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q-linear Functions and Algebraic Independence

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Abstract. We define q-linear arithmetical functions and $-q$-linear ones and show the algebraic independence over $\mathbf{C}(z)$ of their generating functions.

1. Introduction.

Algebraic independence of power series and their values defined by digital properties of integers have been investigated by many authors (cf. [4], [6], [9], [10], [11]).

Let $q \geq 2$ be an integer. An arithmetical function $a(n) : \mathbf{N} \to \mathbf{C}$ is called *strongly q-additive* if $a(nq + r) = a(n) + a(r)$ $(n \geq 0, 0 \leq r < q)$, where $\mathbf{N} = \{0, 1, 2, \cdots\}$. Its generating function $f(z) = \sum_{n=0}^{\infty} a(n)z^n$ $(|z| < 1)$ satisfies the functional equation

$$f(z) = \frac{1 - z^q}{1 - z} f(z^q) + \frac{1}{1 - z^q} \sum_{r=1}^{q-1} a(r)z^r .$$

Toshimitsu [10] proved that, if $a_1(n), \cdots, a_m(n)$ are strongly q-additive functions, the functions $g_k(z) = \sum_{n=0}^{\infty} a_k(n)z^n$ $(1 \leq k \leq m)$ are algebraically independent over $\mathbf{C}(z)$ if and only if $(a_k(1), \cdots, a_k(q-1)) \in \mathbf{C}^{q-1}$ $(1 \leq k \leq m)$ are linearly independent over $\mathbf{C}$. As a corollary, the algebraic independence of the values $g_k(\alpha)$ $(1 \leq k \leq m)$ for any fixed algebraic number α with $0 < |\alpha| < 1$ can be deduced. A typical example of a strongly q-additive function is the sum of digits function $s_q(n) = \sum_{h=0}^{k} d_h$, where

$$n = \sum_{h=0}^{k} d_h q^h , \quad d_h \in \{0, 1, \cdots, q-1\} , \quad d_k \neq 0 \quad \text{if } n \neq 0 \tag{1}$$

is the q-adic expansion of $n \in \mathbf{N}$. The sum $\sum_{n \leq x} s_q(n)$ and also the power sum $\sum_{n \leq x} s_q(n)^l$ $(l \geq 1)$ have been extensively studied (cf. [1], [8], [9]).

In this paper we introduce *q-linear* functions and *$-q$-linear* ones and prove the algebraic independence of the generating functions and their values. Our method of proof is to apply two basic theorems in transcendence theory of Mahler functions (see Lemmas 2.1 and 2.2 below).

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An arithmetical function $a(n) : \mathbf{N} \to \mathbf{C}$ is called q-linear, if there is an $\alpha \in \mathbf{C}^{\times}$ such that

$$a(nq + r) = \alpha a(n) + a(r) \tag{2}$$

for any integers $n \geq 0$ and $0 \leq r < q$. By definition $a(0) = 0$. A q-linear function $a(n)$ is uniquely determined by the coefficient α and the initial vector $\boldsymbol{a} := (a(1), \cdots, a(q-1))$; namely

$$a(n) = \sum_{h=0}^{k} a(d_h)\alpha^h ,$$

where d_h are as in (1). Especially, $a(n)$ is not identically zero if $\boldsymbol{a} \neq (0, \cdots, 0)$.

Next we define $-q$-linearlity. An arithmetical function $b(n) : \mathbf{Z} \to \mathbf{C}$ is called $-q$-linear, if there is a $\beta \in \mathbf{C}^{\times}$ such that

$$b(n(-q) + r) = \beta b(n) + b(r) \tag{3}$$

for any integers n and $0 \leq r < q$. We note that $b(0) = 0$. A $-q$-linear function $b(n)$ is determined uniquely by β and $\boldsymbol{b} = (b(1), b(2), \cdots, b(q-1))$. Every $n \in \mathbf{Z}$ can be expanded uniquely as

$$n = \sum_{h=0}^{k} e_h(-q)^h , \quad e_h \in \{0, 1, \cdots, q-1\} , \quad e_k \neq 0 \quad \text{if } n \neq 0 \tag{4}$$

(cf. [3, Chap. 4]). We note that $n > 0$ if and only if k is even. Then we have

$$b(n) = \sum_{h=0}^{k} b(e_h)\beta^h .$$

EXAMPLES. We give some examples of q-linear functions and $-q$-linear ones.

1. The strongly q-additive function defined above is q-linear with $\alpha = 1$. In particular the sum of digits function $s_q(n)$ is q-linear with $\alpha = 1$ and $\boldsymbol{a} = (1, 2, \cdots, q-1)$, and the sum of digits function in base $-q$, i.e. $s_{-q}(n) = \sum_{h=0}^{k} e_h$ ($n \in \mathbf{Z}$) where e_h are given by (4) is $-q$ -linear with $\beta = 1$ and $\boldsymbol{b} = (1, 2, \cdots, q-1)$. The sum $\sum_{n \leq x} s_{-q}(n)$ behaves similarly as the sum $\sum_{n \leq x} s_q(n)$ mentioned above (cf. [2]).

2. The radical inverse function $\phi_q(n)$ defined by $\phi_q(n) = \sum_{h=0}^{k} d_h q^{-h-1} = 0.d_0 d_1 \cdots d_k$, (cf. [5, Chap. 3]) is q-linear with $\alpha = q^{-1}$ and $\boldsymbol{a} = q^{-1}(1, 2, \cdots, q-1)$, where d_h are given by (1). The radical inverse function in base $-q$ defined similarly as above by $\phi_{-q}(n) = \sum_{h=0}^{k} e_h(-q)^{-h-1}$ ($n \in \mathbf{Z}$) is $-q$-linear with $\beta = -q^{-1}$ and $\boldsymbol{b} = -q^{-1}(1, 2, \cdots, q-1)$. Moreover, the generalized radical inverse function $\phi_q^{\sigma}(n)$ is defined by $\phi_q^{\sigma}(n) = \sum_{h=0}^{k} d_h^{\sigma} q^{-h-1} = 0.d_0^{\sigma} d_1^{\sigma} \cdots d_k^{\sigma}$, where σ is a permutation of $\{0, 1, \cdots, q-1\}$ with $0^{\sigma} = 0$, which is also q-linear with $\alpha = q^{-1}$ and $\boldsymbol{a} = q^{-1}(1^{\sigma}, 2^{\sigma}, \cdots, (q-1)^{\sigma})$. Similarly, $\phi_{-q}^{\sigma}(n)$ can be defined.

3. The bases change function $\gamma_q^p(n)$. For any $p \in \mathbf{Z}$ with $|p| \geq q$, the bases change function $\gamma_q^p(n) : \mathbf{N} \to \mathbf{Z}$ defined by $\gamma_q^p(n) = \sum_{h=0}^{k} d_h p^h$ is q-linear with $\alpha = p$ and $\boldsymbol{a} = (1, 2, \cdots, q-1)$, and the bases change function $\gamma_{-q}^p(n) : \mathbf{Z} \to \mathbf{Z}$ defined by $\gamma_{-q}^p(n) = \sum_{h=0}^{k} e_h p^h$ is $-q$-linear with $\beta = p$ and $\boldsymbol{b} = (1, 2, \cdots, q-1)$.

4. We note that the linear function $a(n) = cn$ ($c \in \mathbf{C}^\times$) is q-linear with $\alpha = q$ and $\boldsymbol{a} = c(1, 2, \cdots, q-1)$, and $b(n) = cn$ ($n \in \mathbf{Z}$, $c \in \mathbf{C}^\times$) is $-q$-linear with $\beta = -q$ with $\boldsymbol{b} = c(1, 2, \cdots, q-1)$.

Let $a(n)$ be a q-linear function with the coefficient α. Then

$$f(z) = \sum_{n=1}^{\infty} a(n) z^n$$

converges in $|z| < 1$ by the definition (2) of q-linearlity and satisfies the functional equation

$$f(z) = \alpha \frac{1-z^q}{1-z} f(z^q) + \frac{1}{1-z^q} \sum_{r=1}^{q-1} a(r) z^r \,,$$

since

$$\begin{aligned} f(z) &= \sum_{r=0}^{q-1} \sum_{n=0}^{\infty} a(nq+r) z^{nq+r} \\ &= \alpha \sum_{r=0}^{q-1} \sum_{n=0}^{\infty} a(n) z^{nq+r} + \sum_{r=0}^{q-1} \sum_{n=0}^{\infty} a(r) z^{nq+r} \,. \end{aligned}$$

We note that for $a(n) = cn$ ($c \in \mathbf{C}^\times$) in Example 4

$$f(z) = \frac{cz}{(1-z)^2} \in \mathbf{C}(z) \,.$$

Let $b(n)$ be a $-q$-linear function with the coefficient β and let

$$g(z) = \sum_{n=1}^{\infty} b(n) z^n \,, \quad g^*(z) = \sum_{n=1}^{\infty} b(-n) z^n \,.$$

These power series converge in $|z| < 1$ by (3) and satisfy the functional equations

$$g(z) = \beta \frac{1-z^q}{1-z} g^*(z^q) + \frac{1}{1-z^q} \sum_{r=1}^{q-1} b(r) z^r,$$

$$g^*(z) = \beta z^{-q+1} \frac{1-z^q}{1-z} g(z^q) + \frac{1}{1-z^q} \sum_{r=1}^{q-1} b(q-r) z^r.$$

Indeed, we have using (3)

$$\begin{aligned}
g(z) &= \sum_{r=0}^{q-1}\sum_{n=0}^{\infty} b(nq+r)z^{nq+r} \\
&= \sum_{r=0}^{q-1}\sum_{n=0}^{\infty} b((-n)(-q)+r)z^{nq+r} \\
&= \beta\sum_{r=0}^{q-1}\sum_{n=0}^{\infty} b(-n)z^{nq+r} + \sum_{r=0}^{q-1}\sum_{n=0}^{\infty} b(r)z^{nq+r}\,,
\end{aligned}$$

$$\begin{aligned}
g^*(z) &= \sum_{r=0}^{q-1}\sum_{n=0}^{\infty} b(-(nq+r))z^{nq+r} \\
&= \sum_{n=1}^{\infty} b(n(-q))z^{nq} + \sum_{r=1}^{q-1}\sum_{n=1}^{\infty} b(n(-q)+r)z^{nq-r} \\
&= \beta\sum_{r=0}^{q-1}\sum_{n=1}^{\infty} b(n)z^{nq-r} + \sum_{r=1}^{q-1}\sum_{n=1}^{\infty} b(r)z^{nq-r}\,.
\end{aligned}$$

We note that for $b(n) = cn$ ($c \in \mathbf{C}^\times$) in Example 4,

$$g(z) = \frac{cz}{(1-z)^2}\,, \quad g^*(z) = -\frac{cz}{(1-z)^2} \in \mathbf{C}(z)\,.$$

Putting

$$F(z) = (1-z)f(z)\,, \quad G(z) = (1-z)g(z)\,, \quad G^*(z) = (1-z)g^*(z)\,,$$

we have the system of functional equations

$$\begin{pmatrix} F(z^q) \\ G(z^q) \\ G^*(z^q) \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ 0 & 0 & \beta^{-1}z^{q-1} \\ 0 & \beta^{-1} & 0 \end{pmatrix}\begin{pmatrix} F(z) \\ G(z) \\ G^*(z) \end{pmatrix} - \frac{1}{\varphi(z)}\begin{pmatrix} \alpha^{-1}\rho(z) \\ \beta^{-1}z^{q-1}\tau(z) \\ \beta^{-1}\sigma(z) \end{pmatrix}\,, \quad (5)$$

where $\rho(z) = \sum_{r=1}^{q-1} a(r)z^r$, $\sigma(z) = \sum_{r=1}^{q-1} b(r)z^r$, $\tau(z) = \sum_{r=1}^{q-1} b(q-r)z^r$, and

$$\varphi(z) = \sum_{r=0}^{q-1} z^r = \frac{1-z^q}{1-z}\,. \quad (6)$$

We state our theorem. Let $a_{ij}(n)$ $(1 \le i \le h, 1 \le j \le m(i))$ be q-linear functions with the coefficient $\alpha_i \in \mathbf{C}^\times$ and let $b_{ij}(n)$ $(1 \le i \le h, 1 \le j \le n(i))$ be $-q$-linear functions with the coefficient $\beta_i \in \mathbf{C}^\times$, where

$$\alpha_i \ne \alpha_k\,, \quad \beta_i \ne \beta_k \quad (i \ne k)\,. \quad (7)$$

We put $\boldsymbol{a}_{ij} = (a_{ij}(1), a_{ij}(2), \cdots, a_{ij}(q-1)), \boldsymbol{b}_{ij} = (b_{ij}(1), b_{ij}(2), \cdots, b_{ij}(q-1))$, and

$$f_{ij}(z) = \sum_{n=1}^{\infty} a_{ij}(n) z^n\,, \quad g_{ij}(z) = \sum_{n=1}^{\infty} b_{ij}(n) z^n\,, \quad g_{ij}^*(z) = \sum_{n=1}^{\infty} b_{ij}(-n) z^n\,.$$

THEOREM 1.1. *Let $f_{ij}(z), g_{ij}(z)$, and $g^*_{ij}(z)$ be as above. Then the functions $f_{ij}(z)$ $(1 \le i \le h,\ 1 \le j \le m(i))$, $g_{ij}(z)$ and $g^*_{ij}(z)$ $(1 \le i \le h, 1 \le j \le n(i))$ are algebraically independent over $\mathbf{C}(z)$ if and only if each of $2h$ sets $\{\boldsymbol{a}_{ij};\, 1 \le j \le m(i)\}$, $\{\boldsymbol{b}_{ij};\, 1 \le j \le n(i)\}$ $(1 \le i \le h)$ are linearly independent over $\mathbf{C}$ and*

$$(1, 2, \cdots, q-1) \notin Span_{\mathbf{C}}\{\boldsymbol{a}_{ij};\, 1 \le j \le m(i)\} \quad if\ \alpha_i = q\,,$$

$$(1, 2, \cdots, q-1) \notin Span_{\mathbf{C}}\{\boldsymbol{b}_{ij};\, 1 \le j \le n(i)\} \quad if\ \beta_i = -q\,.$$

REMARK 1.1. The linear independency of $\boldsymbol{a}_{ij}$ $(1 \le j \le m(i))$ and that of $\boldsymbol{b}_{ij}$ $(1 \le j \le n(i))$ imply that $m(i) < q$ and if $m(i) = q-1$ then $\alpha_i \ne q$, and also $n(i) < q$ and if $n(i) = q-1$ then $\beta_i \ne -q$.

COROLLARY 1.1. *Let $f_{ij}(z), g_{ij}(z)$, and $g^*_{ij}(z)$ be as in Theorem 1.1. Assume that $\alpha_i, \beta_i, a_{ij}(n), b_{ij}(n)$ belong to an algebraic number field $\boldsymbol{K}$ for all i, j and $1 \le n < q$. If α is an algebraic number with $0 < |\alpha| < 1$, then $f_{ij}(\alpha)$ $(1 \le i \le h, 1 \le j \le m(i))$, $g_{ij}(\alpha)$ and $g^*_{ij}(\alpha)$ $(1 \le i \le h, 1 \le j \le n(i))$ are algebraically independent.*

EXAMPLES. We give some examples of Theorem 1.1.

1. The generating functions of the sum of digits functions $\sum_{n\ge1} s_q(n)z^n$, $\sum_{n\ge1} s_{-q}(n)z^n$, and $\sum_{n\ge1} s_{-q}(-n)z^n$ are algebraically independent over $\mathbf{C}(z)$. Let $\mathrm{ord}_q m$ be defined by $m = aq^{\mathrm{ord}_q m}$ with $q \nmid a$. We remark that, if q is a prime, the functions $\sum_{n\ge1} \mathrm{ord}_q n! z^n$ and $\sum_{n\ge1} s_q(n)z^n$ are linearly dependent over $\mathbf{Q}$ mod $\mathbf{Q}(z)$, since $\mathrm{ord}_q n! = (n - s_q(n))/(q-1)$.

2. Let σ be the cyclic permutation of $\{1, 2, \cdots, q-1\}$ and let ϕ^{σ^i} be the generalized radical inverse functions. Then the functions $\sum_{n\ge1} \phi_q^{\sigma^i}(n)z^n$, $\sum_{n\ge1} \phi_{-q}^{\sigma^i}(n)z^n$, and $\sum_{n\ge1} \phi_{-q}^{\sigma^i}(-n)z^n$ $(0 \le i \le q-2)$ are algebraically independent, since the initial vectors $(1, 2, \cdots, q-1)^{\sigma^i}$ $(i = 0, 1, \cdots, q-2)$ are linearly independent over $\mathbf{C}$, because

$$\det \begin{pmatrix} 1 & 2 & \cdots & q-1 \\ q-1 & 1 & \cdots & q-2 \\ \vdots & & \ddots & \vdots \\ 2 & 3 & \cdots & 1 \end{pmatrix} \ne 0\,.$$

3. The generating functions of bases change functions $\sum_{n\ge1} \gamma_q^p(n)z^n$, $\sum_{n\ge1} \gamma_{-q}^p(n)z^n$, and $\sum_{n\ge1} \gamma_{-q}^p(-n)z^n$ $(p \in \mathbf{Z}\,, |p| > q)$ are algebraically independent.

Furthermore, all these functions except $\sum_{n\geq 1} \mathrm{ord} n! z^n$ in Example 1 are algebraically independent over $\mathbf{C}(z)$ and their values at algebraic α $(0 < |\alpha| < 1)$ are algebraically independent.

2. Two lemmas.

The proof depends on the following lemmas.

LEMMA 2.1 (cf. [7, Theorem 3.5]). *Let $d > 1$ be an integer. Let $f_{ij}(z) \in \mathbf{C}[[z]]$ $(1 \leq i \leq h, 1 \leq j \leq n(i))$ satisfy the functional equations*

$$f_{ij}(z^d) = a_i(z) f_{ij}(z) + b_{ij}(z)\,, \tag{8}$$

where $a_i(z), b_{ij}(z) \in \mathbf{C}(z)^\times$. Suppose that

(i) $a_i(z)/a_j(z) \notin H := \{g(z^d)/g(z);\ g(z) \in \mathbf{C}(z)^\times\}$ $(i \neq j)$,

(ii) *if $c_{ij} \in \mathbf{C}$ $(1 \leq j \leq n(i))$ are not all zero, there is no $f(z) \in \mathbf{C}(z)$ such that*

$$f(z^d) = a_i(z) f(z) + \sum_{j=1}^{n(i)} c_{ij} b_{ij}(z)\,.$$

Then the functions $f_{ij}(z)$ $(1 \leq i \leq h, 1 \leq j \leq n(i))$ are algebraically independent over $\mathbf{C}(z)$.

LEMMA 2.2 (cf. [7, Theorem 4.2.1]). *Let $\boldsymbol{K}$ be an algebraic number field, and let $f_1(z), \cdots, f_m(z) \in \boldsymbol{K}[[z]]$ converge in a disc $U \subset \{|z| < 1\}$. Suppose that for an integer $d > 1$, the functional equation*

$$\begin{pmatrix} f_1(z^d) \\ \vdots \\ f_m(z^d) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix} + B(z) \tag{9}$$

is fulfiled, where $A(z)$ is an $m \times m$ matrix with entries in $\boldsymbol{K}(z)$ and B(z) is an m-dimensional vector with entries in $\boldsymbol{K}(z)$. If $\alpha \in U$ is a nonzero algebraic number such that α^{d^k} is not a pole of $A(z), B(z)$ for any $k \geq 0$, then

$$\textit{Trans. deg}_{\mathbf{Q}} \mathbf{Q}(f_1(\alpha), \cdots, f_m(\alpha)) \geq \textit{Trans. deg}_{\boldsymbol{K}(z)} \boldsymbol{K}(z)(f_1(z), \cdots, f_m(z))\,.$$

Corollary 1.1 follows from Theorem 1.1 and Lemma 2.2, since our functions $f_{ij}(z)$, $g_{ij}(z)$, and $g^*_{ij}(z)$ satisfy the functional equations of the form (9). So we prove only Theorem 1.1. It is enough to show if part, since the converse is trivial. The algebraic independence of functions over $\mathbf{C}(z)$ satisfying (9) can be reduced to linear independence over $\mathbf{C}$ mod $\mathbf{C}(\mathbf{z})$ if all the components of $A(z)$ are in $\mathbf{C}$ (cf. [7, Theorem 3.2.2]). However, in the case in which the components are not all constant, it is not easy to decide the algebraic independence of these functions over $\mathbf{C}(z)$. Unfortunately our functional equation (5) contains non constant components. However, by iterating it, we get functional equations of the form (8) with $d = q^2$. Thus we can apply Lemma 2.1, a criterion of algebraic independence of functions over $\mathbf{C}(z)$.

3. Proof of Theorem 1.1.

We shall prove the algebraic independence over $\mathbf{C}(z)$ of the functions

$$F_{ij}(z) = (1-z)f_{ij}(z) \quad (1 \le i \le h,\ 1 \le j \le m(i)),$$

$$G_{ij}(z) = (1-z)g_{ij}(z), \quad G^*_{ij}(z) = (1-z)g^*_{ij}(z) \quad (1 \le i \le h,\ 1 \le j \le n(i)),$$

which satisfy the functional equation of the form (5). By iterating it, we have

$$\left.\begin{aligned} F_{ij}(z^{q^2}) &= \alpha_i^{-2}F_{ij}(z) - \alpha_i^{-2}\left(\alpha_i\frac{\rho_{ij}(z^q)}{\varphi(z^q)} + \frac{\rho_{ij}(z)}{\varphi(z)}\right), \\ G_{ij}(z^{q^2}) &= \beta_i^{-2}z^{q(q-1)}G_{ij}(z) - \beta_i^{-2}z^{q(q-1)}\left(\beta_i\frac{\tau_{ij}(z^q)}{\varphi(z^q)} + \frac{\sigma_{ij}(z)}{\varphi(z)}\right), \\ G^*_{ij}(z^{q^2}) &= \beta_i^{-2}z^{q-1}G^*_{ij}(z) - \beta_i^{-2}\left(\beta_i\frac{\sigma_{ij}(z^q)}{\varphi(z^q)} + z^{q-1}\frac{\tau_{ij}(z)}{\varphi(z)}\right), \end{aligned}\right\} \tag{10}$$

the functional equations of the same forms as (8) with $d = q^2$. However, since $a_i(z)$ in (8) are α_i^{-2}, $\beta_i^{-2}z^{q(q-1)}$, and $\beta_i^{-2}z^{q-1}$ in (10), it may happen even under the assumption (7) of Theorem 1.1 that $a_i(z) = a_k(z)$ for some $i \ne k$, namely if $\alpha_k = -\alpha_i$ or $\beta_k = -\beta_i$. In such cases, we denote for example α_k (or β_k) by $-\alpha_i$ (or $-\beta_i$) and put $m_0(i) = m(i), m_1(i) = m_0(i) + m(k)$ (or $n_0(i) = n(i), n_1(i) = n_0(i) + n(k)$). For notational convenience we assume that $\alpha_i > 0$ if α_i is real and $\beta_i < 0$ if β_i is real. After these change of the subscript if necessarily, we have the stronger assumption

$$\alpha_i^2 \ne \alpha_k^2, \quad \beta_i^2 \ne \beta_k^2 \quad (i \ne k), \tag{11}$$

under which $a_i(z) \ne a_k(z)$ for all $i \ne k$ and the functional equations take the following form

$$\left.\begin{aligned} F_{ij}(z^{q^2}) &= A_i(z)F_{ij}(z) + P_{ij}(z)\ (1 \le i \le h, 1 \le j \le m_1(i)), \\ G_{ij}(z^{q^2}) &= B_i(z)G_{ij}(z) + Q_{ij}(z)\ (1 \le i \le h, 1 \le j \le n_1(i)), \\ G^*_{ij}(z^{q^2}) &= C_i(z)G^*_{ij}(z) + R_{ij}(z)\ (1 \le i \le h, 1 \le j \le n_1(i)), \end{aligned}\right\} \tag{12}$$

where

$$A_i(z) = \alpha_i^{-2}, \quad B_i(z) = \beta_i^{-2}z^{q(q-1)}, \quad C_i(z) = \beta_i^{-2}z^{q-1}, \tag{13}$$

$$P_{ij}(z) = \begin{cases} -\alpha_i^{-2}\left(\alpha_i \dfrac{\rho_{ij}(z^q)}{\varphi(z^q)} + \dfrac{\rho_{ij}(z)}{\varphi(z)}\right) & (1 \le j \le m_0(i)) , \\ -\alpha_i^{-2}\left(-\alpha_i \dfrac{\rho_{ij}(z^q)}{\varphi(z^q)} + \dfrac{\rho_{ij}(z)}{\varphi(z)}\right) & (m_0(i) < j \le m_1(i)) , \end{cases}$$

$$Q_{ij}(z) = \begin{cases} -\beta_i^{-2} z^{q(q-1)}\left(\beta_i \dfrac{\tau_{ij}(z^q)}{\varphi(z^q)} + \dfrac{\sigma_{ij}(z)}{\varphi(z)}\right) & (1 \le j \le n_0(i)) , \\ -\beta_i^{-2} z^{q(q-1)}\left(-\beta_i \dfrac{\tau_{ij}(z^q)}{\varphi(z^q)} + \dfrac{\sigma_{ij}(z)}{\varphi(z)}\right) & (n_0(i) < j \le n_1(i)) , \end{cases}$$

$$R_{ij}(z) = \begin{cases} -\beta_i^{-2}\left(\beta_i \dfrac{\sigma_{ij}(z^q)}{\varphi(z^q)} + z^{q-1} \dfrac{\tau_{ij}(z)}{\varphi(z)}\right) & (1 \le j \le n_0(i)) , \\ -\beta_i^{-2}\left(-\beta_i \dfrac{\sigma_{ij}(z^q)}{\varphi(z^q)} + z^{q-1} \dfrac{\tau_{ij}(z)}{\varphi(z)}\right) & (n_0(i) < j \le n_1(i)) , \end{cases}$$

$$\rho_{ij}(z) = \sum_{r=1}^{q-1} a_{ij}(r) z^r , \quad \sigma_{ij}(z) = \sum_{r=1}^{q-1} b_{ij}(r) z^r , \quad \tau_{ij}(z) = \sum_{r=1}^{q-1} b_{ij}(q-r) z^r .$$

Then we assume that, for each i, each of the sets $\{\boldsymbol{a}_{ij};\, 1 \le j \le m_0(i)\}$, $\{\boldsymbol{a}_{ij};\, m_0(i) < j \le m_1(i)\}$, $\{\boldsymbol{b}_{ij};\, 1 \le j \le n_0(i)\}$, and $\{\boldsymbol{b}_{ij};\, n_0(i) < j \le n_1(i)\}$ are linearly independent over $\mathbf{C}$ and

$$(1, 2, \cdots , q-1) \notin \mathrm{Span}_{\mathbf{C}}\{\boldsymbol{a}_{ij};\, 1 \le j \le m_0(i)\} \quad \text{if } \alpha_i = q , \tag{14}$$

$$(1, 2, \cdots , q-1) \notin \mathrm{Span}_{\mathbf{C}}\{\boldsymbol{b}_{ij};\, 1 \le j \le n_0(i)\} \quad \text{if } \beta_i = -q . \tag{15}$$

Now we apply Lemma 2.1 to $F_{ij}(z)$, $G_{ij}(z)$, and $G^*_{ij}(z)$ satisfying the functional equations (12). Clearly $A_i(z), B_i(z), C_i(z) \in \mathbf{C}(z)^\times$, and $P_{ij}(z), Q_{ij}(z), R_{ij}(z) \in \mathbf{C}(z)^\times$ follow from Lemma 3.2, bellow.

We first prove the property (i) in Lemma 2.1. It follows from (11) with (13) that $A_i(z)/A_k(z)$, $B_i(z)/B_k(z)$, $C_i(z)/C_k(z) \in \mathbf{C} \setminus \{1\}$ for any $i \ne k$. So they are not contained in H, since $H \cap \mathbf{C} = \{1\}$. Since H is a subgroup of $\mathbf{C}(z)^\times$ and $A_i(z)/B_k(z)$, $B_i(z)/C_k(z)$, $C_i(z)/A_k(z) \in \{cz^l \mid c \in \mathbf{C}^\times, 1 \le |l| \le q(q-1)\}$ for any i and k, it is enough to show that $cz^l \notin H$ for any $c \in \mathbf{C}^\times$ and $1 \le l \le q(q-1)$. Assume that $cz^l = g(z^{q^2})/g(z)$, where $g(z) = A(z)/B(z)$ with coprime $A(z), B(z) \in \mathbf{C}[z]$. Then $cz^l B(z^{q^2})A(z) = A(z^{q^2})B(z)$, so that $B(z^{q^2})$ devides $B(z)$, and hence $B(z) \in \mathbf{C}^\times$. Thus we get $cz^l A(z) = A(z^{q^2})$. Comparing the degrees of both sides , we have a contradiction.

In the rest of the proof, our arguments will be independent of the subscript i $(1 \le i \le h)$. So we fix i and omit it from our notations. From now on, we denote $a_{ij}(n), b_{ij}(n), F_{ij}(z), \cdots$ by $a_j(z), b_j(z), F_j(z), \cdots$. In particular, $\alpha_i, \beta_i, m_0(i), m_1(i), A_i(z), \cdots$ will be written as $\alpha, \beta, m_0, m_1, A(z), \cdots$.

To prove the property (ii), we prepare some lemmas.

LEMMA 3.1. *Let $f(z) \in \mathbf{C}(z)$ satisfy*

$$f(z^{q^2}) = D(z)f(z) + \frac{E(z)}{\varphi(z)\varphi(z^q)}\,, \tag{16}$$

where $\varphi(z)$ is defined by (6) *and $D(z), E(z) \in \mathbf{C}[z] \setminus \{0\}$ are such that*

$$D(z) \in \mathbf{C}^{\times} \text{or } D(0) = 0,\ \deg D(z) \le q(q-1)\,, \tag{17}$$

$$E(0) = 0\,, \quad \deg E(z) \le 2q^2 - q - 1\,. \tag{18}$$

Then $D(1) = q^{-2}$ and

$$f(z) = \frac{cz}{1-z} \quad (c \in \mathbf{C}^{\times})\,. \tag{19}$$

PROOF. We put $f(z) = a(z)/b(z)$ with coprime $a(z), b(z) \in \mathbf{C}[z]$. Then it follows from (16) that

$$(a(z^{q^2})b(z) - D(z)a(z)b(z^{q^2}))\varphi(z)\varphi(z^q) = E(z)b(z)b(z^{q^2})\,. \tag{20}$$

We shall prove that

$$b(z) = 1 - z\,. \tag{21}$$

Since $a(z^{q^2})$ and $b(z^{q^2})$ are coprime, we have

$$b(z^{q^2}) \mid b(z)\varphi(z)\varphi(z^q)\,, \tag{22}$$

which implies $\deg b(z) \le 1$. Suppose that $\deg b(z) = 0$. Then we can put $b(z) = 1$, so that by (20)

$$(a(z^{q^2}) - D(z)a(z))\varphi(z)\varphi(z^q) = E(z)\,. \tag{23}$$

If $\deg a(z) = 0$, say $a(z) = c \in \mathbf{C}^{\times}$, we have $c(1 - D(z))\varphi(z)\varphi(z^q) = E(z)$. Putting $z = 0$, we get $D(0) = 1$, since $E(0) = 0$. Hence $D(0) \neq 0$, and so $D(z) = 1$ by (17). Thus we have $E(z) = 0$, which contradicts the assumption. Hence $s = \deg a(z) \ge 1$. This implies $\deg a(z^{q^2}) > \deg D(z)a(z)$, and so comparing the degrees of both sides of (23), we get $sq^2 + q^2 - 1 \le 2q^2 - q - 1$ using (18), which yields $s < 1$, a contradiction. Therefore $\deg b(z) = 1$. We can put $b(z) = c_0 - z$ ($c_0 \in \mathbf{C}$), so that we have by (22) $c_0 - z^{q^2} = c_1(c_0 - z)\varphi(z)\varphi(z^q)$ ($c_1 \in \mathbf{C}^{\times}$). Comparing the coefficients of both sides we find $c_1 = 1$, and putting $z = 1$ we obtain $c_0 = 1$, and (21) follows.

To prove (19) it remains to show that

$$a(z) = cz \quad (c \in \mathbf{C}^{\times})\,. \tag{24}$$

It follows from (20) and (21) that

$$a(z^{q^2}) - D(z)a(z)\varphi(z)\varphi(z^q) = (1-z)E(z)\,. \tag{25}$$

Putting $z = 0$ in (25), we have $D(0) = 1$ or $a(0) = 0$. If $D(0) = 1, D(z) = 1 \in \mathbf{C}^\times$ by assumption (17). Then (25) with $z = 1$ yields $a(1) = 0$. This contradicts (21) and $(a(z), b(z)) = 1$. Hence we have $a(0) = 0$. If $\deg a(z) = 0$, we have a contradiction as above. Suppose that $s = \deg a(z) \geq 2$. Then $\deg a(z^{q^2}) > \deg D(z)a(z)\varphi(z)\varphi(z^q)$ by (17). So comparing the degrees of both sides of (25) using (18), we have $sq^2 \leq 2q^2 - q$, which is impossible. Therefore $\deg a(z) = 1$, which with $a(0) = 0$ implies (24).

It follows from (24) and (25) that $cz^{q^2} - czD(z)\varphi(z)\varphi(z^q) = (1-z)E(z)$. Putting $z = 1$, we get $c(1 - D(1)q^2) = 0$ with $c \neq 0$, so that $D(1) = q^{-2}$; and the lemma is proved.

We shall use the following notations. We put for $c_1, c_2, \cdots \in \mathbf{C}$

$$S(z) = \sum_{j=1}^{m_0} c_j \rho_j(z) - \sum_{j=m_0+1}^{m_1} c_j \rho_j(z) = \sum_{r=1}^{q-1} s(r) z^r \,,$$

$$T(z) = \sum_{j=1}^{m_1} c_j \rho_j(z) = \sum_{r=1}^{q-1} t(r) z^r \,,$$

where $s(r) = \sum_{j=1}^{m_0} c_j a_j(r) - \sum_{j=m_0+1}^{m_1} c_j a_j(r),\ t(r) = \sum_{j=1}^{m_1} c_j a_j(r)$,

$$U(z) = \sum_{j=1}^{n_0} c_j \tau_j(z) - \sum_{j=n_0+1}^{n_1} c_j \tau_j(z) = \sum_{r=1}^{q-1} u(r) z^r \,,$$

$$V(z) = \sum_{j=1}^{n_1} c_j \sigma_j(z) = \sum_{r=1}^{q-1} v(r) z^r \,,$$

where $u(r) = \sum_{j=1}^{n_0} c_j b_j(q-r) - \sum_{j=n_0+1}^{n_1} c_j b_j(q-r),\ v(r) = \sum_{j=1}^{n_1} c_j b_j(r)$,

$$X(z) = \sum_{j=1}^{n_0} c_j \sigma_j(z) - \sum_{j=n_0+1}^{n_1} c_j \sigma_j(z) = \sum_{r=1}^{q-1} x(r) z^r \,,$$

$$Y(z) = \sum_{j=1}^{n_1} c_j \tau_j(z) = \sum_{r=1}^{q-1} y(r) z^r \,,$$

where $x(r) = \sum_{j=1}^{n_0} c_j b_j(r) - \sum_{j=n_0+1}^{n_1} c_j b_j(r),\ y(r) = \sum_{j=1}^{n_1} c_j b_j(q-r)$.
Then it follows from the definitions that

$$\sum_{j=1}^{m_1} c_j P_j(z) = -\alpha^{-2} \frac{\alpha S(z^q)\varphi(z) + T(z)\varphi(z^q)}{\varphi(z)\varphi(z^q)} \,, \tag{26}$$

$$\sum_{j=1}^{n_1} c_j Q_j(z) = -\beta^{-2} z^{q(q-1)} \frac{\beta U(z^q)\varphi(z) + V(z)\varphi(z^q)}{\varphi(z)\varphi(z^q)} \,, \tag{27}$$

$$\sum_{j=1}^{n_1} c_j R_j(z) = -\beta^{-2}\frac{\beta X(z^q)\varphi(z) + z^{q-1}Y(z)\varphi(z^q)}{\varphi(z)\varphi(z^q)}\,. \tag{28}$$

LEMMA 3.2. *Each of the sets* $\{P_j(z); 1 \le j \le m_1\}$, $\{Q_j(z); 1 \le j \le n_1\}$, *and* $\{R_j(z); 1 \le j \le n_1\}$ *is linearly independent over* $\mathbf{C}$.

PROOF. We prove that $P_1(z), \cdots, P_{m_1}(z)$ are linearly independent over $\mathbf{C}$. Suppose that $\sum_{j=1}^{m_1} c_j P_j(z) = 0$ for some $c_1, \cdots, c_{m_1} \in \mathbf{C}$ not all zero. Then by (26)

$$\alpha S(z^q)\varphi(z) = -T(z)\varphi(z^q)\,. \tag{29}$$

If $T(z) = 0$, then we have $S(z) = 0$. Hence $s(r) = t(r) = 0$ $(1 \le r < q)$, and so

$$\sum_{j=1}^{m_0} c_j a_j(r) = \sum_{j=m_0+1}^{m_1} c_j a_j(r) = 0 \quad (1 \le r < q)\,,$$

which contradicts the assumption that each of the sets $\{\boldsymbol{a}_j; 1 \le j \le m_0\}$ and $\{\boldsymbol{a}_j; m_0 < j \le m_1\}$ are linearly independent over $\mathbf{C}$ and $c_1, \cdots, c_{m_1}$ not all zero. Now if $T(z) \ne 0$, then by (29) $S(z) \ne 0$, which together with $T(0) = 0$ and $S(0) = 0$ imply that $\mathrm{ord}_{z=0} S(z^q) \ge q$ and $1 \le \deg T(z) < q$, a contradiction.

Similary we can prove that $Q_1(z), \cdots, Q_{n_1}(z)$ are linearly independent over $\mathbf{C}$. To prove the linear independency of $R_1(z), \cdots, R_{n_1}(z)$ over $\mathbf{C}$, we assume that $\sum_{n=1}^{n_1} c_j R_j(z) = 0$ for some $c_1, \cdots, c_{n_1} \in \mathbf{C}$ not all zero. Then it follows from (28) that

$$\beta X(z^q)\varphi(z) = -z^{q-1}Y(z)\varphi(z^q)\,.$$

Comparing the degrees of both sides, we have $q(q-1)+q-1 \ge q-1+\deg Y(z)+q(q-1)$, so that $\deg Y(z) = 0$. Since $Y(0) = 0$, we get $Y(z) = 0$ and so $X(z) = 0$. Hence $x(r) = y(r) = 0$ $(1 \le r < q)$, which contradicts the linear independency over $\mathbf{C}$ of each of the sets $\{\boldsymbol{b}_j; 1 \le j \le n_0\}$ and $\{\boldsymbol{b}_j; n_0 < j \le n_1\}$; and the lemma is proved.

LEMMA 3.3. *If* $P(z) = \sum_{r=1}^{q-1} p(r)z^r \in \mathbf{C}[z]$ *satisfies*

$$(1-\zeta)P(\zeta) + \gamma = 0 \tag{30}$$

for some $\gamma \in \mathbf{C}$ *and any* $\zeta \ne 1$ *with* $\zeta^q = 1$, *then* $\gamma = qp(1)$ *and*

$$P(z) = p(1)\sum_{r=1}^{q-1} rz^r = p(1)\frac{z\varphi(z) - qz^q}{1-z}\,,$$

where $\varphi(z)$ *is defined by* (6).

PROOF. It follows from (30) that

$$\sum_{r=1}^{q-1}(p(r) - p(r-1))\zeta^r - p(q-1) + \gamma = 0\,, \quad p(0) = 0\,.$$

If we put

$$\xi(z) = \sum_{r=1}^{q-1}(p(r) - p(r-1))z^r - p(q-1) + \gamma\,,$$

$$\eta(z) = (p(q-1) - p(q-2))\sum_{r=0}^{q-1} z^r\,,$$

we have $\xi(z) = \eta(z)$, since they have $q-1$ common distinct roots. Comparing the coefficients, we get $p(r) = rp(1)$ $(1 \le r < q)$ and $\gamma = qp(1)$, and the lemma is proved.

Now we verify the property (ii) in Lemma 2.1 using Lemmas 3.1–3.3. We have to prove under the assumptions of the theorem that if $c_1, \cdots, c_{m_1} \in \mathbf{C}$ are not all zero, there is no $f(z) \in \mathbf{C}(z)$ satisfying

$$f(z^{q^2}) = A(z)f(z) + \sum_{j=1}^{m_1} c_j P_j(z)\,, \tag{31}$$

and if $c_1, \cdots, c_{n_1} \in \mathbf{C}$ are not all zero, there is no $f(z) \in \mathbf{C}(z)$ satisfying

$$f(z^{q^2}) = B(z)f(z) + \sum_{j=1}^{n_1} c_j Q_j(z) \tag{32}$$

or

$$f(z^{q^2}) = C(z)f(z) + \sum_{j=1}^{n_1} c_j R_j(z)\,. \tag{33}$$

Suppose first that there is $f(z) \in \mathbf{C}(\mathbf{z})$ satisfying (31). Recalling (26) we have the equation (16) with $D(z) = A(z) = \alpha^{-2} \in \mathbf{C}^\times$,

$$E(z) = -\alpha^{-2}(\alpha S(z^q)\varphi(z) + T(z)\varphi(z^q)) \neq 0$$

(by Lemma 3.2), and (18). Hence we can apply Lemma 3.1 and get (19) and $D(1) = \alpha^{-2} = q^{-2}$, so that $\alpha = q$ by our convention. Thus we have by (26) and (31)

$$c(z^{q^2} - q^{-2}z\varphi(z)\varphi(z^q)) = -q^{-2}(1-z)(qS(z^q)\varphi(z) + T(z)\varphi(z^q)) \tag{34}$$

$(c \in \mathbf{C}^\times)$. Putting $z = \zeta \neq 1$ with $\zeta^q = 1$, we get $(1-\zeta)T(\zeta) + qc = 0$, and hence

$$T(z) = c\frac{z\varphi(z) - qz^q}{1-z}$$

by Lemma 3.3, substituting this to (34), we find $c(qz^{q^2} - z^q\varphi(z^q)) = -(1-z)S(z^q)\varphi(z)$, which yields $T(z) = S(z)$. Comparing the coefficients of both sides, we obtain

$$\sum_{j=1}^{m_0} c_j a_j(r) = cr \quad (c \in \mathbf{C}^\times,\ 1 \le r < q)\,,$$

which contradicts the assumption (14).

Next we suppose that $f(z) \in \mathbf{C}(z)$ satisfies (32). Recalling (27) we have (16) with $D(z) = \beta^{-2} z^{q(q-1)}$,

$$E(z) = -\beta^{-2} z^{q(q-1)} (\beta U(z^q)\varphi(z) + V(z)\varphi(z^q)) \neq 0$$

(by Lemma 3.2), and (18). Then we apply Lemma 3.1 and get (19) and $\beta = -q$. Thus we have by (27) and (32)

$$c(z^{q^2} - q^{-2} z^{q(q-1)} z\varphi(z)\varphi(z^q)) = -q^{-2} z^{q(q-1)} (1-z)(-qU(z^q)\varphi(z) + V(z)\varphi(z^q)) \quad (35)$$

($c \in \mathbf{C}^{\times}$). Putting $z = \zeta \neq 1$ with $\zeta^q = 1$, we have $(1-\zeta)V(\zeta) + qc = 0$, and hence

$$V(z) = c\frac{z\varphi(z) - qz^q}{1-z}\,. \quad (36)$$

by Lemma 3.3, putting this into (35) , we get $U(z^q) = c(qz^q - z^q\varphi(z^q))/(1-z^q)$, which implies

$$U(z) = c\sum_{r=1}^{q-1} (q-r)z^r\,, \quad (37)$$

since

$$\sum_{r=1}^{q-1} (q-r)z^r = \frac{qz - z\varphi(z)}{1-z}\,. \quad (38)$$

It follows from (36) and (37) that $u(q-r) = v(r) = cr$, and hence

$$\sum_{j=1}^{n_0} c_j b_j(r) = cr \quad (c \in \mathbf{C}^{\times},\ 1 \leq r < q)\,,$$

which contradicts the assumption (15).

Finally we assume that $f(z) \in \mathbf{C}(z)$ satisfies (33). Similarly as in the previous case, we have using (28) $\beta = -q$ and

$$c(z^{q^2} - q^{-2} z^{q-1} z\varphi(z)\varphi(z^q)) = -q^{-2}(1-z)(-qX(z^q)\varphi(z) + z^{q-1}Y(z)\varphi(z^q)) \quad (39)$$

($c \in \mathbf{C}^{\times}$). Putting $z = \zeta^{1/q}$, where $\zeta \neq 1$ satisfies $\zeta^q = 1$, we get $(1-\zeta)X(\zeta) - qc = 0$. Then by Lemma 3.3

$$X(z) = -c\sum_{r=1}^{q-1} rz^r = -c\frac{z\varphi(z) - qz^q}{1-z}\,.$$

Substituting this to (39) and using (38) we have

$$Y(z) = -c\frac{qz - z\varphi(z)}{1-z} = -c\sum_{r=1}^{q-1} (q-r)z^r\,.$$

Hence $x(r) = y(q - r) = -cr$ $(1 \le r < q)$, and therefore

$$\sum_{j=1}^{n_0} c_j b_j(r) = -cr \quad (c \in \mathbf{C}^\times,\ 1 \le r < q)\,,$$

which contradicts (15). The proof of Theorem 1.1 is now completed.

References

[1] H. DELANGE, Sur la fonction sommatoire de la fonction "somme des chiffres", Enseign. Math. (2) **21** (1975), 31–47.

[2] P. J. GRABNER and J. M. THUSWALDNER, On the sum of digits function for number systems with negative bases, Ramanujan J. **4** (2000), 201–220.

[3] D. E. KNUTH, *The Art of Computer Programming*, **2**, Addison Wesley (1981).

[4] K. MAHLER, On the generating function of the integers with a missing digit, J. Indian Math. Soc. **15A** (1951), 33-40.

[5] H. NIEDERREITER, *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conf. Ser. in Appl. Math. **63**, Philadelphia, Pennsylvania, (1992).

[6] KE. NISHIOKAand KU. NISHIOKA, Algebraic independence of functions satisfying a certain system of functional equations, Funkcial. Ekvac. **37** (1994), 195–209.

[7] KU. NISHIOKA, *Mahler functions and Transcendence*, LNM **1631** (1996), Springer.

[8] T. OKADA, T. SEKIGUCHI and Y. SHIOTA, Applications of binomial measures to power sums of digital sums, J. Number Theory **52** (1995), 256–266.

[9] T. TOSHIMITSU, q-additive functions and algebraic independence, Arch. Math. **69** (1997), 112–119.

[10] T. TOSHIMITSU, Strongly q-additive functions and algebraic independence, Tokyo J. Math. **21** (1998), 107–113.

[11] Y. UCHIDA, Algebraic independence of the power series defind by blocks of digits, J. Number Theory **78** (1999), 107–118.

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