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q-Bessel Functions and Rogers-Ramanujan Type Identities

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Abstract

We evaluate q-Bessel functions at an infinite sequence of points and introduce a generalization of the Ramanujan function and give an extension of the m-version of the Rogers-Ramanujan identities. We also prove several generating functions for Stieltjes-Wigert polynomials with argument depending on the degree. In addition we give several Rogers-Ramanujan type identities.

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1 Introduction

The Rogers–Ramanujan identities are

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}&=\frac{1}{(q,q^4;q^5)_\infty}\\ \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}&=\frac{1}{(q^2,q^3;q^5)_\infty},\end{aligned}\tag{1.1}$$

where the notation for the q-shifted factorials is the standard notation in [10], [12]. References for the Rogers-Ramanujan identities, their origins and many of their applications are in [1], [2], and [4]. In particular we recall the partition theoretic interpretation of the first Rogers–Ramanujan identity as the partitions of an integer n into parts $\equiv$ 1or 4(mod 5) are equinumerous with the part ions of n into parts where any two parts differ by at least 2.

Garrett, Ismail, and Stanton [9] proved the m-version of the Rogers-Ramanujan identities

$$\sum_{n=0}^{\infty}\frac{q^{n^2+mn}}{(q;q)_n}=\frac{(-1)^m q^{-\binom{m}{2}}a_m(q)}{(q,q^4;q^5)_\infty}-\frac{(-1)^m q^{-\binom{m}{2}}b_m(q)}{(q^2,q^3;q^5)_\infty},\tag{1.2}$$

where

$$\begin{aligned}a_m(q)&=\sum_j q^{j^2+j}\begin{bmatrix}m-j-2\\ j\end{bmatrix}_q,\\ b_m(q)&=\sum_j q^{j^2}\begin{bmatrix}m-j-1\\ j\end{bmatrix}_q.\end{aligned}\tag{1.3}$$

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The polynomials $a_m(q)$ and $b_m(q)$ were considered by Schur in conjunction with his proof of the Rogers–Ramanujan identities, see [1] and [9] for details. We shall refer to $a_m(q)$ and $b_m(q)$ as the Schur polynomials. The closed form expressions for a_m and b_m in (1.3) were given by Andrews in [3], where he also gave a polynomial generalization of the Rogers–Ramanujan identities. In this paper we give a family of Rogers–Ramanujan type identities involving the evaluation of q-Bessel and allied functions at special points. We also give the partition theoretic interpretation of these identities. In Section 2 we define the functions and polynomials used in our analysis. In Section 3 we present our Rogers–Ramanujantype identities. They resemble the m form in (1.2).

In a series of papers from 1903 till 1905 F. H. Jackson introduced q-analogues of Bessel functions. The modern notation for the modified q-Bessel functions, that is q-Bessel functions with imaginary argument, is, [11],

$$I_\nu^{(1)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^\infty \frac{(z/2)^{\nu+2n}}{(q,q^{\nu+1};q)_n}, \quad |z|<2, \tag{1.4}$$

$$I_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^\infty \frac{q^{n(n+\nu)}}{(q,q^{\nu+1};q)_n}(z/2)^{\nu+2n}, \tag{1.5}$$

$$I_\nu^{(3)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^\infty \frac{q^{\binom{n}{2}}}{(q,q^{\nu+1};q)_n}(z/2)^{\nu+2n}. \tag{1.6}$$

The functions $I_\nu^{(1)}$ and $I_\nu^{(2)}$ are related via

$$I_\nu^{(1)}(z;q) = \frac{I_\nu^{(2)}(z;q)}{(z^2/4;q)_\infty}, \tag{1.7}$$

[12, Theorem 14.1.3].

$$S_n(x;q) = \frac{1}{(q;q)_n}\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2}(-x)^k = \frac{1}{(q;q)_n}\sum_{k=0}^n \frac{(q^{-n};q)_k}{(q;q)_k} q^{\binom{k+1}{2}}(xq^n)^k, \tag{1.8}$$

respectively. Ismail and C. Zhang [14] proved the following symmetry relation for the Stieltjes–Wigert polynomials

$$q^{n^2}(-t)^n S_n(q^{-2n}/t;q) = S_n(t;q). \tag{1.9}$$

Section 2 contanis the evaluation of $I_\nu^{(2)}$ at an infinite number of special points. These new sums seem to be new. In Section 3 we introduce a generalization of the Ramanujan function

$$A_q(z) := \sum_{n=0}^\infty \frac{(-z)^n}{(q;q)_n} q^{n^2}, \tag{1.10}$$

which S. Ramanujan introduced and studied many of its properties In the lost note book [18]. It was later realized that this is an analogue of the Airy function. In Section 4 we introduce a function B_q^α prove some identities it satisfies then use them to derive several Rogers-Ramanujan type identities. The function B_q^α is also a generalization of the Ramanujan function and is expected to lead to numerous new Rogers-Ramanujan type identities. The Stieltjes-Wigert polynomials satisfy a second order q-difference equation of polynomial coefficients of the for

$$f(x)y(qx) + g(x)y(x) + h(x)y(x/q) = 0.$$

In Section 5 we evaluate $y(q^n x)$ in terms of $y(x$ and $y(x/q)$ with explicit coefficients. Section 6 contains misclaneous properties of the Stieltjes-Wigert polynomials.

2 q-Bessel Sums

Our first result is the following theorem.

Theorem 2.1. *The function $I_\nu^{(2)}$ has the represetation*

$$I_\nu^{(2)}(2z;q) = \frac{z^\nu}{(q;q)_\infty}\, {}_1\phi_1(z^2;0;q,q^{\nu+1}). \tag{2.1}$$

In particular $I_\nu^{(2)}$ takes the special values

$$I_\nu^{(2)}\left(2q^{-n/2};q\right) = \frac{q^{\nu n/2} S_n\left(-q^{-\nu-n};q\right)}{\left(q^{n+1};q\right)_\infty}, \tag{2.2}$$

and

$$I_\nu^{(2)}\left(2q^{-n/2};q\right) = \frac{q^{-\nu n/2} S_n\left(-q^{\nu-n};q\right)}{\left(q^{n+1};q\right)_\infty} \tag{2.3}$$

Proof. Recall the Heine transformation [10, (III.2)]

$${}_2\phi_1\left(\begin{matrix}A,B\\ C\end{matrix}\,\middle|\, q,Z\right) \quad = \quad \frac{(C/B,BZ;q)_\infty}{(C,Z;q)_\infty}\, {}_2\phi_1\left(\begin{matrix}ABZ/C,B\\ BZ\end{matrix}\,\middle|\, q,\frac{C}{B}\right). \tag{2.4}$$

The left-hand side of (2.1) is

$$\begin{aligned}\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} z^\nu \sum_{k=0}^{\infty} \frac{q^{k^2+k\nu} z^{2k}}{(q^{\nu+1},q;q)_k} &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} z^\nu \lim_{a,b\to\infty} {}_2\phi_1\left(\begin{matrix}a,b\\ q^{\nu+1}\end{matrix}\,\middle|\, q,\frac{q^{\nu+1}z^2}{ab}\right)\\ &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} z^\nu \frac{1}{(q^{\nu+1};q)_\infty} \lim_{a,b\to\infty} {}_2\phi_1\left(\begin{matrix}z^2,b\\ z^2q^{\nu+1}/a\end{matrix}\,\middle|\, q,\frac{q^{\nu+1}}{b}\right)\end{aligned}$$

which implies (2.1). When $z = q^{-n/2}$ and in view of (1.8), the left-hand side of (2.2) equals its right-hand side. Formula (2.3) follows from the symmetry relation (1.9) $\square$

The results (2.2)–(2.3) of Theorem 2.1 when written as a series becomes

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{q^{k(k+\nu-n)}}{(q,q^{\nu+1};q)_k} &= \frac{q^{n\nu}}{(q^{\nu+1};q)_\infty} \sum_{k=0}^{n} \begin{bmatrix}n\\ k\end{bmatrix}_q q^{k^2} q^{-k(\nu+n)}\\ &= \frac{1}{(q^{\nu+1};q)_\infty} \sum_{k=0}^{n} \begin{bmatrix}n\\ k\end{bmatrix}_q q^{k^2} q^{k(\nu-n)}.\end{aligned} \tag{2.5}$$

Another way to prove (2.2) for integer ν is to use the generating function

$$\sum_{-\infty}^{\infty} q^{\binom{m}{2}} I_m^{(2)}(z;q) t^m = (-tz/2,-qz/2t;q)_\infty. \tag{2.6}$$

Carlitz [6] did this for $n = 0,1$ and used this to give another proof of the Rogers–Ramanujan identities.

Theorem 2.2. [1] *The q-binomial coefficient $\begin{bmatrix}n\\ k\end{bmatrix}_q$ is the generating function for integer partitions whose Ferrers diagrams fit inside a $k\times(n-k)$ rectangle.*

Recall that

$$I_\nu^{(j)}(z;q) = e^{-i\nu\pi/2} J_\nu^{(j)}(e^{i\pi/2}z;q), j = 1,2. \tag{2.7}$$

Chen, Ismail, and Muttalib [8] established an asymptotic series for $J_\nu^{(2)}(z;q)$. Their main term for $r > 0$ is

$$\begin{aligned} I_\nu^{(2)}(r;q) &= (r/2)^\nu \frac{(q^{1/2};q)_\infty}{2(q;q)_\infty} \\ &\times \left[(rq^{(\nu+1/2)/2}/2;q^{1/2})_\infty + (-rq^{(\nu+1/2)/2}/2;q^{1/2})_\infty, \right] \end{aligned} \tag{2.8}$$

as $r \to +\infty$. This determines the large r behavior of the maximum modulus of $I_\nu^{(2)}$.

We next derive a Mittag–Leffler expansion for $I_\nu^{(1)}$.

Theorem 2.3. *We have the expansion*

$$I_\nu^{(1)}(z;q) = \frac{\left(\frac{z}{2}\right)^\nu}{(q;q)_\infty^2} \sum_{n=0}^\infty \frac{(-1)^n q^{\binom{n+1}{2}} S_n(-q^{\nu-n};q)}{(1 - z^2q^n/4)}. \tag{2.9}$$

Using residues it is easy to see that the difference between $I_\nu^{(1)}(z;q)/(z^2;q)_\infty$ and the right-hand side of (2.9) is entire. We give a direct proof that this difference is zero.

Proof of Theorem 2.3. Use (1.8) to see that the sum on the right-hand side of (2.9) is

$$\begin{aligned} &\sum_{n=0}^\infty \frac{(-1)^n q^{\binom{n+1}{2}}}{(1 - z^2q^n/4)} \sum_{k=0}^n \frac{q^{k^2+k(\nu-n)}}{(q;q)_k (q;q)_{n-k}} \\ &= \sum_{k=0}^\infty \frac{(-1)^k q^{k(\nu+(k+1)/2)}}{(q;q)_k (1 - z^2q^k/4)} \, {}_1\phi_1(z^2q^k/4; z^2q^{k+1}/4; q, q). \end{aligned}$$

Now apply (III.4) of [10] with $a = z^2q^k/4, b = 1, c = 0, z = q$ to see that the above sum is $(q;q)_\infty/(z^2q^{k+1}/4;q)_\infty$. This shows that the right-hand side of (2.9) is given by

$$\frac{(z/2)^\nu}{(q, z^2/4; q)_\infty} \, {}_1\phi_1(z^2/4; 0; q, q^{\nu+1}),$$

and the result follows from (2.1) and (1.7). □

3 A Generalization of the Ramanujan Function

The Rogers-Ramanujan identities evaluate A_q at $z = -1, -q$, The result (1.2) evaluates $A_q(-q^m)$. This motivated us to consider the function

$$u_m(a,q) := \sum_{n=-\infty}^\infty \frac{q^{n^2+mn}}{(aq;q)_n}, \tag{3.1}$$

as a function of q^m. When $a = 1$ we get the Rogers-Ramanujan function. It is clear that

$$q^{m+1} u_{m+2}(a,q) = \sum_{n=-\infty}^\infty \frac{(1 - aq^n)}{(aq;q)_n} q^{n^2+mn}$$

Therefore

$$q^{m+1}u_{m+2}(a,q) = u_m(a,q) - au_{m+1}(a,q). \tag{3.2}$$

Let $u_m(a,q) = q^{-\binom{m}{2}}(-1)^m\, \tilde{u}_m(a,q)$. Then $\{\tilde{u}_m(a,q)\}$ satisfy the difference equation

$$y_{m+1} = q^{m-1}\, y_{m-1} + ay_m, \quad m = 0, \pm 1, \cdots. \tag{3.3}$$

We now solve (3.3) for $m \geq 0$ using generating functions. The generating function $Y(t) := \sum_{n=0}^{\infty} y_n t^n$ satisfies

$$Y(t) = \frac{y_0 + t(y_1 - ay_0)}{1 - at} + \frac{t^2}{1 - at}\, Y(qt),$$

whose solution is

$$Y(t) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}t^{2n}}{(at;q)_{n+1}}\, [y_0 + tq^n(y_1 - ay_0)].$$

We now need two initial conditions, so choose two solutions $\{c_m(a,q)\}$ and $\{d_m(a,q)\}$

$$c_0(a,q) = 1, c_1(a,q) = 0, \;\; d_0(a,q) = 0, d_1(a,q) = 1. \tag{3.4}$$

Theorem 3.1. *The polynomials $\{c_m(a,q)\}$ and $\{d_m(a,q)\}$ have the generating functions*

$$\sum_{n=0}^{\infty} c_n(a,q)t^n = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{(at;q)_n}\, t^{2n}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} d_n(a,q)t^n = \sum_{n=0}^{\infty} \frac{q^{n^2}t^{2n+1}}{(at;q)_{n+1}}. \tag{3.6}$$

It is clear from the initial conditions (3.4) and the recurrence relation (3.3) that both $\{c_n(a,q)\}$ and $\{d_n(a,q)\}$ are polynomials in a and in q.

Theorem 3.2. *The polynomials $\{c_n(a,q)\}$ and $\{d_n(a,q)\}$ have the explicit form*

$$c_n(a,q) = \sum_{j=0}^{\lfloor (n-2)/2 \rfloor} q^{j(j+1)} \begin{bmatrix} n-j-2 \\ j \end{bmatrix}_q a^{n-2j-2}, \tag{3.7}$$

$$d_n(a,q) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} q^{j^2} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_q a^{n-2j-1}. \tag{3.8}$$

The proof follows form equations (3.5) and (3.6); and the q-binomial theorem.

Theorem 3.3. *We have*

$$\sum_{n=-\infty}^{\infty} \frac{q^{n^2+mn}}{(aq;q)_n} = (-1)^m q^{-\binom{m}{2}}$$
$$\times \left[c_m(a,q) \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(aq;q)_n} + d_m(a,q) \sum_{n=-\infty}^{\infty} \frac{q^{n^2+n}}{(aq;q)_n} \right]$$

The case $a = 1$ is the m-version of the Rogers-Ramanujan identities in (1.2) first proved by Garret, Ismail, and Stanton [9].

We now solve (3.3) for $m \leq 0$. From the initial conditions (3.4) it is clear that

$$c_{-1}(a,q) = -aq, \quad d_{-1}(a,q) = q. \tag{3.9}$$

We follow the same generating function technique and conclude that

$$\sum_{n=0}^{\infty} c_{-n}(a,q)t^n = \sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}} a^n t^n (qt/a;q)_n, \tag{3.10}$$

$$\sum_{n=0}^{\infty} d_{-n}(a,q)t^n = qt\sum_{n=0}^{\infty}(-1)^n a^n t^n (qt/a;q)_n q^{\binom{n+2}{2}}. \tag{3.11}$$

This establishes the following theorem.

Theorem 3.4. *For $n \geq 0$ we have*

$$c_{-n}(a,q) = (-1)^n q^{\binom{n+1}{2}} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k(k-n)} \begin{bmatrix} n-k \\ k \end{bmatrix}_q a^{n-2k}, \tag{3.12}$$

$$d_{-n-1}(a,q) = (-1)^n q^{\binom{n+2}{2}} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k(k-n)} \begin{bmatrix} n-k \\ k \end{bmatrix}_q a^{n-2k} \tag{3.13}$$

The polynomials $\{c_n(a,q)\}$ and $\{d_n(a,q)\}$ first appeared in Carlitz's paper [7] where he introduced them as Fibonacci polynomials. Štampach [20] observed that they are orthogonal polynomials and studied their moment problem including computing the corresponding Nevanlinna matrix. In Štampach's notation

$$d_n(x,a) = \phi_n(x;q), \quad T_n(x;q) = q^{-n} c_{n+2}(xq,q), n = 0, 1, \cdots. \tag{3.14}$$

He did not consider the case $n < 0$.

4 Rogers-Ramanujan Type Identities

In this section we prove several identities of Rogers-Ramanujan type. One of the proofs uses the Ramanujan ${}_1\psi_1$ sum [10, (II.29)]

$$\sum_{-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(q, b/a, az, q/az;q)_\infty}{(b, q/a, z, b/az;q)_\infty}, \quad \left|\frac{b}{a}\right| < |z| < 1. \tag{4.1}$$

Throughout this section we define ρ by

$$\rho = e^{2\pi i/3}. \tag{4.2}$$

Lemma 4.1. *For nonnegative integer j, k, ℓ, m, n and $\rho = e^{2\pi i/3}$ we have*

$$\sum_{k=0}^{n} \frac{(a;q)_k (a;q)_{n-k} (-1)^k}{(q;q)_k (q;q)_{n-k}} = \begin{cases} 0 & n = 2m+1 \\ \frac{(a^2;q^2)_m}{(q^2;q^2)_m} & n = 2m \end{cases}, \tag{4.3}$$

and

$$\sum_{\substack{j+k+\ell=n \\ j,k,\ell \geq 0}} \frac{(a;q)_j (a;q)_k (a;q)_\ell}{(q;q)_j (q;q)_k (q;q)_\ell} \rho^{k+2\ell} = \begin{cases} 0 & 3 \nmid n \\ \frac{(a^3;q^3)_m}{(q^3;q^3)_m} & n = 3m \end{cases}. \tag{4.4}$$

For $j,k,m,\ell,n \in \mathbb{Z}$, we have

$$\sum_{j+k=n} \frac{(a;q)_j\,(a;q)_k\,(-1)^k}{(b;q)_j\,(b;q)_k} = \begin{cases} 0 & n=2m+1 \\ \frac{(q,b/a,-b,-q/a;q)_\infty}{(-q,-b/a,b,q/a;q)_\infty}\frac{\left(a^2;q^2\right)_m}{\left(b^2;q^2\right)_m} & n=2m \end{cases} \tag{4.5}$$

and

$$\sum_{j+k+\ell=n}^{\infty} \frac{(a;q)_j\,(a;q)_k\,(a;q)_\ell\,\rho^{k+2\ell}}{(b;q)_j\,(b;q)_k\,(b;q)_\ell} = 0 \tag{4.6}$$

for $3 \nmid n$,

$$\begin{aligned} &\sum_{j+k+\ell=3m}^{\infty} \frac{(a;q)_j\,(a;q)_k\,(a;q)_\ell\,\rho^{k+2\ell}}{(b;q)_j\,(b;q)_k\,(b;q)_\ell} \\ &= \frac{(q,b/a;q)_\infty^3}{(b,q/a;q)_\infty^3}\frac{\left(b^3,q^3a^{-3};q^3\right)_\infty}{\left(q^3,b^3a^{-3};q^3\right)}\frac{\left(a^3;q^3\right)_m}{\left(b^3;q^3\right)_m}. \end{aligned} \tag{4.7}$$

Proof. Formula (4.3) follows from

$$\frac{(at;q)_\infty}{(t;q)_\infty}\frac{(-at;q)_\infty}{(-t;q)_\infty} = \frac{\left(a^2t^2;q^2\right)_\infty}{\left(t^2;q^2\right)_\infty}, \quad |t|<1,$$

while (4.4) follows from

$$\frac{(at;q)_\infty}{(t;q)_\infty}\frac{(a\rho t;q)_\infty}{(\rho t;q)_\infty}\frac{\left(a\rho^2 t;q\right)_\infty}{\left(\rho^2 t;q\right)_\infty} = \frac{\left(a^3t^3;q^3\right)}{\left(t^3;q^3\right)}, \quad |t|<1.$$

For $\left|ba^{-1}\right| < |x| < 1$, apply the Ramanujan ${}_1\psi_1$ sum (4.1) to the identity

$$\frac{(ax,q/(ax);q)_\infty}{(x,b/(ax);q)_\infty}\frac{(-ax,-q/(ax);q)_\infty}{(-x,-b/(ax);q)_\infty} = \frac{\left(a^2x^2,q^2/\left(a^2x^2\right);q^2\right)_\infty}{\left(x^2,b^2/\left(a^2x^2\right);q^2\right)_\infty},$$

to derive (4.5). Similarly we apply (4.1) to

$$\begin{aligned} &\frac{\left(a^3x^3,q^3/\left(a^3x^3\right);q^3\right)_\infty}{\left(x^3,b^3/\left(a^3x^3\right);q^3\right)_\infty} \\ &= \frac{\left(ax\rho^2,q/\left(ax\rho^2\right);q\right)_\infty}{\left(x\rho^2,b/\left(ax\rho^2\right);q\right)_\infty}\frac{(ax\rho,-q/(ax\rho);q)_\infty}{(x\rho,-b/(ax\rho);q)_\infty}\frac{(ax,q/(ax);q)_\infty}{(x,b/(ax);q)_\infty}, \end{aligned}$$

and establish (4.6)-(4.7). □

It must be noted that (4.3) is essentially the evaluation of a continuous q-ultraspherical polynomial at $x=0$, [12, (12.2.19)].

For $\alpha > 0$, let

$$A_q^{(\alpha)}(a;t) = \sum_{n=0}^{\infty} \frac{(a;q)_n\,q^{\alpha n^2}t^n}{(q;q)_n}, \tag{4.8}$$

in particular,

$$A_q^{(1)}(q;t) = \omega(t;q), \quad A_{q^2}^{(2)}\left(q^2;t^2\right) = \omega\left(t^2;q^4\right), \quad A_q^{(1)}(0;t) = A_q(-t),$$

where

$$\omega(v;q) = \sum_{n=0}^{\infty} q^{n^2}v^n.$$

Theorem 4.2. *Let $\alpha \geq 0$, then*

$$A_{q^2}^{(2\alpha)}\left(a^2;t^2\right) = \sum_{j=0}^{\infty} \frac{(a;q)_j\, q^{\alpha j^2}\,(-t)^j}{(q;q)_j} A_q^{(\alpha)}\left(a;tq^{2\alpha j}\right). \tag{4.9}$$

For $\rho = e^{2\pi i/3}$ we have

$$A_{q^3}^{(3\alpha)}\left(a^3;t^3\right) = \sum_{j,k=0}^{\infty} \frac{(a;q)_j\,(a;q)_k\,\rho^k q^{\alpha(j+k)^2} t^{j+k}}{(q;q)_j\,(q;q)_k} A_q^{(\alpha)}\left(a;\rho^2 q^{2\alpha(j+k)}t\right). \tag{4.10}$$

Proof. These two identities can be proved by applying (4.3) and (4.4) and straightforward series manipulation. □

Corollary 4.3. *For any $m = 0,1,\ldots$ we have*

$$\begin{aligned} &\frac{q^{-\binom{m}{2}}a_m(q^2)}{(q^2,q^8;q^{10})_\infty} - \frac{q^{-\binom{m}{2}}b_m(q^2)}{(q^4,q^6;q^{10})_\infty} \\ &= \sum_{j=0}^{\infty} \frac{(-q^{1-m})^j}{(q;q)_j\, q^{j^2}} \left\{ \frac{a_{m+2j}(q)}{(q,q^4;q^5)_\infty} - \frac{b_{m+2j}(q)}{(q^2,q^3;q^5)_\infty} \right\}. \end{aligned} \tag{4.11}$$

In particular for $m = 0,1$ we have

$$\frac{1}{(q^2,q^8;q^{10})_\infty} = \sum_{j=0}^{\infty} \frac{(-q)^j}{(q;q)_j\, q^{j^2}} \left\{ \frac{a_{2j}(q)}{(q,q^4;q^5)_\infty} - \frac{b_{2j}(q)}{(q^2,q^3;q^5)_\infty} \right\} \tag{4.12}$$

and

$$\frac{1}{(q^4,q^6;q^{10})_\infty} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(q;q)_j\, q^{j^2}} \left\{ \frac{b_{2j+1}(q)}{(q^2,q^3;q^5)_\infty} - \frac{a_{2j+1}(q)}{(q^1,q^4;q^5)_\infty} \right\}. \tag{4.13}$$

Proof. Observe that

$$A_q^{(1)}(0;z) = A_q(-z), \quad A_{q^2}^{(2)}\left(0,z^2\right) = A_{q^2}\left(-z^2\right)$$

and

$$A_{q^2}\left(-t^2\right) = A_{q^2}^{(2)}\left(0;t^2\right) = \sum_{j=0}^{\infty} \frac{q^{j^2}(-t)^j}{(q;q)_j} A_q^{(1)}\left(0;tq^{2j}\right) = \sum_{j=0}^{\infty} \frac{q^{j^2}(-t)^j}{(q;q)_j} A_q\left(-tq^{2j}\right).$$

We rewrite (1.2) to get

$$\frac{a_m(q)}{(q,q^4;q^5)_\infty} - \frac{b_m(q)}{(q^2,q^3;q^5)_\infty} = (-1)^m q^{\binom{m}{2}} \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n}$$

and

$$\frac{a_{m+2j}(q)}{(q,q^4;q^5)_\infty} - \frac{b_{m+2j}(q)}{(q^2,q^3;q^5)_\infty} = \mathcal{O}\left(q^{\binom{m+2j}{2}}\right) \tag{4.14}$$

as $j \to \infty$. Then for any nonnegative integer m we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{2n^2+2mn}}{(q^2;q^2)_n} &= \frac{(-1)^m q^{-2\binom{m}{2}} a_m(q^2)}{(q^2,q^8;q^{10})_\infty} - \frac{(-1)^m q^{-2\binom{m}{2}} b_m(q^2)}{(q^4,q^6;q^{10})_\infty} \\ = \sum_{j=0}^{\infty} \frac{(-1)^j q^{j^2+mj}}{(q;q)_j} A_q(-q^{m+2j}) &= \sum_{j=0}^{\infty} \frac{(-1)^{j+m} q^{j^2+mj}}{(q;q)_j\, q^{\binom{m+2j}{2}}} \left\{ \frac{a_{m+2j}(q)}{(q,q^4;q^5)_\infty} - \frac{b_{m+2j}(q)}{(q^2,q^3;q^5)_\infty} \right\}, \end{aligned}$$

which simplifies to (4.11). From (4.14) it is clear that the above series actually converges very fast. □

We now consider the following generalization of the ${}_1\psi_1$ function. For $\alpha \ge 0$, define $B_q^{(\alpha)}$ by

$$B_q^{(\alpha)}(a,b;x) = \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} q^{\alpha n^2} x^n, \tag{4.15}$$

Theorem 4.4. *We have*

$$\frac{(-b,-q/a,q,b/a;q)_\infty}{(-q,-b/a,b,q/a;q)_\infty} B_{q^2}^{(2\alpha)}\left(a^2,b^2;x^2\right) = \sum_{j=-\infty}^{\infty} \frac{(a;q)_j\, q^{\alpha j^2}(-x)^j}{(b;q)_j} B_q^{(\alpha)}\left(a,b;xq^{2\alpha j}\right). \tag{4.16}$$

and

$$\begin{aligned} B_{q^3}^{(3\alpha)}\left(a^3,b^3;x^3\right) &= \frac{(b,q/a;q)_\infty^3}{(q,b/a;q)_\infty^3} \frac{\left(q^3,b^3a^{-3};q^3\right)}{(b^3,q^3a^{-3};q^3)_\infty} \\ &\times \sum_{j,k=-\infty}^{\infty} \frac{(a;q)_j\,(a;q)_k\,\rho^k q^{\alpha(j+k)^2} x^{j+k}}{(b;q)_j\,(b;q)_k} B_q^{(\alpha)}\left(a,b;xq^{2\alpha(j+k)}\right). \end{aligned} \tag{4.17}$$

The proof follows from (4.5), (4.6) and (4.7) and straightforward series manipulation.

Corollary 4.5. *The following Rogers-Ramanujan type identities hold*

$$\frac{(-a,-q/a,q,q;q)_\infty}{(a,q/a,-q,-q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{4n^2}x^{2n}}{1-a^2q^{2n}} = \sum_{j,k=-\infty}^{\infty} \frac{q^{(j+k)^2}(-1)^j x^{j+k}}{(1-aq^j)(1-aq^k)}, \tag{4.18}$$

$$\frac{(q,q;q)_\infty}{(-q,-q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{4n^2}x^{2n}}{1+q^{2n+1}} = \sum_{j,k=-\infty}^{\infty} \frac{q^{(j+k)^2}(-1)^j x^{j+k}}{\left(1+iq^{j+1/2}\right)\left(1+iq^{k+1/2}\right)}. \tag{4.19}$$

Proof. Formula (4.18) is the special case $\alpha = 1$ and $b = aq$ of (4.16) while (4.19) is the speical case $a = -q^{1/2}i$ of (4.18). □

The special choice $\alpha = 1$ and $b = aq$ in (4.17) establishes

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{q^{9n^2}x^{3n}}{1-a^3q^{3n}} &= \frac{\left(q^3;q^3\right)_\infty^2}{(q;q)_\infty^6} \frac{(a,q/a;q)_\infty^3}{(a^3,q^3a^{-3};q^3)_\infty} \\ &\times \sum_{j,k,\ell=-\infty}^{\infty} \frac{\rho^{k+2\ell} q^{(j+k+\ell)^2} x^{j+k+\ell}}{(1-aq^j)(1-aq^k)(1-aq^\ell)}. \end{aligned} \tag{4.20}$$

Two special case of (4.20) are worth noting. First when $a = q^{1/3}$ we find that

$$\begin{aligned} &\frac{(q;q)_\infty^7}{(q^3;q^3)_\infty^3\left(q^{1/3},q^{2/3};q\right)_\infty^3} \sum_{n=-\infty}^{\infty} \frac{q^{9n^2}x^{3n}}{1-q^{3n+1}} \\ &= \sum_{j,k,\ell=-\infty}^{\infty} \frac{\rho^{k+2\ell} q^{(j+k+\ell)^2} x^{j+k+\ell}}{\left(1-q^{j+1/3}\right)\left(1-q^{k+1/3}\right)\left(1-q^{\ell+1/3}\right)}. \end{aligned} \tag{4.21}$$

With $a = -q^{1/3}$ in (4.20) we conclude that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{q^{9n^2}x^{3n}}{1+q^{3n+1}} &= \frac{\left(q^3;q^3\right)_\infty^2}{(q;q)_\infty^6} \frac{\left(-q^{1/3},-q^{2/3};q\right)_\infty^3}{(-q^2,-q;q^3)_\infty} \\ &\times \sum_{j,k,\ell=-\infty}^{\infty} \frac{\rho^{k+2\ell} q^{(j+k+\ell)^2} x^{j+k+\ell}}{\left(1+q^{j+1/3}\right)\left(1+q^{k+1/3}\right)\left(1+q^{\ell+1/3}\right)}. \end{aligned} \tag{4.22}$$

It is clear that one can generate other identities by specializing the parameters in the master formulas.

5 q-Lommel Polynomials

Iterating the three term recurrence relation of the q-Bessel function leads to

$$q^{n\nu+n(n-1)/2}J^{(2)}_{\nu+n}(x;q) = h_{n,\nu}\left(\frac{1}{x};q\right)J^{(2)}_{\nu}(x;q) - h_{n-1,\nu+1}\left(\frac{1}{x};q\right)J^{(2)}_{\nu-1}(x;q), \tag{5.1}$$

where $h_{n,\nu}(x;q)$ are the q-Lommel polynomials introduced in [11], [12, §14.4]. It is more convenient to use the polynomials

$$p_{n,\nu}(x;q) := e^{-i\pi n/2}h_{n,\nu}(ix) = \sum_{j=0}^{\lfloor n/2\rfloor}\frac{(q^\nu,q;q)_{n-j}}{(q,q^\nu;q)_j(q;q)_{n-2j}}(2x)^{n-2j}q^{j(j+\nu-1)}. \tag{5.2}$$

The identity (5.1) expressed in terms of I_ν's is

$$\begin{aligned}&(-1)^n q^{n\nu+n(n-1)/2}I^{(2)}_{\nu+n}(x;q)\\ &= p_{n,\nu}(1/x;q)I^{(2)}_{\nu}(x;q) - p_{n-1,\nu+1}(1/x;q)I^{(2)}_{\nu-1}(x;q),\end{aligned} \tag{5.3}$$

When $x = 2q^{-k/2}$ we obtain, after replacing ν by $\nu+k$,

$$\begin{aligned}(-1)^n q^{n(n+2\nu+k-1)/2}S_k\left(-q^{\nu+n};q\right) &= p_{n,\nu+k}(q^{k/2}/2;q)S_k\left(-q^{\nu};q\right)\\ &-q^{k/2}p_{n-1,\nu+k+1}(q^{k/2}/2;q)S_k\left(-q^{\nu-1};q\right).\end{aligned}$$

We now rewrite this as a functional equation in the form

$$\begin{aligned}y^n q^{n(n+k-1)/2}S_k\left(yq^n;q\right) &= u_n(q^{k/2},-yq^k;q)S_k\left(y;q\right)\\ &-q^{k/2}u_{n-1}(q^{k/2},-yq^{k+1};q)S_k\left(y/q;q\right).\end{aligned} \tag{5.4}$$

with

$$u_n(x,y) = \sum_{j=0}^{\lfloor n/2\rfloor}\frac{(y,q;q)_{n-j}}{(q,y;q)_j(q;q)_{n-2j}}x^{n-2j}. \tag{5.5}$$

Therefore

$$\begin{aligned}S_k(y;q) &= \frac{y^n q^{n(n+k-1)/2}u_n(q^{k/2},-yq^{k+1};q)}{\Delta_n}S_k(yq^n;q)\\ &-\frac{y^{n+1}q^{(n+1)(n+k)/2}u_{n+1}(q^{k/2},-yq^{k+1};q)}{\Delta_n}S_k(-q^{\nu+n+1};q),\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}\Delta_n &= u_n(q^{k/2},-yq^{k+1};q)u_n(q^{k/2},-yq^k;q)\\ &-u_{n+1}(q^{k/2},-yq^k;q)u_{n-1}(q^{k/2},-yq^{k+1};q).\end{aligned} \tag{5.7}$$

6 Identities Involving Stieltjs–Wigert Polynomials

In this section we state several identities involving Stieltjes–Wigert polynomials and the Ramanujan function.

(6.1) $$(xt, -t; q)_\infty = \sum_{n=0}^{\infty} q^{\binom{n}{2}} t^n S_n\left(xq^{-n}; q\right).$$

(6.2) $$\frac{q^{\binom{n}{2}} x^n}{(q;q)_n} = \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_{n-k}} S_k\left(xq^{-k}; q\right),$$

(6.3) $$S_n(x) = \sum_{k=0}^{\infty} \frac{q^{\binom{k+1}{2}} (xq^n)^k A_q\left(xq^k\right)}{(q;q)_n (q;q)_k},$$

(6.4) $$S_n(ab; q) = b^n \sum_{k=0}^{n} \frac{\left(b^{-1}; q\right)_k \left(-q^{1-n}\right)^k q^{\binom{k}{2}}}{(q;q)_k} S_{n-k}\left(aq^k; q\right),$$

(6.5) $$S_n(a; q) = \frac{(-aq;q)_\infty}{(q, -aq; q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2} (-a)^k}{(q, -aq^{n+1}; q)_k},$$

(6.6) $$S_{2n+1}\left(q^{-2n-1}; q\right) = 0, \quad S_{2n}\left(q^{-2n}; q\right) = \frac{(-1)^n q^{-n^2}}{(q^2; q^2)_n}.$$

(6.7) $$S_n\left(-q^{-n+1/2}; q\right) = \frac{q^{-\left(n^2-n\right)/4}}{\left(q^{1/2}; q^{1/2}\right)_n},$$

(6.8) $$S_n\left(-q^{-n-1/2}; q\right) = \frac{q^{-\left(n^2+n\right)/4}}{\left(q^{1/2}; q^{1/2}\right)_n},$$

(6.9) $$A_q(wz) = (wq;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} w^n}{(wq;q)_n} S_n\left(zq^{-n}; q\right).$$

(6.10) $$A_q(z) = (q;q)_m \sum_{n=0}^{\infty} \frac{q^{n^2+mn} (-z)^n}{(q;q)_n} S_m\left(zq^n; q\right).$$

Proofs. Formula (6.1) follows from the definition (1.8) and Euler's identities. Dividing both sides of (6.1) by $(-t;q)_\infty$ then expand $1/(-t;q)_\infty$ on the right-hand side implies (6.2). The expansion (6.3) follows from (1.10), and the q-binomial theorem in the form

(6.11) $$(x;q)_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-x)^j q^{\binom{k}{2}}.$$

To prove (6.4) start with (6.1) as

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} t^n S_n(abq^{-n}; q) = (abt, -t; q)_\infty = (abt, -bt; q)_\infty \frac{(-t;q)_\infty}{(-bt;q)_\infty},$$

then expand the first product in $S_k(aq^{-k}; q)$ and the second term using the q-binomial theorem. The proof of (6.5) consists of writing $(-aq;q)_\infty/(-aq;q)_n(-aq^{n+1};q)_k$ as $-aq^{n+k+1};q)_\infty$ then expand this infinite product and use (6.11). The special values in (6.6) follow from letting $x = 1$ in (6.1) then equate like powers of t. Similarly the special values in (6.7) and (6.8) follow from putting $x = -q^{\mp 1/2}$ in (6.1). Replace x by z in then multiply by $(-w)^n q^{\binom{n+1}{2}}$ and sum to prove (6.9). To prove (6.10) we expand the right-hand side in powers of z and realize that the coefficient of $(-z)^n$ is

$$\frac{q^{n^2+mn}}{(q;q)_n} {}_2\phi_1(q^{-m}, q^{-n}; 0, ; q, q).$$

By the q-Chu-Vandermonde sum [10, (II.6)] the ${}_2\phi_1$ equals q^{-mn}. $\square$

We note that the polynomials $\{S_n(xq^{-n};q)\}$ are related to the q^{-1}-Hermite polynomials, [5], [13], which are defined by

$$h_n(\sinh\xi \mid q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} (-1)^k q^{k(k-n)} e^{(n-2k)\xi}. \tag{6.12}$$

Indeed

$$S_n(e^{-2\xi}q^{-n};q) = \frac{1}{(q;q)_n} h_n(\sinh\xi \mid q). \tag{6.13}$$

In fact (6.1) is equivalent to the generating function for the q^{-1}-Hermite polynomials, [12]. Moreover (6.13) and the generating function [12, Theorem 21.3.1] lead to

$$\sum_{n=0}^{\infty} \frac{(q;q)_n q^{n^2/4}}{(\sqrt{q};\sqrt{q})_n} t^n S_n(zq^{-n};q) = \frac{(-tq^{1/4}, -tq^{1/4}z; \sqrt{q})_\infty}{(-t^2z;q)_\infty}. \tag{6.14}$$

The Poisson kernel of q^{-1}-Hermite polynomials, [12, Theorem 21.2.3] implies

$$\sum_{n=0}^{\infty} (q;q)_n q^{\binom{n}{2}} t^n S_n(zq^{-n};q) S_n(\zeta q^{-n};q) = \frac{(-t, -tz\zeta, tz, t\zeta; q)_\infty}{(t^2 z\zeta/q;q)_\infty}. \tag{6.15}$$

Similarly one can derive other generating relations.

It must be noted that (6.7) and (6.8) when written in terms of the q^{-1}-Hermite polynomials are the evaluation of $h_n(0|q)$, see [12, Corollary 21.2.2]. It is easy to see that the evaluations (6.7) and (6.8) are equivalent to the identity in the following theorem.

Theorem 6.1. *We have*

$$A_{q^2}\left(-b^2\right) = (b\sqrt{q};q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2/2} b^n}{\left(q, b\sqrt{q};q\right)_n}, \tag{6.16}$$

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