BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 2, March 1983

Q VALUED FUNCTIONS MINIMIZING DIRICHLET'S INTEGRAL AND THE REGULARITY OF AREA MINIMIZING RECTIFIABLE CURRENTS UP TO CODIMENSION TWO

F. J. ALMGREN, JR.[1]

We announce several results of an extensive study [**A**] of the size of singular sets in oriented m dimensional surfaces which are area minimizing in $m+l$ dimensional Riemannian manifolds. Our principal result is that the Hausdorff dimension of such a singular set does not exceed $m-2$. Examples show this is the best possible such general estimate when $l \geq 2$, i.e., when branching singularities are possible. The general existence of such surfaces of least area is well known in a variety of settings [**F**, 5.1.6].

In order to obtain estimates on branching of area minimizing surfaces we were led to use Taylor's expansion in terms of first derivatives at 0 to approximate the nonparametric area integrand by Dirichlet's integrand. Accordingly, we study branched coverings of regions in $\mathbf{R}^m$ which are graphs of multiple valued functions minimizing the integral of Dirichlet's integrand. As a central estimate in our analysis of area minimizing surfaces we show that the Hausdorff dimension of the branch set of such a minimizing covering does not exceed $m-2$.

To state several results in more detail we use the terminology of [**F**]. Suppose that A is a bounded open subset of $\mathbf{R}^m$ with smooth boundary, and let k, l, m, n, Q be positive integers with $k \geq 3$, $l \leq n$, and $m \geq 2$.

INTERIOR REGULARITY OF ORIENTED AREA MINIMIZING SURFACES. *Suppose N is an $m+l$ dimensional submanifold of $\mathbf{R}^{m+n}$ of class $k+2$ and that T is an m dimensional rectifiable current in $\mathbf{R}^{m+n}$ which is absolutely area minimizing with respect to N. Then there is an open subset U of $\mathbf{R}^{m+n}$ such that $\operatorname{spt} T \cap U$ is an m dimensional minimal submanifold of N of class k and the Hausdorff dimension of $\operatorname{spt} T \sim (U \cup \operatorname{spt} \partial T)$ does not exceed $m-2$.*

For such area minimizing T we have additionally

SINGULARITY STRATIFICATION BY TANGENT CONE TYPE. *Whenever $p \in \operatorname{spt} T \sim \operatorname{spt} \partial T$ and S is an oriented tangent cone to T at p then*

$$P(S) = \mathbf{R}^{m+n} \cap \{x : \theta^m(\|S\|, x) = \theta^m(\|S\|, 0) = \theta^m(\|T\|, p)\}$$

Received by the editors November 11, 1982.

1980 *Mathematics Subject Classification*. Primary 45F22; Secondary 53A10.

[1]This work was supported in part by grants from the National Science Foundation and from the Institute for Advanced Study in Princeton.

© 1983 American Mathematical Society
0273-0979/82/0000-1203/$01.50

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

is either the point $\{0\}$ *or a linear subspace of* $\mathbf{R}^{m+n}$ *with* $m-1 \neq \dim P(S) \leq m$. *Furthermore, for each* $j \in \{0,1,\ldots,m-2,m\}$, *the Hausdorff dimension of*

$$(\operatorname{spt} T \sim \operatorname{spt} \partial T) \cap \{p : j = \sup\{\dim P(S) \colon S \text{ is an oriented tangent cone to } T \text{ at } p\}\}$$

does not exceed j.

We denote by $\mathbf{Q}$ the space of all 0 dimensional integral currents V in $\mathbf{R}^n$ for which $Q = \mathbf{M}(V) = \langle V, 1\rangle$ with metric given by setting

$$\begin{aligned}&\operatorname{dist}([\![p(1)]\!] + \cdots + [\![p(Q)]\!], [\![q(1)]\!] + \cdots + [\![q(Q)]\!]) \\ &= \inf\left\{\left(\sum_{i=1}^{Q} |p(i) - q(\sigma(i))|^2\right)^{1/2} : \sigma \text{ is a permutation of } \{1,\ldots,Q\}\right\}\end{aligned}$$

whenever $p(1),\ldots,p(Q)$, $q(1),\ldots,q(Q) \in \mathbf{R}^n$. For Lipschitz $\mathbf{Q}$ valued functions we show a Lipschitz extension theorem analogous to Kirszbraun's theorem, an almost everywhere Q fold affine approximation theorem analogous to Rademacher's theorem, and also show that each Lipschitz function $A \to \mathbf{Q}$ induces a natural chain mapping of degree 0 from the chain complex of real flat chains having supports in A into the chain complex of real flat chains in $\mathbf{R}^n$. In terms of Dirichlet's integral naturally defined for appropriate functions $A \to \mathbf{Q}$ we have the following central results.

EXISTENCE AND REGULARITY OF DIRICHLET INTEGRAL MINIMIZING $\mathbf{Q}$ VALUED FUNCTIONS. *For each appropriate function* $g \colon \partial A \to \mathbf{Q}$ *there exists a* (*strictly defined but not necessarily unique*) *function* $f \colon A \to \mathbf{Q}$ *having boundary values* g *and of least Dirichlet integral among such functions. Furthermore, each such minimizing* f *is Hölder continuous, and* $A \times \mathbf{R}^n \cap \{(x,y) : y \in \operatorname{spt}(f(x))\}$ *is an* m *dimensional real analytic* (*harmonic*) *submanifold of* $A \times \mathbf{R}^n$ *except possibly for a closed set of Hausdorff dimension not exceeding* $m-2$.

Assuming that m and n and even integers and the usual complex identifications have been made, we show that the $\mathbf{Q}$ valued function produced by projection mapping slicing of a complex holomorphic chain in $A \times \mathbf{R}^n$ associated with a Q fold analytic branched covering of A is uniquely Dirichlet integral minimizing. Our Hausdorff codimension two singularity estimate for Dirichlet integral minimizing $\mathbf{Q}$ valued functions is thus the best possible.

REFERENCES

[A] F. Almgren, $\mathbf{Q}$ *valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension* 2 (being typed, approximately 1500 manuscript pages).

[F] H. Federer, *Geometric measure theory*, Die Grundlehren der math. Wissenschaften, Band 153, Springer-Verlag, Berlin, Heidelberg and New York, 1969.

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08544

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use