Commun. Math. Phys. 134, 421–431 (1990)

Communications in
Mathematical
Physics

q-Weyl Group and a Multiplicative Formula for Universal R-Matrices*

A. N. Kirillov[1] and N. Reshetikhin[2]

[1] LOMI, Fontanka 27, SU-191011 Leningrad, USSR
[2] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA

Received May 11, 1990

Abstract. We define the q-version of the Weyl group for quantized universal enveloping algebras of simple Lie group and we find explicit multiplicative formulas for the universal R-matrix.

1. For any semisimple complex Lie algebra $\mathscr{G}$ there is a natural deformation of its universal enveloping algebra $U\mathscr{G}$ as a Hopf algebra over the formal power series over $\mathbf{C}$ [D1, J]. This deformation $U_h\mathscr{G}$ is called a quantum universal enveloping algebra or quantum group [D1]. These algebras are important in the theory of quantum integrable systems [F] because with each $U_h\mathscr{G}$ one can associate a certain canonical element R in $(U_h\mathscr{G})^{\otimes 2}$ which satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}\,.$$

Here $R_{ij}\in U_h\mathscr{G}^{\otimes 3}$, $R_{12}=R\otimes 1$, $R_{23}=1\otimes R$ and $R_{13}=\sum_i \alpha_i\otimes 1\otimes\beta_i$ if we rewrite R as $R=\sum_i \alpha_i\otimes\beta_i$, α_i, $\beta_i\in U_h\mathscr{G}$.

But up to now there was no explicit formula for R, except for the cases $g=sl_2$ [D1], $\mathscr{G}=sl_n$ [Ro2]. Drinfeld (private communication) conjectured that there is a relation between the Weyl group and the universal R-matrix for general simple Lie algebras. In this paper we define a completion $U_h\mathscr{G}$ by the Weyl elements of sl_2 triples corresponding to simple roots. This completion gives us a description of the quantum Weyl group as well as explicit formulas for the element R.

2. Let $\mathscr{G}$ be a semisimple Lie algebra of rank n, a_{ij} its Cartan matrix, and d_i the length of the i-th root (then $d_i a_{ij}=a_{ji}d_j$).

* Supported in part by the Department of Energy under Grant DE-FG02-88ER25065

Let h be a formal variable. For integers n and m we use the notations:

$$[n]_h=\frac{sh\left(\frac{nh}{2}\right)}{sh\left(\frac{h}{2}\right)},\qquad [n]_h!=[n]_h[n-1]_h\dots[1]_h,$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_h=\frac{[h]_h!}{[m]_h!\,[n-m]_h!}.$$

Following [D1, J] we consider an algebra $U_h\mathcal{G}$ over $\mathbf{C}[\![h]\!]$ with generators H_i, X_i, Y_i and relations:

$$[H_i,H_j]=0,\qquad [H_i,H_j]=a_{ij}X_j,\tag{1}$$

$$[H_i,Y_j]=-a_{ij}Y_j,\qquad [X_iY_j]=\delta_{ij}\frac{sh\left(\frac{d_jH_j}{2}\right)}{sh\left(\frac{hd_i}{2}\right)}\delta_{ij},$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{d_ih} X_i^kX_jX_i^{1-a_{ij}-k}=0,\qquad i\neq j,$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{d_ih} Y_i^kY_jY_i^{1-a_{ij}-k}=0,\qquad i\neq j.$$

This is a Hopf algebra with comultiplication $\Delta: U_n\mathcal{G}\to(U_h\mathcal{G})^{\otimes 2}$:

$$\Delta H_i=H_i\otimes 1+1\otimes H_i,\qquad \Delta X_i=X_i\otimes e^{\frac{hH_id_i}{4}}+e^{-\frac{hH_id_i}{4}}\otimes X_i,$$

$$\Delta Y_i=Y_i\otimes e^{\frac{hH_id_i}{4}}+e^{-\frac{hH_id_i}{4}}\otimes Y_i.$$

An antipode S and counit ε is defined by the Hopf algebra axioms:

$$S(H_i)=-H_i,\qquad S(X_i)=-e^{\frac{hd_i}{2}}X_i,$$

$$S(Y_i)=-e^{-\frac{hd_i}{2}}Y_i,$$

$$\varepsilon(H_i)=\varepsilon(Y_i)=\varepsilon(X_i)=0.$$

In $U_h\mathcal{G}$ there are important Hopf subalgebras U_hb_+ generated by 1, H_i, X_i and U_hb_- generated by 1, H_i, Y_i. They are dual to each other over $\mathbf{C}[\![h^{-1},h]\!]$ with respect to the pairing

$$\langle H_i,H_j\rangle=\frac{2}{h}d_ia_{ij},\qquad \langle X_i,Y_j\rangle=\delta_{ij}(1-e^{-hd_i})^{-1},\tag{2}$$

defined on the generators. The pairing between other elements can be found from the Hopf algebra structure on $U_hb_\pm$,

$$\langle a\otimes b,\Delta(c)\rangle=\langle ba,c\rangle, a,b\in U_hb_+, c\in U_hb_-,$$

$$\langle \Delta a,c\otimes b\rangle=\langle a,cb\rangle, a\in U_hb_+, b,c\in U_hb_-.$$

The algebras $U_h\mathcal{G}$ are quasitriangular Hopf algebras, i.e. for each $\mathcal{G}$ there exists an element R belonging to an appropriate completion of $(U_h\mathcal{G})^{\otimes 2}$ in h-adic topology satisfying the relations:

$$\Delta'(a)=R\Delta(a)R^{-1}, \quad (\Delta\otimes\mathrm{id})R=R_{13}R_{23}, \quad (\mathrm{id}\otimes\Delta)R=R_{13}R_{12}.$$

From the description of $U_h\mathcal{G}$ as a double of U_hb_+ it follows that this element is unique and it is the canonical element under the pairing (2) between U_hb_+ and U_hb_-. The first coefficient in the expansion of R in powers of X_i, $Y_i(B_{ij}=d_ia_{ij})$ has the form

$$\begin{aligned} R&=\exp\left(\frac{h}{2}(B^{-1})_{ij}H_i\otimes H\right)\\ &\quad\times\left(1+\sum_{i=1}^{n}2sh\left(\frac{hd_i}{2}\right)e^{-\frac{hd_i}{2}}e^{\frac{hH_i}{4}}X_i\otimes e^{-\frac{hH_i}{4}}Y_i+\dots\right)\\ &=\left(1+\sum_{i=1}^{n}2sh\left(\frac{hd_i}{2}\right)e^{-\frac{hd_i}{2}}e^{\frac{-hd_iH_i}{4}}X_i\otimes e^{\frac{hH_id_i}{4}}Y_i+\dots\right)\\ &\quad\times\exp\left(\frac{h}{2}(B^{-1})_{ij}H_i\otimes H_j\right). \end{aligned}$$

For any Hopf algebra A one can define the adjoint action of A on itself by

$$a\circ b=\sum_i a^ibS(a_i), \tag{3}$$

where a^i and a_i are the components of $\Delta(a):\Delta(a)=\sum_i a^i\otimes a_i$. The action

$$a\bullet b=S^{-1}(a\circ S(b))=\sum_i a_ibS^{-1}(a^i)$$

defines another adjoint action on A is not equivalent to (3) for noncommutative Hopf algebras. For $A=U_h\mathcal{G}$ we have

$$H_i\circ a=[H_i,a], \tag{4}$$

$$X_i\circ a=X_ia\exp\left(\frac{hH_id_i}{4}\right)-e^{\frac{hd_i}{2}}\exp\left(-\frac{hH_id_i}{4}\right)aX_i, \tag{5}$$

$$Y_i\circ a=Y_ia\exp\left(\frac{hH_id_i}{4}\right)-e^{-\frac{hd_i}{2}}\exp\left(-\frac{hH_id_i}{4}\right)aY_i. \tag{6}$$

Remark. Let $\mathcal{G}_h$ be the minimal nontrivial orbit in $U_h\mathcal{G}$ under the adjoint action (4–6). Because $\mathcal{G}_h$ is an irreducible representation of $U_h\mathcal{G}$ and at $h=0$ this is the adjoint representation of $\mathcal{G}$, we have $\dim\mathcal{G}_h=\dim\mathcal{G}$ [L, Ro1]. Fix e_i a linear basis in $\mathcal{G}_h$, then the action of these elements on itself defines the quantum version of Lie brackets on $\mathcal{G}$.

In quasitriangular Hopf algebras an important role is played by the element

$$u=\sum_i S(\beta_i)\alpha_i,$$

where α_i and β_i are coordinates of the element $R:R=\sum_i\alpha_i\otimes\beta_i$. One can show [D2] that

$$S^2(a)=uau^{-1}$$

and for $U_h\mathcal{G}$ we have

$$v=u\exp\left(-\frac{hH_\varrho}{2}\right)\in\text{center of }U_h\mathcal{G}.$$

Here H_ϱ is an element corresponding to the half of the sum of positive roots in Cartan subalgebra $U(H)\subset U_h\mathcal{G}$ generated by elements H_i, $i=1,\dots n$.

3. According to the decomposition (3) let us introduce regular generators on $U_h\mathcal{G}$:

$$E_i=e^{\frac{hd_iH_i}{4}}X_i,\quad F_i=e^{-\frac{hd_iH_i}{4}}Y_i,$$
$$\bar{E}_i=e^{-\frac{hd_iH_i}{4}}X_i,\quad \bar{F}_i=e^{\frac{hd_iH_i}{4}}Y_i.$$

Proposition 1. 1. *The maps φ and $\bar{\varphi}$*

$$\varphi(H_i)=H_i,\quad \varphi(X_i)=E_i,\quad \varphi(Y_i)=F_i,$$
$$\Psi(H_i)=H_i,\quad \Psi(X_i)=\bar{E}_i,\quad \Psi(Y_i)=\bar{F}_i$$

preserve the relations (2).

2. $$E_i\bar{F}_j=q_i^{\frac{a_{ij}}{2}}\bar{F}_jE_i,\quad \bar{E}_iF_j=q_i^{\frac{a_{ij}}{2}}F_j\bar{E}_i,$$

where $q_i=\exp(hd_i)$.

Let us define now the q-commutator as

$$[A,B]_q=ABq-BAq^{-1}.$$

Proposition 2.

$$(F_i)^n\circ F_j=q_i^{1/4(na_{ij}+n(n-1))}\left[F_i,\dots\left[F_i,[F_i,F_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\right]_{q_i^{\frac{a_{ij}+2(n-1)}{4}}}$$
$$(\bar{E}_i)^n\circ \bar{E}_j=q_i^{1/4(na_{ij}+n(n-1))}\left[\bar{E}_i,\dots\left[\bar{E}_i,[\bar{E}_i,\bar{E}_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\right]_{q_i^{\frac{a_{ij}+2n-2}{4}}}$$

The proof follows from (7) by induction in n

Proposition 3. *The q-Serre relations* (2) *are equivalent to the following ones:*

$$(F_i)^{-a_{ij}+1}\circ F_j=0,\quad (\bar{E}_i)^{-a_{ij}+1}\circ \bar{E}_j=0.$$

The adjoint action of regular generators has the following form:

$$\begin{aligned}\bar{E}_i\circ b&=\bar{E}_ib-K_i^{-2}bK_i^2\bar{E}_i,\\ \bar{F}_i\circ b&=(\bar{F}_ib-b\bar{F}_i)K_i^{-2},\\ E_i\circ b&=(E_ib-bE_i)K_i^{-2},\\ F_i\circ b&=F_ib-K_i^{-2}bK_i^2F_i.\end{aligned}\tag{7}$$

Representations of $U_h\mathcal{G}$ are isomorphic as a linear spaces to corresponding representations $U\mathcal{G}$. If V^λ is a representation of $U_h\mathcal{G}$ with highest weight λ, then

$$vV^\lambda=\exp(-h(\lambda\,|\,\lambda+2\varrho))V^\lambda.$$

4. Let $\mathcal{G}=sl_2$. An irreducible finite dimensional representation V^j of U_hsl_2 is parametrised by half integers $j=0,\frac{1}{2},1,\dots$. The action of generators H,X,Y, in the weight basis e_m^j, $m=-j,\ -j+1,\dots j$ of the space V^j has the following form:

$$He_m^j=me_m^j,\quad Xe_m^j=\sqrt{[j-m]\,[j+m+1]}\,e_{m+1}^j,$$
$$Ye_m^j=\sqrt{[j+m]\,[j-m+1]}\,e_{m-1}^j.$$

The universal R-matrix for $U_h sl_2$ has the following form

$$R=R(H,X,Y\,|\,h)=\exp\left(\frac{h}{2}H\otimes H\right)\sum_{n\geqq 0}\frac{(1-q^{-1})^n}{[n]_h!}q^{\frac{n(n-1)}{4}}(e^{\frac{hH}{4}}X)^n\otimes(e^{-\frac{hH}{4}}Y)^n$$

$$=\left(\sum_{n\geqq 0}\frac{(1-q^{-1})^n}{[n]_h!}q^{\frac{n(n-1)}{4}}(e^{-\frac{hH}{4}}X)^n\otimes(e^{\frac{hH}{4}}Y)^n\right)\exp\left(\frac{h}{4}H\otimes H\right). \tag{8}$$

It is easy to check that this is the canonical element in $U_h b_+\otimes U_h b_-$ with pairing (2). The algebra $U_h sl_2$ can be completed by the element w, defined in each irreducible representation as

$$we^j_m=(-1)^{j-m}e^{-h\frac{j(j+1)}{2}+\frac{mh}{2}}e^j_{-m}. \tag{9}$$

Let us denote this completion by $\overline{U_h sl_2}$.

Theorem [KR].

1. *The element w satisfies the relation*

$$wXw^{-1}=-q^{1/2}Y,\qquad wYw^{-1}=-q^{-\frac{1}{2}}X,\qquad wHw^{-1}=-H. \tag{10}$$

2. *$\overline{U_h sl_2}$ is a Hopf algebra with*

$$\Delta w=R^{-1}w\otimes w,\qquad s(w)=we^{\frac{hH}{2}},\qquad \varepsilon(w)=1,$$

where R is the universal R-matrix for $U_h sl_2$.

3. *Let $u=\sum_i S(\beta_i)\alpha_i$ be the element describing the square of the antipode, then*

$$w^2=v\varepsilon=u\varepsilon^{\frac{hH}{2}}\varepsilon,$$

where ε is the unipotent central element $\varepsilon^2=1$, $\varepsilon V^j=(-1)^{2j}V^j$.

The element w has another interesting interpretation [VS] in representation theory of dual Hopf algebra to $U_h sl_2$.

5. In each $U_h\mathscr{G}$ module we can define the action of the Weyl elements of sl_2 – triples corresponding to simple roots of $\mathscr{G}$. Because $U_h\mathscr{G}$ is a semisimple algebra it is enough to define the action of $\check{w}_i$ in irreducible representations. Let $V^\lambda=\overset{\oplus}{_j}(W_j^\lambda\otimes V^j)$ be the decomposition of V^λ into irreducible $(U_h sl_2)_i$ submodules. Define the action of w_i in V^λ as $w_i=\overset{\oplus}{_j}(I^\lambda_{w_j}\otimes(w_i)_j)$, where $(w_i)_j$ is the action of $\check{w}$ in V^j, (see (9)).

Let us denote the algebra $U_h\mathscr{G}$ extended by w_i, $i=1,\dots,$ rank $\mathscr{G}$ as $\overline{U_h\mathscr{G}}$. The definition of w_i implies the following relations in $U_h\mathscr{G}$:

$$w_iH_jw_i^{-1}=H_j-a_{ij}H_i,\qquad w_iX_iw_i^{-1}=-Y_iq_i^{1/2},\qquad w_iY_iw_i^{-1}=-X_iq_i^{-1/2}. \tag{11}$$

also,

$$\Delta w_i=R(i)^{-1}w_i\otimes w_i,$$

where $R(i)\equiv R(H_i,X_i,Y_i|hd_i)$ and $R(H,X,Y|h)$ is defined by (8).

Theorem 1. *The following relations hold in the algebra* $\bar{U}_h\mathcal{G}$:

$$w_i\bar{E}_jK_i^{a_{ij}}w_i^{-1}=(-1)^{a_{ij}}q^{\frac{a_{ij}}{4}-\frac{a_{ij}(a_{ij}-2)}{8}}\frac{1}{[-a_{ij}]_{hdi}!}(\bar{E}_i)^{-a_{ij}}\circ\bar{E}_j, \tag{Ad1}$$

$$w_iS(F_j)K_i^{-a_{ij}}w_i^{-1}=q_i^{-\frac{a_{ij}}{4}-\frac{a_{ij}(a_{ij}-2)}{8}}\frac{1}{[-a_{ij}]_{hd_i}!}S((F_i)^{-a_{ij}}\circ F_j). \tag{Ad2}$$

Proof. Let us first prove two auxiliary lemmas.

Lemma 1.

$$w_i\circ F_j=S(w_i)^{-1}K_i^{a_{ij}}F_jS(w_i),$$
$$w_i\circ\bar{E}_j=w_i\bar{E}_jK_i^{a_{ij}}w_i^{-1}.$$

Proof. Let α_k and β_k be coordinates of $R(H_i,X_i,Y_i\,|\,hd_i)=\sum_k\alpha_k\otimes\beta_k$,

$$\begin{aligned}
w_i\circ\bar{E}_j&=\sum_k S(\alpha_k)w_i\bar{E}_jS(w_i)S(\beta_k)=\sum_k\alpha_kw_i\bar{E}_jS(w_i)\beta_k\\
&=\sum_{n,m\geqq 0}a_m\frac{\left(\frac{hd_i}{4}\right)^n}{n!}w_i(\bar{F}_i)^mH_i^n\bar{E}_j(\bar{E}_i)^mH_i^nq_i^mS(w_i)\\
&=\sum_{n,m\geqq 0}a_m\frac{\left(\frac{hd_i}{4}\right)^n}{n!}w_i\bar{E}_j\bar{F}_i^m(H_i+a_{ij})^n\bar{E}_i^mH_i^nq_i^mS(w_i)\\
&=w_i\bar{E}_j\sum_k\beta_kS^2(\alpha_k)K_i^{a_{ij}}S(w_i)=w_i\bar{E}_jK_i^{a_{ij}}u_i^{-1}S(w_i)\\
&=w_i\bar{E}_jK_i^{a_{ij}}w_i^{-1}.
\end{aligned}$$

The similar calculations give us the action of w_i on F_j:

$$\begin{aligned}
w_i\circ F_j&=\sum_k S(\alpha_k)w_iF_jS(w_i)S(\beta_k)\\
&=\sum_{n,m\geqq 0}a_m\frac{\left(\frac{h_id_i}{4}\right)^n}{n!}H_i^nE_i^nw_iF_jS(w_i)H_i^nF_i^n\\
&=\sum_{n,m}a_m\frac{(hd_i)}{n!}w_iH_i^nF_i^mq_i^mF_jH_i^nE_i^mS(w_i)\\
&=\sum_{n,m\geqq 0}a_m\frac{\left(\frac{hd_i}{4}\right)}{n!}w_iF_i^m(H_i-2m)^n(H_i+a_{ij})^nE_i^mq_i^mF_jS(w_i)\\
&=w_i\sum_{m\geqq 0}a_m\exp\left(\frac{hd_i}{4}(H_i^2+2mH_i+H_ia_{ij}0)\right)F_i^mE_i^mq_i^mF_jS(w_i)\\
&=w_iK_i^{a_{ij}}\sum_k\beta_kS^2(\alpha_k)F_jS(w_i)=w_iw^{-1}K_i^{a_{ij}}F_jS(w_i)\\
&=S(w_i)^{-1}K_i^{a_{ij}}F_jS(w_i).
\end{aligned}$$

Lemma 2. *The linear spaces* $V_{ij}=\{(F_i)^n\circ F_j\}_{n=0}^{-a_{ij}}$, $\bar{V}_{ij}=\{(\bar{E}_i)^n\circ \bar{E}_j\}_{n=0}^{-a_{ij}}$ *are irreducible* $(U_h sl_2)_i$ *modules with highest weight* $-a_{ij}$.

Proof. From relations (1) and from Proposition 1 we obtain the following structure of the adjoint action of $(U_h sl_2)_i$ in these spaces:

$$
\begin{aligned}
F_i\circ(F_i^n\circ F_j)&=F_i^{n+1}\circ F_j\,,\\
E_i\circ((F_i)^n\circ F_j)&=[-a_{ij}+1-n]_{hd_i}[n]_{hd_i}F_i^{n-1}\circ F_j\,,\\
H_i\circ(F_i^nF_j)&=(-a_{ij}-2n)F_i^n\circ F_j\,,\\
\bar{E}_i\circ(\bar{E}_i^n\circ \bar{E}_j)&=E_i^{n+1}\circ \bar{E}_j\,,\\
\bar{F}_i\circ(\bar{E}_i^n\circ \bar{E}_j)&=[-a_{ij}+1-n]_{hd_i}[n]_{hd_i}\bar{E}_i^{n-1}\circ \bar{E}_j\,,\\
H_i\circ(\bar{E}_i^n\circ \bar{F}_j)&=(a_{ij}+2n)\bar{E}_i^n\circ \bar{F}_j\,.
\end{aligned}
$$

The maps

$$
\sigma(F_i^n\circ F_j)=\sqrt{\frac{[n]_{hd_i}!}{[-a_{ij}-n]_{hd_i}!}}\;e_{\frac{a_{ij}}{2}-n}^{-\frac{a_{ij}}{s}}\,,
$$

$$
\tau(\bar{E}_i^n\circ \bar{E}_j)=\sqrt{\frac{[-a_{ij}-n]_{hd_i}!}{[n]_{hd_i}!}}\;e_{-\frac{a_{ij}}{2}+n}^{-\frac{a_{ij}}{2}}
$$

obviously define an isomorphism between V_{ij}, $\bar{V}_{ij}$, and $V^{-a_{ij}}$.

Now, to prove Theorem 1 let us combine these two lemmas with the explicit action of the Weyl element for $U_h sl_2$ and we immediately obtain relations (Ad1, Ad2).

Theorem 2. *The elements* w_i *satisfy the following relations:*

$$
\begin{aligned}
w_iw_jw_i&=w_jw_iw_j\,, && a_{ij}=-1\,,\\
w_iw_jw_iw_j&=w_jw_iw_jw_i\,, && a_{ij}=-2\,,\\
w_iw_jw_iw_jw_i&=w_jw_iw_jw_iw_j\,, && a_{ij}=-3\,.
\end{aligned}
\tag{12}
$$

To prove this theorem it is sufficient to consider only rank $\mathcal{G}=2$ cases. From the relations (Ad1, Ad2) it follows that the left-hand side and right-hand side parts of (12) can differ only by a central element (in the appropriate rank 2 algebra, A_2 for $a_{ij}=-1$, B_2 for $a_{ij}=-2$, G_2 for $a_{ij}=-3$). Acting by left-hand side and right-hand side parts on the h.w. vector we immediately obtain that this central element is unit.

The following two lemmas are useful for simplification of formulas (Ad1, Ad2).

Lemma 3.

$$
\bar{E}_i^n\circ \bar{E}_j=K_i^{-n}K_j^{-1}\Big[X_i,\dots[X_i,X_j]_{q_i^{\frac{a_{ij}}{4}}}\dots\Big]_{q_i^{\frac{a_{ij}+2n-2}{4}}}\,,
$$

$$
F_i^n\circ F_j=K_i^{-n}K_j^{-1}\Big[Y_i,\dots[Y_i,Y_j]_{q_i^{-\frac{a_{ij}}{4}}}\dots\Big]_{q_i^{-\frac{a_{ij}+2n-2}{4}}}\,.
$$

Lemma 4.

$$S([Y_i,\dots,[Y_i,Y_j]_{q^{-n}}]_{q^{-n+2}}\dots]_{q^{n-2}})=-q_i^{-n/2}q_j^{-1/2}[Y_i,\dots[Y_iY_j]_{q^n}]_{q^{n-2}}\dots]_{q^{-n+2}}.$$

Now, we can rewrite relations (Ad1, Ad2) in the following more explicit form:

$$w_iX_jw_i^{-1}=(-1)^{a_{ij}}q^{\frac{a_{ij}}{8}+\frac{a_{ij}}{2}}\frac{1}{[a_{-ij}]_{hd_i}!}\left[\left[X_i,\dots,[X_i,X_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\right]_{q_i^{\frac{-a_{ij}-2}{4}}}K_i^{a_{ij}},$$

$$w_iY_jw_i^{-1}=q_i^{-\frac{a_{ij}}{8}-\frac{a_{ij}}{2}}\frac{1}{[-a_{ij}]_{hd_i}!}\left[\left[Y_i,\dots,[Y_i,Y_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\dots\right]_{q_i^{\frac{-a_{ij}-2}{4}}}K_i^{-a_{ij}},$$

6. Consider elements

$$w_i=\check{w}_iq_i^{\frac{H_i^2}{8}}$$

and define automorphisms

$$T_i(a)=\check{w}_i^{-1}a\check{w}_i.$$

From the relations between w_i and generators of $U_h\mathcal{G}$ we obtain

$$\begin{aligned}
&T_i(K_j)=K_jK_i^{-a_{ij}},\quad T_i(X_i)=Y_iK_i^{-2},\quad T_i(Y_i)=-K_i^2X_i,\\
&T_i(X_j)=(-1)^{a_{ij}}\frac{1}{[-a_{ij}]!}\left[\left[X_i,\dots[X_i,X_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\dots\right]_{q_i^{\frac{-a_{ij}-2}{4}}},\\
&T_i(Y_j)=\frac{1}{[-a_{ij}]!}\left[\left[Y_i,\dots[Y_i,Y_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\dots\right]_{q_i^{\frac{-a_{ij}-2}{4}}},
\end{aligned}\qquad(13)$$

which coincides with Lusztig's automorphisms [L].

Lemma 5. *The elements $\check{w}_i$ satisfy the Weyl group relations:*

$$\underbrace{\check{w}_i\check{w}_j\check{w}_i\dots}_{-a_{ij}+2}=\underbrace{\check{w}_j\check{w}_i\check{w}_j\dots}_{-a_{ij}+2}.$$

It follows from Theorem 2 and relations (11).

7. From the definition of $\check{w}_i$ we obtain the action of the comultiplication on the elements $\check{w}_i$:

$$\Delta\check{w}_i=\tilde{R}^{-1}(i)\check{w}_i\otimes\check{w}_i,$$

where

$$\tilde{R}(i)=\sum_{n\geq 0}\frac{(1-q_i^{-1})^n}{[n]_{hd_i}!}q_i^{\frac{n(n-1)}{4}}E_i^n\otimes F_i^n.$$

Let $s_0=s_{i_1}\dots s_{i_k}$ be a decomposition of the element of Weyl group with maximal length in the minimal product of elementary reflections.

From relation Lemma 5 follows that the element

$$\check{w}_0=\check{w}_{i_1}\dots\check{w}_{i_k}$$

is well defined and does not depend on the choice of decomposition of s_0.

Theorem 3. *The universal R-matrix for $U_h\mathcal{G}$ has the following form:*

$$R=\exp\left(\frac{h}{2}\sum_{ij=1}^{n}(B^{-1})_{ij}H_i\otimes H_j\right)(\check{w}_0\otimes\check{w}_0)\Delta(\check{w}_0)^{-1}$$

or

$$R=\exp\left(\frac{h}{2}\sum_{ij=1}^{n}(B^{-1})_{ij}H_i\otimes H_j\right), \quad \tilde{R}(i_k\,|\,s_{i_1}\ldots s_{i_{k-1}})\ldots\tilde{R}(i_2\,|\,s_{i_1})\tilde{R}(i_1), \tag{14}$$

where

$$\tilde{R}(i_l\,|\,s_{i_1}\ldots s_{i_{l-1}})=(T_{i_1}^{-1}\otimes T_{i_1}^{-1})\ldots(T_{i_{l-1}}^{-1}\otimes T_{i_{l-1}}^{-1})\tilde{R}(i_l)$$

and T_i are the authomorphisms in (14).

To prove this theorem it is convenient to introduce the following enumeration of positive roots. Let $s_0=s_{i_1}\ldots s_{i_k}$ be the decomposition of the maximal element of the Weyl group. The set of positive roots Δ_+ can be considered as a set of roots α_{i_1}, $s_{i_1}\alpha_{i_2},\ldots, s_{i_1}\ldots s_{i_{k-1}}\alpha_{i_k}$ [B, L]. According to this enumeration introduce elements

$$E(p)=T_{i_1}^{-1}\ldots T_{i_{p-1}}^{-1}E_{ip}, \quad F(p)=T_{i_1}^{-1}\ldots T_{i_{p-1}}^{-1}F_{i_p}.$$

From relations in $U_h\mathcal{G}$ it follows (see [L] for details) that the elements

$$H_1^{m_1}\ldots H_n^{m_n} \quad E(1)^{n_1}\ldots E(k)^{b_k}, \tag{15}$$

$$(H_1^v)^{m_1}\ldots(H_n^v)^{m_n} \quad F(1)^{n_1}\ldots F(k)^{n_k}, \tag{16}$$

where

$$H_i^v=\frac{h}{2}\sum_j(B^{-1})_{ij}H_j$$

form the bases in U_hb_+ and U_ab_- respectively.

Lemma 6. *With respect to the pairing* (2) *we have:*

$$\langle E(s),F(t)\rangle=\delta_{st}(1-e^{-hd_{i_s}})^{-1}. \tag{17}$$

It can be derived from the pairing (2) and from the definition of $E(p)$, $F(p)$. From the formula for the action of comultiplication on $\check{w}_i$ and from the definition of T_i it follows

$$\Delta(T_i^{-1}(a))=\tilde{R}(i)^{-1}((T_i^{-1}\otimes T_i^{-1})\Delta(s))\tilde{R}(i).$$

This formula gives us the action of comultiplication on elements $E(i)$.

Lemma 7. *Bases* (16) *and* (17) *are dual with respect to the pairing* (2) *between U_hb_+ and U_hb_-:*

$$\langle H_1^{m_1}\ldots H_n^{m_n}E(1)^{n_1}\ldots E(k)^{n_k},(H_1^v)^{m'_1}\ldots(H_n^v)^{m'_n}F(1)^{n'_1}\ldots F(k)^{n'_k}\rangle$$
$$=\prod_{j=1}^{n}\delta_{m_jm'_j}m_j!\prod_{p=1}^{k}\delta_{n_pn'_p}\frac{[n_p]_{hd_{ip}}!}{(1-e^{-hd_{i_p}})^{n_p}}e^{-\frac{hn_p(n_p-1)}{4}d_{i_p}}.$$

The proof follows from the lemma and formula (18).

So for the canonical element R we have the representation (15).
8. Let us describe more precisely authomorphisms T_i as an authomorphism of Hopf algebras.

Theorem 4. *Let z be an invertible element of the quasitriangular Hopf algebra A. Then the triple $(A, \Delta^{(z)}, R^{(z)})$, where*

$$\Delta^{(z)}(a) = (z \otimes z)\Delta(z^{-1}az)z^{-1} \otimes z^{-1},$$
$$R^{(z)} = z^{-1} \otimes z^{-1} R z \otimes z$$

also forms a quasitriangular Hopf algebra.

Proof. Associativity of $\Delta^{(z)}$ is a consequence of the following equalities:

$$(\Delta^{(z)} \otimes \mathrm{id})\Delta^{(z)}(a) = (z \otimes z \otimes z)(\Delta \otimes \mathrm{id})\Delta(a) z^{-1} \otimes z^{-1} \otimes z^{-1},$$
$$(\mathrm{id} \otimes \Delta^{(z)})\Delta^{(z)}(a) = (z \otimes z \otimes z)(\mathrm{id} \otimes \Delta)\Delta(a)(z^{-1} \otimes z^{-1} \otimes z^{-1}).$$

From the definition of $R^{(z)}$ we have the relation

$$\Delta^{(z)}(a)' = R^{(z)}\Delta^{(z)}(a) R^{(z)-1}.$$

The quasitriangular relations also follow from the structure of $R^{(z)}$ and from quasitriangularity of A.

Consider $z = \check{w}_{i_1}^{-1} \dots \check{w}_{i_{k-1}}^{-1} \equiv \check{w}$ and denote the corresponding Hopf algebra structure on $U_y\mathcal{G}$ by $(U_h\mathcal{G})_w$. As an algebra this is $U_h\mathcal{G}$ but the comultiplication now differs from the previous one for $U_h\mathcal{G}$ and has the form:

$$\Delta^{(w)}(a) = (T_w \otimes T_w)(\Delta(T_w^{-1}(a))),$$

where $T_w(a) = \check{w} a \check{w}^{-1}$.

So, we see that automorphisms T_i are not automorphisms of $U_h\mathcal{G}$ as a Hopf algebra, $T_i^{-1} : (U_h\mathcal{G})_{\check{w}} \to (U_h\mathcal{G})_{\check{w}_i\check{w}}$. But they are automorphisms of the Hopf algebra $U_h\mathcal{G}$ in the sense of the Theorem 4.

9. *Remark 1.* The same construction gives us the quantum version of a Weyl group for Kac-Moody algebras. The relations (14) are still true.

Remark 2. Elements $\check{w}_{i_1} \dots \check{w}$ describes irreducible representations of the quantized algebra of algebraic functions over G [S]. The multiplicative formula for the R-matrix together with the construction of the dual double given in [RST] make explicit the way for a description of cell decomposition of $\mathbf{C}_q(G)$.

10. The authors would like to thank V. Drinfeld and M. A. Semenov-Tian-Shansky for stimulating discussions. One of us (N.R.) would like to thank D. Kazhdan for useful remarks and Sarah Warren for help in typing.

When this work was completed one of us (N.R.) received the work by S. Z. Levendorskiî and Ya. S. Soybelman where similar results are obtained.

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Communicated by A. Jaffe