q-distributions in complex systems: a brief review

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(Received on 26 March, 2009)

The nonextensive statistical mechanics proposed by Tsallis is today an intense and growing research field. Probability distributions which emerges from the nonextensive formalism (q-distributions) have been applied to an impressive variety of problems. In particular, the role of q-distributions in the interdisciplinary field of complex systems has been expanding. Here, we make a brief review of q-exponential, q-Gaussian and q-Weibull distributions focusing some of their basic properties and recent applications. The richness of systems analyzed may indicate future directions in this field.

Keywords: q-exponential, q-Gaussian, q-Weibull, Nonextensive statistics

1. INTRODUCTION

Common characteristics of complex systems include long-range correlations, multifractality and non-Gaussian distributions with asymptotic power law behavior. Typically, such systems are not well described by approaches based on the usual statistical mechanics. In this scenario, a new formalism capable of providing a better description of complex systems is welcome. This is the case of the generalized (nonextensive) statistical mechanics proposed by Tsallis - nowadays, an intense and growing research field[1–4].

Concepts related with nonextensive statistical mechanics have found applications in a variety of disciplines including physics, chemistry, biology, mathematics, geography, economics, medicine, informatics, linguistics among others[5–7]. Probability distributions which emerge from the nonextensive formalism - also called q-distributions - have been applied to an impressive variety of problems in diverse research areas including the interdisciplinary field of complex systems.

In the present work we focus on q-exponential, q-Gaussian and q-Weibull distributions. We summarized some of their basic properties and provide useful references of recent applications. The richness of systems analyzed may indicate future directions in this research line.

2. q-EXPONENTIAL DISTRIBUTION

The q-exponential distribution is given by the probability density function (pdf)

$$p_{qe}(x) = p_0 \left[1-(1-q)\,\frac{x}{x_0}\right]^{1/(1-q)} \tag{1}$$

for $1-(1-q)x/x_0 \geq 0$. If $p_0 = (2-q)/x_0$, eq. (1) is normalized.

In the limit $q \to 1$, eq. (1) recovers the usual exponential distribution in the same way in which the q-exponential function, defined as $e_q^{-x} \equiv [1-(1-q)x]^{1/(1-q)}$, recovers exponential function in the limit $q \to 1$ ($e_1^{-x} \equiv e^{-x}$). If $q < 1$, eq. (1) has a finite value for any finite real value of x since, by definition, $p_{qe}(x) = 0$ for $1-(1-q)x/a < 0$. If $q > 1$, eq. (1) exhibits power law asymptotic behavior,

$$p_{qe}(x) \sim x^{-1/(q-1)}. \tag{2}$$

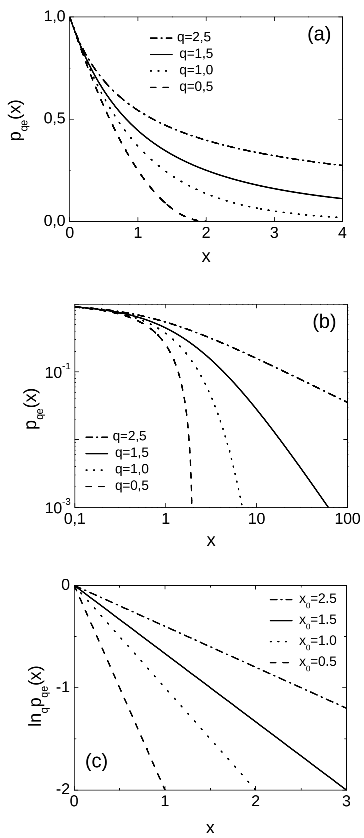

FIG. 1: **q-exponential distribution**. a) Plot of $p_{qe}(x)$ versus x, with $p_0 = x_0 = 1$ and typical values of q. b) Log-log plot of the curves in a). c) $\ln_q p_{qe}(x)$ versus x for $p_0 = 1$ and typical values of x_0.

Note also that $p_{qe}(x) \simeq 1+x$ for small x, independently of the q value. Figures 1a and 1b show $p_{qe}(x)$ versus x for typical values of q.

The q-exponential distribution, for $q > 1$, corresponds to the Zipf-Mandelbrot law[8] and a Burr-type distribution[9]. In this sense, the q-exponential is a generalization of these distributions for $q < 1$. Thus, by choosing suitable values for q, q-exponentials may be used to represent both short and long tailed distributions. This feature also holds for the other q-distributions.

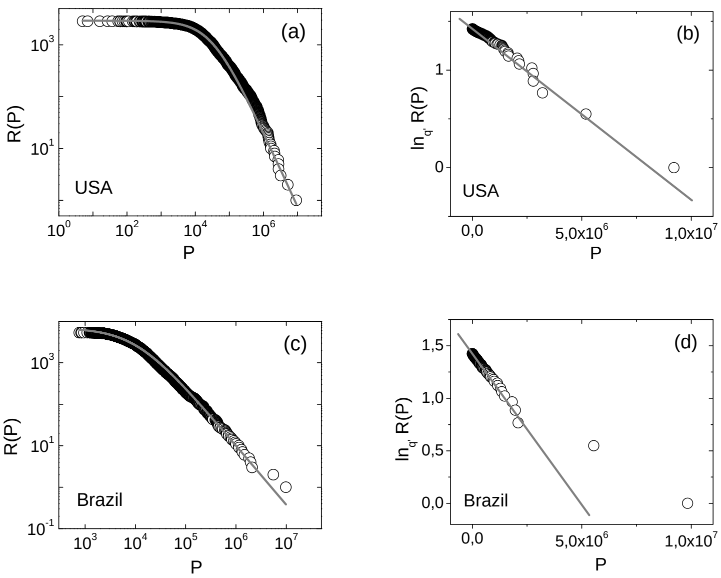

FIG. 2: **Population of cities.** a) Empirical cdf $R(P)$, where P is the population of USA cities. The solid line is a q-exponential, given by eq. (3), with $q' = 1.7$ ($q \simeq 1.4$), $x'_0 = 21,250$ and $c' = 2,919$. b) $\ln_{q'} R(P)$ versus P, with $q' = 1.7$, for the same data shown in (a). The solid line is a linear fit to the data. c) Empirical cdf $R(P)$, where P is the population of Brazilian cities. The solid line is a q-exponential, given by eq. (3), with $q' = 1.7$ ($q \simeq 1.4$), $x'_0 = 7,073$ and $c' = 6,968$. d) $\ln_{q'} R(P)$ versus P, with $q' = 1.7$, for the same data shown in (c). The solid line is a linear fit to the data.

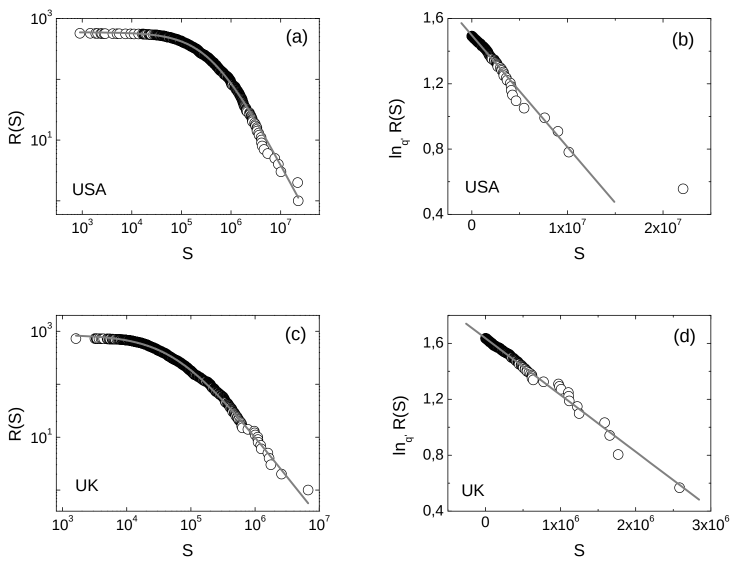

FIG. 3: **Circulation of magazines.** a) Empirical cdf $R(S)$, where S is the circulation of 570 USA magazines in 2004. The solid line is a q-exponential, given by eq. (3), with $q' = 1.65$ ($q \simeq 1.4$), $x'_0 = 255,204$ and $c' = 594$. b) $\ln_{q'} R(S)$ versus S, with $q' = 1.65$, for the same data shown in (a). The solid line is a linear fit to the data. c) Empirical cdf $R(S)$, where S is the circulation of 727 UK magazines in 2005. The solid line is a q-exponential, given by eq. (3), with $q' = 1.65$ ($q \simeq 1.4$), $x'_0 = 37,493$ and $c' = 860$. b) $\ln_{q'} R(S)$ versus S, with $q' = 1.65$, for the same data shown in (c). The solid line is a linear fit to the data.

The cumulative distribution function (cdf) associated to eq. (1) is given by

$$\begin{aligned} R_{qe}(x) &= \int_x^{\infty} p_{qe}(y)\,dy \\ &= p'_0 \left[1-(1-q')\frac{x}{x'_0}\right]^{1/(1-q')}, \end{aligned} \qquad (3)$$

defined for $q < 2$, with $q' = 1/(2-q)$, $x'_0 = x_0/(2-q)$ and $p'_0 = p_0 x_0/(2-q)$. Observe that $R_{qe}(x)$ and $p_{qe}(x)$ exhibit the same mathematical form.

It is possible to visualize q-exponential distributions as straight lines in graphs with appropriate scales. Applying the q-logarithm function, defined as $\ln_q x \equiv [x^{(1-q)} - 1]/(1-q)$, with $\ln_1 x \equiv \ln(x)$, in both sides of eq. (1), we have

$$\ln_q p_{qe}(x) = \ln_q p_0 - [1+(1-q)\ln_q p_0]\frac{x}{x_0}. \qquad (4)$$

A similar result holds for $R_{qe}(x)$. Figure 1c shows $\ln_q p_{qe}(x)$ versus x for typical values of x_0.

The q-exponential function given by eq. (1) has been employed in a growing number of theoretical and empirical works on a large variety of themes. Examples include scale-free networks[10–14], dynamical systems[15–27], algebraic structures[28–31] among other topics in statistical physcics[32–36].

As specific examples of q-exponential distributions in complex systems, let us consider results on population of cities[37] and circulation of magazines[38]. Figure 2 shows the cumulative distribution of the population of cities in the USA and Brazil. Figure 3 shows the cumulative distribution of circulation of magazines in the USA and UK. In both cases - population of cities and circulation of magazines - the empirical data are consistent with a q-exponential distribution, with $q \simeq 1.4$.

q-exponential distributions have also been applied in the empirical study of stock markets[39–42], DNA sequences[43], family names[44], human behavior[45–47], geomagnetic records[48, 49], train delays[50], reaction kinetics[51], air networks[52], hydrological phenomena[53], fossil register[54], basketball[55], earthquakes[56–58], world track records[59], voting processes[60], internet[61], individual success[62], citations of scientific papers[63, 64], football[65], linguistics[66, 67] and solar neutrinos[68, 69].

3. q-GAUSSIAN DISTRIBUTION

The q-Gaussian distribution is specified by the pdf

$$p_{qg}(x) = p_0 \left[1-(1-q)\left(\frac{x}{x_0}\right)^2\right]^{1/(1-q)}, \qquad (5)$$

for $1-(1-q)(x/x_0)^2 \geq 0$ and $p_{qg}(x) = 0$ otherwise. It is normalized if $p_0 = (2/x_0)\sqrt{(q-1)/\pi}\,\Gamma[1/(q-1)]/\Gamma[1/(q-1)-1/2]$. In addition, eq. (5) presents unit variance if $x_0^2 = 5-3q$, with $q < 5/3$.

In the limit $q \to 1$, eq. (5) recovers the usual Gaussian distribution, so $q \neq 1$ indicates a departure from Gaussian statistics. For $q > 1$, the tails of q-Gaussian decrease as power laws,

$$p_{qg}(|x|) \sim |x|^{-2/(q-1)}. \qquad (6)$$

Figures 4a and 4b show $p_{qg}(x)$ for typical values of q.

Applying the q-logarithm function in both sides of eq. (5), we have

$$\ln_q p_{qg}(x) = \ln_q p_0 - [1+(1-q)\ln_q p_0]\left(\frac{x}{x_0}\right)^2. \qquad (7)$$

Figure 4c shows $\ln_q p_{qg}(x)$ versus x^2 for typical values of x_0.

Recent works have been focused on the study of mathematical properties of q-Gaussian functions[70–78], including methods for generating random numbers which follow q-Gaussian distributions[79, 80]. q-Gaussians have been employed in the study of a wide range of themes including probabilistic models[81, 82], stellar plasmas[83], porous-medium equation[84], Bose-condensed gases[85–87],

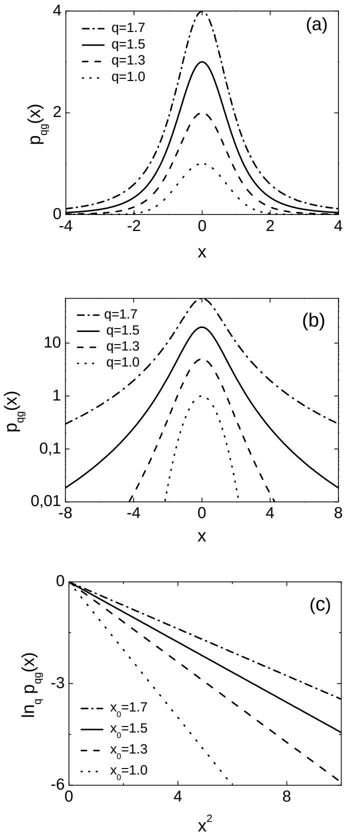

FIG. 4: **q-Gaussian distribution**. a) Plot of $p_{qg}(x)$ versus x, with $p_0 = x_0 = 1$, for typical values of q. Some curves were vertically shifted for a better visualization. b) The same curves shown in a), but for mono-log scale. Some curves were also shifted . c) $\ln_q p_{qg}(x)$ versus x^2 for $p_0 = 1$ and typical values of x_0.

dynamical systems[88–90], polymeric networks[91], small-world networks[92], fingering processes[93], processes with stochastic volatility[94, 95] and nonlinear diffusion[96, 97].

In order to illustrate a recent application of q-Gaussian distributions in complex systems, we mention here results on the dynamics of earthquakes[98]. Figure 5 shows the distribution of energy differences between successive earthquakes at the San Andreas Fault. The empirical data is consistent with a q-Gaussian distribution, with $q = 1.75$.

Other recent applications of q-Gaussian distribution include stock markets[99–107], DNA molecules[108], the solar wind[109–111], galaxies[112], optical lattices[113], cellular aggregates[114] and the atmosphere[115].

4. q-WEIBULL DISTRIBUTION

The q-Weibull distribution is given by the pdf

$$p_{qw}(x) = p_0 \frac{r x^{r-1}}{x_0^r}\left[1-(1-q)\left(\frac{x}{x_0}\right)^r\right]^{1/(1-q)}, \qquad (8)$$

for $1-(1-q)(x/x_0)^r \geq 0$ and $p_{qw}(x) = 0$ otherwise. Eq. (8) is normalized if $p_0 = 2-q$.

In the limits $q \to 1$, $r \to 1$, and $q \to 1$, $r \to 1$, eq. (8) recovers Weibull, q-exponential and exponential distributions, re-

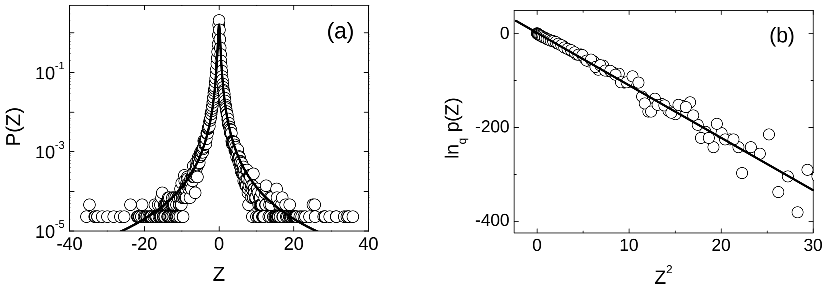

FIG. 5: **Earthquakes**. a) Empirical pdf $P(Z)$, where $Z = E(t+1) - E(t)$ is the energy difference between successive earthquakes at the San Andreas Fault in the period 1966-2006. The solid line is a q-Gaussian, given by eq. (5), with $q = 1.75$, $x_0 = 0.25$ and $p_0 = 1.63$. b) $\ln_q P(Z)$ versus Z^2, with $q = 1.75$, for small values of Z. The solid line is a linear fit to the data.

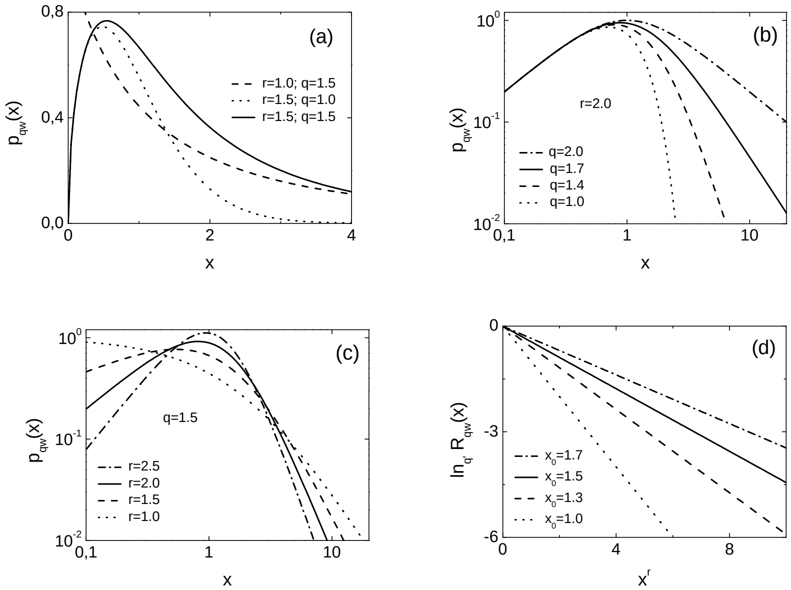

FIG. 6: q-**Weibull distribution**. a) Plot of $p_{qw}(x)$ versus x, with $p_0 = x_0 = 1$, and typical values of q and r. b) Log-log plot of $p_{qw}(x)$ versus x, with $p_0 = x_0 = 1$, $r = 2$ and typical values of q. c) a) Log-log plot of $p_{qw}(x)$ versus x, with $p_0 = x_0 = 1$, $q = 1.5$ and typical values of r. d) $\ln_{q'} R_{qw}(x)$ versus x^r for $p'_0 = 1$ and typical values of x_0.

spectively. If $q < 1$, $p_{qw}(x)$ has a finite limit since $p_{qw}(x) = 0$ for $1-(1-q)(x/x_0)^r < 0$. If $q > 1$, $p_{qw}(x)$ exhibits power law behavior both for small and large values of x. More specifically,

$$p_{qw}(x) \sim x^{-\xi}, \tag{9}$$

with $\xi = (1-r)$ for small x and $\xi = r[(2-q)/(q-1)]+1$ for large x. Figures 6a, 6b and 6c show $p_{qw}(x)$ versus x for typical values of q and r.

The cdf associated to $p_{qw}(x)$ is given by

$$R_{qw}(x) = p'_0 \left[1-(1-q')\left(\frac{x}{x'_0}\right)^r\right]^{1/(1-q')}, \tag{10}$$

with $q' = 1/(2-q)$, $(x'_0)^r = x^r_0/(2-q)$ and $p'_0 = p_0/(2-q)$. Applying the q-logarithm function in both sides of the cdf R_{qw}, we have

$$\ln_{q'} R_{qw}(x) = \ln_{q'} p'_0 - [1+(1-q)\ln_{q'} p'_0]\left(\frac{x}{x_0}\right)^r. \tag{11}$$

Figure 6c shows $\ln_{q'} R_{qW}(x)$ versus x^r for typical values of x_0.

If $p_{qw}(x)$ is normalized ($p_0 = 2-q$), Eq. (11) reduces to $\ln_{q'} R_{qw}(x) = -(x/x_0)^r$. In this case,

$$\ln[-\ln_{q'}(R_{qw}(x))] = r\ln x - r\ln x_0. \tag{12}$$

As specific example of q-Weibull distribution in complex systems, we now consider results on citations in scientific journals[116]. Figure 7 shows the distribution of the impact factor of scientific journals in comparison with a q-Weibull curve. The empirical data is consistent with a q-Weibull distribution, with $q = 1.45$ and $r = 1.50$.

Other recent works have been related to q-Weibull distributions. For example, new classes of generalized asymmetric distributions have been introduced which include q-Weibull as a special case[117, 118]. q-Weibull has also been applied in the study of fractal kinetics[119], dieletric breakdown in oxides[120], relaxation in heterogeneous systems[121], ciclone victims and highway lengths[55] among others.

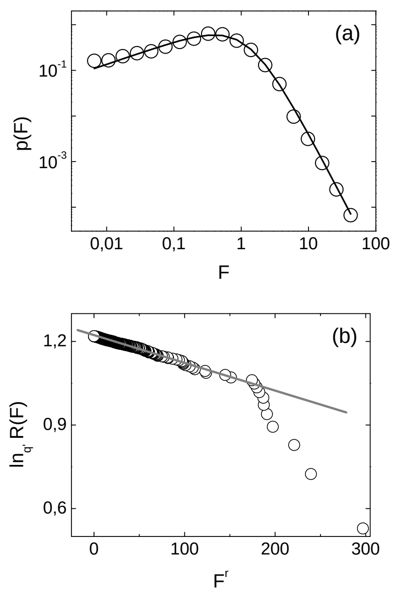

FIG. 7: **Citations in scientific journals.** a) Empirical pdf $p(F)$, where F is the 2004 impact factor for 5912 scientific journals. The solid line is a q-Weibull distribution, given by eq. (8), with $r = 1.5$, $q = 1.45$, $x_0 = 0.74$ and $p_0 = 0.58$. b) $\ln_{q'} R(F)$ versus F^r, with $q' = 1.82$ ($q = 1.45$) and $r = 1.5$. The solid line is a linear fit to the data.

5. BASIS FOR q-DISTRIBUTIONS

¿From the viewpoint of the principle of the maximum entropy, some q-distributions optimizes generalized entropies - more general entropic measures than the standard Boltzmann-Gibbs entropy. A striking example is the q-entropy proposed by Tsallis[1]

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}, \tag{13}$$

where W is the total number of microstates of the system, p_i are the occupation probabilities and q is a real parameter. The standard Boltzmann-Gibbs entropy is recovered in the limit $q \to 1$.

The maximization of S_q subject to specific constraints generates occupation probabilities following a q-exponential distribution. The q-exponential optimizes other generalized entropic measures such as the Renyi and normalized Tsallis entropies. However, only Tsallis entropy can provide an appropriate basis for the q-exponential distribution since it presents several properties essential for an entropy[122, 123]. Changing the constraints, the maximization of S_q also generates occupation probabilities following a q-Gaussian distribution.

Formally, q-distributions can arise when the exponential function of the original distribution is replaced by a q-exponential function. For example, this basic procedure applied in standard exponential, Gaussian and Weibull distributions leads to q-exponential, q-Gaussian and q-Weibull, respectively[55]. This viewpoint suggests the consideration of other q-distributions which could be obtained by simply replacing its exponential function by a q-exponential one.

q-distributions can also emerge from compound distributions[124]

$$p_q(x) = \int_0^\infty p(x,\lambda) f(\lambda) d\lambda, \tag{14}$$

where $f(\lambda)$ is a Gamma function. For example, if $p(x,\lambda)$ is a Weibull distribution, $p_q(x)$ is given by a q-Weibull distribution[120]. Naturally, other forms for $f(\lambda)$ may be considered to obtain alternative distributions. In a physical context, this scenario has been explored with success in superstatistics where nonequilibrium situations with local fluctuations of the environment are taken into account[125–127].

The generalized distributions considered here can also be obtained from the following ordinary differential equation:

$$\frac{dy}{dx} = \rho y^q. \tag{15}$$

In fact, if ρ is constant, the solution of eq. (15) is a q-exponential; if $\rho \propto x$, the solution is a q-Gaussian. If y is the cdf and $\rho \propto x^r$, we have a q-Weibull. By considering further terms in eq. (15), other q-distributions can be obtained[128]. q-distributions can also emerge in other contexts. For instance, q-Gaussian arises from the non-linear diffusion (porous media) equation[84] and from a generalization of the central limit theorem[3]. Another example is the q-lognormal distribution which emerges from generalized cascades[28].

6. CONCLUSION

The present work presents a brief overview of recent applications of some q-distributions largely used in the context of Tsallis statistics. It illustrates how q-exponential, q-Gaussian and q-Weibull distributions have been applied in the study of a wide variety of systems in several fields.

The success of q-distributions in describing diverse systems is in part due to its ability of exhibit heavy-tails and model power law phenomena - a typical characteristic of complex systems. The positive and exciting results obtained with q-distributions also indicate possible applications of Tsallis nonextensive statistical mechanics. Naturally, further work may be necessary to explore possible relations between the analyzed systems and the present theory.

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