overview of many of them can be found in the book by Elmar Wolfstetter (1999).

[5] This property is labeled "complete properness" by John W. Pratt and Richard Zeckhauser (1987). This class of utility functions also was examined independently by Patrick L. Brocket and Linda L. Golden (1987).

[6] One notable exception is the paper by D. L. Hanson and Menezes (1971), who more than 30 years ago had made this exact same observation. To the best of our knowledge, the first direct look at the utility premium was the work of Milton Friedman and Leonard J. Savage (1948).

[7] For instance, if preferences are defined only over positive levels of final wealth, we assume throughout the paper that all changes to wealth, be it by subtracting a fixed wealth or adding a random wealth term, are chosen so as to preserve wealth to be positive.

[8] Strict-preference analogs follow, but require significantly more-complex modelling, with little extra in the way of economic insight.

[9] John P. Bigelow and Menezes (1995) essentially show that our lottery preference as defined below is equivalent to $u''' \geq 0$. Our main distinction here is to use this lottery preference relation as the *definition* of prudence.

[10] More formally, if F_y and F_c denote the (marginal) distribution functions of random variables $\widetilde{y}$ and C respectively, then the distribution over the sum of these random variables $\widetilde{y} + C$ is given by the convolution of these distribution functions, $F_y * F_c$.

[11] We do not particularly like introducing new terminology, but one overarching goal is to have a generalized concept that can be extended to various orders, much along the lines of stochastic dominance. By apportioning harms within a lottery, we wish to mitigate their detrimental effects. Hence the terminology "risk apportionment." For orders 1 and 2, this makes less sense, but we include the terminology to have consistency in our general results. Obviously risk apportionment of order 3 is already well known as "prudence" and "temperance" in the extant sense is equivalent in our definition to risk apportionment of order 4.

[12] For a random $\widetilde{x}$, we can simply replace utility u with the derived utility function $\widehat{u}(y) = Eu(y + \widetilde{x})$, as defined by David Nachman (1982). It follows trivially that the signs of the n^{th} derivatives of u and $\widehat{u}$ with respect to y will all be the same.

[13] This is the negative of how the utility premium is often defined, in the scant literature on the topic. However, one very notable exception is Friedman and Savage (1948). Defining it in this manner helps to facilitate our discussions that follow.

[14] For example, absolute risk aversion and relative risk aversion are respectively the decay rate and elasticity of changes in marginal utility with respect to increases in wealth. Note, however, that if preferences are not required to be "smooth," such as allowing non-differentiability of u at some wealth levels, risk aversion might also be a first-order effect, as pointed out by Uzi Segal and Avia Spivak (1990).

[15] For example, the reader can easily verify that, under the common assumption of constant absolute risk aversion (CARA), the level of "pain" associated with adding the risk $\widetilde{\varepsilon}$ is actually decreasing in wealth, whereas the willingness to pay to remove a unit of "pain" is increasing in wealth. Of course, under CARA, these two effects exactly offset one another.

[16] Actually, this lottery formulation is not presented by Pratt and Zeckhauser (1997) themselves, but rather by a reformulation of their result by Kimball (2003).

[17] These same arguments have been taken up to the fifth-order recently by Fatma Lajeri-Chaherli (2004), who also provides a nice summary of the fourth-order concepts of properness, risk vulnerability and standard risk aversion. Her fifth-order effect of "standard prudence" relates to precautionary savings in the presence of a background risk.

[18] See, for example, Johnathan E. Ingersoll (1987).

[19] They show equivalence for their more general formulation of increased outer risk. The lottery they consider as an illustration is the same as the one we present here, with $\widetilde{\varepsilon}_2$ being restricted as $\widetilde{\varepsilon}_2 = [-1, +1]$.

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Figure 1: Risk Apportionment of Order 6, $B_6 \succcurlyeq A_6$