

QED Effective Action Revisited

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Abstract The derivation of a convergent series representation for the quantum electrodynamic effective action obtained by two of us (S.R.V. and D.R.L.) in [Can. J. Phys. **71**, 389 (1993)] is reexamined. We present more details of our original derivation. Moreover, we discuss the relation of the electric-magnetic duality to the integral representation for the effective action, and we consider the application of nonlinear convergence acceleration techniques which permit the efficient and reliable numerical evaluation of the quantum correction to the Maxwell Lagrangian.

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1 Introduction

Maxwell’s equations receive corrections from virtual excitations of the charged quantum fields (notably electrons and positrons). This leads to interesting effects [1]: light-by-light scattering, photon splitting, modification of the speed of light in the presence of strong electromagnetic fields, and – last, but not least – pair production.

When the heavy degrees of freedom are integrated out (in this case, the “heavy particles” are the electrons and positrons), an effective theory results. The corrections can be described by an effective interaction, the so-called quantum electrodynamic (QED) effective Lagrangian. The dominant effect for electromagnetic fields that vary slowly with respect to the Compton wavelength (frequencies $\omega \ll 2mc^2/\hbar$) is described by the one-loop quantum electrodynamic effective (so-called “Heisenberg–Euler”) Lagrangian which is known to all orders in the electromagnetic field [1–6].

The Heisenberg–Euler Lagrangian $\Delta\mathcal{L}$, which constitutes a quantum correction to the Maxwell Lagrangian, is usually expressed as a one-dimensional proper-time integral [see e.g. Eq. (3.43) in [1], the notation is clarified in Sec. 2 below]:

$$\Delta\mathcal{L} = -\frac{e^2}{8\pi^2} \lim_{\epsilon, \eta \rightarrow 0^+} \int_{i\eta}^{\infty+i\eta} \frac{ds}{s} e^{-(m^2-i\epsilon)s} \left[ab \coth(eas) \cot(ebs) - \frac{a^2 - b^2}{3} - \frac{1}{(es)^2} \right]. \quad (1)$$

Because there are no singularities in the first quadrant of the complex plane and because Jordan’s Lemma may be applied to the integral, it is possible to exchange the lower and upper limit of integration by η and $\eta + i\infty$, respectively.

Although the proper-time integral (1) can be evaluated by numerical quadrature, it is evident that a representation by a convergent series expansion could have certain computational as well as conceptual advantages. An expansion of (1) in terms of special functions has been given by us in [7, 8]. Here, we present a unified series expansion which encompasses both the real and the imaginary part of the Lagrangian (see Sec. 2). Also, we clarify certain technical details concerning the derivation of our previous results [7, 8]. In Sec. 4, we discuss the “electric-magnetic duality” which has recently drawn much attention [9, 10]. In particular, we elucidate different kinds of dual invariance and pursue the question as to whether these invariances are realised by QED effective Lagrangians. Before we expand on these aspects, we would like to provide some general discussion on the general relevance of studies related to the QED effective action (1).

The real part of the Lagrangian (1) can be used to delineate dispersive phenomena such as photon propagation in a magnetic field (see [1] and references therein), photon splitting [11–14], vacuum birefringence and second harmonic generation [16], and light scattering in a vacuum [1]. These applications have a strong relevance to particle astrophysics. The imaginary part of the Lagrangian (1) has been applied to absorption processes such as electron-positron pair creation and dichroism (see e.g. [4, 7, 17–20]). The occurrence of strong fields in storage rings, pulsars, magnetars and high-intensity lasers also motivates our study. The advent of state-of-the-art laser beams and photon detectors may provide a signature of “QED’s nonlinear light”. The optical second harmonic generation (SHG) in vacuo is an interesting second-order magneto-optical effect that occurs if the spatial symmetry of the nonlinearity induced by the effective action (1) is broken e.g. by a strong static magnetic field (in a more general context, higher-harmonic generation by vacuum effects was discussed in [15, 16]). Note that the leading contribution to SHG in vacuo involves a fourth-order effect in contrast to the much weaker 6th-order effect (hexagon graph) which gives the leading contribution to photon splitting. SHG and photon splitting in vacuo may be occurring close to astronomical objects such as white dwarfs, neutron stars and “magnetars” (see [14, 21] and references therein) which have strong magnetic fields up to $B \approx 10^{14}$ G. In all cases, a detailed, realistic description of the experimental conditions and/or the involved

astrophysical objects, especially at extreme field strengths, requires techniques for the reliable numerical evaluation of the QED effective action.

A priori, the construction of a series expansion for (1) constitutes a complete solution of the problem from a theoretical point of view. However, such an expansion does not necessarily provide all answers: Many series expansions are known which either converge extremely slowly or which do not converge at all. Moreover, in the case of the Heisenberg–Euler Lagrangian, there is the additional problem that the terms of the series are represented by special functions which are in most cases defined and computed via series expansions. Again, convergence problems are more likely the rule than the exception. Here, we are concerned with the solution of the principal numerical difficulty associated with the the slow overall convergence of the series expansion derived in [7], whose terms are nonalternating in sign. The modern theory of nonlinear sequence transformations which begins with Wynn’s epsilon algorithm [36, 37] was developed when the first computers became generally available. Padé approximants, however, would not be powerful enough to sum our series for the Heisenberg–Euler Lagrangian derived in [7].

From a mathematical point of view, there is a distant analogy between the expansion of (1) in terms of special functions and the (exact) expansion of certain quantum electrodynamic bound-state effects into partial waves [38–41]. However, the mathematical entities involved in the present decomposition possess a far less involved mathematical structure, and it is difficult to associate a meaningful physical interpretation to each term in the special function representation of (1). As is evident from the discussion in Sec. 3, the terms of the convergent series representation may be interpreted as being generated by a “partial-fraction decomposition” in distant analogy to the “partial-wave decomposition” in bound-state calculations. We point out in Sec. 5 that the convergence of the special function representation can be accelerated by the same technique – the so-called “Combined Nonlinear–Condensation Transformation” (CNCT) [42] – which was used successfully for the acceleration of the convergence of partial-wave decompositions in quantum electrodynamic bound-state calculations [40, 41].

At the same time, we would like to mention that the integral (1), when expanded in powers of the electric and magnetic field strengths, represents a divergent series. The resulting divergent series can be used as a “model laboratory” for resummation methods [22–25], and related investigations [24–27] have led to the development of asymptotically improved resummation algorithms which have recently received interesting applications [28–30] in the highly accurate determination of the strong coupling constant at the Z pole and other investigations on nonperturbative effects in gauge theories [31–34]. These investigations are related to the fundamental question of how to “make sense” of the fact that many perturbation series encountered in physics are divergent, and are *not* meant to provide efficient means for numerical evaluation of the integral (1).

2 Representation of the QED Effective Action by Special Functions

The QED effective Lagrangian can be expressed as a function of the Lorentz invariants \mathcal{F} and \mathcal{G} which are given by

$$\mathcal{F} = 1/4 F_{\mu\nu} F^{\mu\nu} = 1/2 (\mathbf{B}^2 - \mathbf{E}^2) = 1/2 (a^2 - b^2) , \quad (2)$$

$$\mathcal{G} = 1/4 F_{\mu\nu} (*F)^{\mu\nu} = -\mathbf{E} \cdot \mathbf{B} = \pm ab , \quad (3)$$

where \mathbf{E} and \mathbf{B} are the electric and magnetic field strengths, $F_{\mu\nu}$ is the field strength tensor, and $(*F)^{\mu\nu}$ denotes the dual field strength tensor $(*F)^{\mu\nu} = (1/2) \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. By a and b we denote

the *secular invariants*,

$$\begin{aligned} a &= \sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}}, \\ b &= \sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}}. \end{aligned} \quad (4)$$

These Lorentz invariants are referred to as secular invariants because they emerge naturally as eigenvalues of the field strength tensor; these eigenvalues are conserved under proper Lorentz transformations of the field strength tensor. There are connections between the different representations [1]: If the relativistic invariant \mathcal{G} is positive, then it is possible to transform to a Lorentz frame in which \mathbf{E} and \mathbf{B} are *antiparallel*. In the case $\mathcal{G} < 0$, it is possible to choose a Lorentz frame in which \mathbf{E} and \mathbf{B} are *parallel*. Irrespective of the sign of \mathcal{G} we have in the specified frame

$$a = |\mathbf{B}| \quad \text{and} \quad b = |\mathbf{E}| \quad \text{if and only if} \quad \mathbf{B} \text{ is (anti-)parallel to } \mathbf{E}. \quad (5)$$

In any case, because a and b are positive definite, we have

$$ab = |\mathbf{E} \cdot \mathbf{B}| > 0 \quad \text{for any Lorentz frame and } \mathcal{G} \neq 0, \quad (6)$$

which clarifies the sign ambiguity in (3). We give in (2) and (3) seemingly redundant definitions, but it will soon become apparent that each of the alternative ‘‘points of view’’ has its applications. The Maxwell Lagrangian is given by

$$\mathcal{L}_{\text{cl}} = -\mathcal{F} = -1/4 F_{\mu\nu} F^{\mu\nu} = 1/2 (\mathbf{E}^2 - \mathbf{B}^2) = 1/2 (b^2 - a^2). \quad (7)$$

As it is obvious from Eq. (1), the correction $\Delta\mathcal{L}$ to the Maxwell Lagrangian is conveniently written in terms of the secular invariants a and b .

The effective action has a dispersive (real) part and an imaginary part which is associated with pair production,

$$\Delta\mathcal{L} = \text{Re } \Delta\mathcal{L} + i \text{Im } \Delta\mathcal{L}. \quad (8)$$

In Eqs. (2)–(6) of [7] we showed that the real part of (1) can be expressed as

$$\begin{aligned} \text{Re } \Delta\mathcal{L} &= -\frac{e^2}{4\pi^3} ab \sum_{n=1}^{\infty} [a_n + d_n], \\ a_n &= \frac{\coth(n\pi b/a)}{n} \left\{ \text{Ci} \left(\frac{n\pi m^2}{ea} \right) \cos \left(\frac{n\pi m^2}{ea} \right) + \text{si} \left(\frac{n\pi m^2}{ea} \right) \sin \left(\frac{n\pi m^2}{ea} \right) \right\}, \\ d_n &= -\frac{\coth(n\pi a/b)}{2n} \left\{ \exp \left(\frac{n\pi m^2}{eb} \right) \text{Ei} \left(-\frac{n\pi m^2}{eb} \right) + \exp \left(-\frac{n\pi m^2}{eb} \right) \text{Ei} \left(\frac{n\pi m^2}{eb} \right) \right\}. \end{aligned} \quad (9)$$

We also derived the following representation for the imaginary part [see Eq. (18) of [7]]:

$$\text{Im } \Delta\mathcal{L} = \frac{e^2 ab}{8\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \coth \left(\frac{n\pi a}{b} \right) \exp \left(-\frac{n\pi m^2}{eb} \right). \quad (10)$$

These results have recently been confirmed in [9]. For the special functions, we use the notation of Abramowitz and Stegun [43]. Here, one might wonder why the cosine and sine integrals appear in an ‘‘asymmetric’’ form (Ci and si instead of ci and si) in the definitions of a_n and d_n in (9). The reason is that the commonly accepted definitions for the cosine and sine integrals

are “asymmetric” in the following sense [see Eqs. (5.2.1), (5.2.2), (5.2.5), (5.2.26), and (5.2.27) of [43]]:

$$\begin{aligned} \text{Ci}(z) &= -\int_z^\infty dt \frac{\cos(t)}{t}, \\ \text{si}(z) &= -\int_z^\infty dt \frac{\sin(t)}{t} = \text{Si}(z) - \frac{\pi}{2}, \\ \text{Si}(z) &= \int_0^z dt \frac{\sin(t)}{t}. \end{aligned} \quad (11)$$

From these formulas, it is evident that “symmetric” integrals with lower limit z and upper limit ∞ require the “asymmetric” occurrence of Ci and si.

Because the imaginary part $\text{Im } \Delta\mathcal{L}$ is generated exclusively by analytic continuation of one of the exponential integrals – specifically, the term

$$\text{Ei}\left(-\frac{n\pi m^2}{eb}\right)$$

in the definition of d_n –, it is obvious how to write down a unified representation for both the real and the imaginary part. We therefore present here the unified representation for both the real and the imaginary part we obtained in Eq. (8) of version 2 of our preprint [44]:

$$\begin{aligned} \Delta\mathcal{L} &= \lim_{\epsilon \rightarrow 0^+} -\frac{e^2}{4\pi^3} ab \sum_{n=1}^{\infty} [b_n + c_n], \\ b_n &= -\frac{\coth(n\pi b/a)}{2n} \left\{ \exp\left(-i\frac{n\pi m^2}{ea}\right) \Gamma\left(0, -i\frac{n\pi m^2}{ea}\right) + \exp\left(i\frac{n\pi m^2}{ea}\right) \Gamma\left(0, i\frac{n\pi m^2}{ea}\right) \right\}, \\ c_n &= \frac{\coth(n\pi a/b)}{2n} \left\{ \exp\left(\frac{n\pi m^2}{eb}\right) \Gamma\left(0, \frac{n\pi m^2}{eb}\right) + \exp\left(-\frac{n\pi m^2}{eb}\right) \Gamma\left(0, -\frac{n\pi m^2}{eb} + i\epsilon\right) \right\}. \end{aligned} \quad (12)$$

It becomes obvious from this representation that the effective action has branch cuts along the positive and negative b axis as well as the positive and negative imaginary a axis. Here, we make extensive use of the incomplete Gamma function defined as [see Eq. (6.5.3) of [43]]

$$\Gamma(a, z) = \int_z^\infty dt e^{-t} t^{a-1}. \quad (13)$$

For $a = 0$, the quantity $\Gamma(0, z)$ as a function of z has a branch cut along the negative real z axis, and we assume

$$\lim_{\epsilon \rightarrow 0^+} \text{Im } \Gamma(0, -x + i\epsilon) = -\pi, \quad x > 0 \quad (14)$$

which follows from the relationships [43, Eq. (5.1.45)] $E_1(z) = \Gamma(0, z)$ and [43, Eq. (5.1.7)] $E_1(-x \pm i0) = -\text{Ei}(x) \mp i\pi$.

A unified expansion in terms of special functions – including infinitesimal imaginary parts – has also been given in the final version of [9]. In this context it is perhaps worth pointing out that it is inconsistent with standard notation to use the exponential integral Ei for such a unified formula. The exponential integral Ei is usually defined only for real argument. It is defined as a Cauchy principal-value integral by Gradshteyn and Ryzhik [45] [see Eqs. (8.211.1) and (8.211.2) *ibid.*] as well as by Abramowitz and Stegun [43] [see Eq. (5.1.2) *ibid.*], and also predominantly in the mathematical literature; see, for example, Olver [46] [see Eq. (3.07) *ibid.*]. In contrast to the exponential integral Ei, the incomplete Gamma function is defined in the entire complex plane with a cut along the negative real axis.

3 An Important Mathematical Identity

W. J. Mielniczuk [47] has outlined a proof of the representation (9) for the real part of the effective action. However, his work suffered from a series of unfortunate typographical errors. Here, we provide details on the intermediate steps used in our calculation [7], and we give, in particular, a corrected version of identity (2.8) of [47]. This corrected version was also used in obtaining the results in Eqs. (9) and (10) above and in Eqs. (2) — (6) and (18) in [7].

The corrected version of identity (2.8) of [47] reads:

$$\begin{aligned} & \tilde{x} \tilde{y} u^2 \coth(\tilde{x} u) \cot(\tilde{y} u) - 1 - \frac{1}{3} (\tilde{x}^2 - \tilde{y}^2) u^2 = \\ & - \frac{2 \tilde{x}^3 \tilde{y} u^4}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{\tilde{x}^2 u^2 + k^2 \pi^2} \coth\left(\frac{\tilde{y}}{\tilde{x}} k \pi\right) \\ & + \frac{2 \tilde{y}^3 \tilde{x} u^4}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{\tilde{y}^2 u^2 - k^2 \pi^2} \coth\left(\frac{\tilde{x}}{\tilde{y}} k \pi\right). \end{aligned} \quad (15)$$

In Ramanujan's notebooks [48], this identity appears as entry (19.3) on p. 271. A proof is given which is based on the repeated application of the well-known identities [see *e.g.* Eq. (19.2) of [48] or Eq. (2.4) of [49]]

$$\pi x \cot(\pi x) = 1 + 2x^2 \sum_{n=1}^{\infty} 1/(x^2 - n^2) \quad (16)$$

and

$$\pi y \coth(\pi y) = 1 + 2y^2 \sum_{n=1}^{\infty} 1/(y^2 + n^2), \quad (17)$$

respectively, to each one of the factors on the left-hand side of (15), and a skillful reformulation of the resulting double sum.

Here, we will derive an alternative, but fully equivalent formulation of (15):

$$\begin{aligned} & ab \coth(az) \cot(bz) - \frac{1}{z^2} - \frac{a^2 - b^2}{3} \\ & = \frac{2abz^2}{\pi} \left\{ \sum_{k=1}^{\infty} \frac{\coth(k\pi a/b)}{k [z^2 - k^2 \pi^2 / b^2]} - \sum_{k=1}^{\infty} \frac{\coth(k\pi b/a)}{k [z^2 + k^2 \pi^2 / a^2]} \right\}. \end{aligned} \quad (18)$$

In our derivation of identity (15) in [7], we used the so-called Partial Fraction Theorem 4.4.5 of [50]. It may be surprising that this identity can be obtained by a straightforward application of this basic theorem to the left-hand side of Eq. (15), especially in view of the fact that the derivation of this result represented a considerable challenge to S. Ramanujan, as it is evident from remarks on p. 271 (top of page) of [48]. This shows that sometimes interesting new results can be obtained by a straightforward application of theorems occurring in standard textbooks.

We quote here the Partial Fraction Theorem [50, Theorem 4.4.5 on pp. 337 - 338]:

Theorem 1. *Suppose that g is meromorphic with simple poles at a_1, a_2, a_3, \dots with $0 \leq |a_1| \leq |a_2| \dots$ and residues b_k at a_k , but analytic at 0. Suppose there is an increasing sequence R_1, R_2, R_3, \dots with $\lim_{n \rightarrow \infty} R_n = \infty$ and simple closed curves C_n satisfying*

(i): $|z| \geq R_n$ for all z on C_n ,

(ii): There is a constant S with $\text{length}(C_n) \leq SR_n$ for all $n \in \mathbb{N}$,

(iii): There is a constant M with $|g(z)| \leq M$ for all z on C_n and for all $n \in \mathbb{N}$.

Then,

$$g(z) = g(0) + \sum_{m=1}^{\infty} \left\{ \frac{b_m}{z - a_m} + \frac{b_m}{a_m} \right\}. \quad (19)$$

Here, it should be noted that normally the series on the right-hand side of (19) does not converge absolutely, which implies that it must *not* be rearranged. In fact, the series with the terms $b_m/(z - a_m)$ may even diverge (compare the remark in the third paragraph on p. 539 of [51]). Thus, the compensatory terms b_m/a_m are necessary to ensure convergence.

Equation (19) implies *eo ipso* – just as the conditions of the partial fraction theorem – that the function $g(z)$ which is expanded into “partial fractions” must necessarily be analytic at zero argument. In order to analyse the behaviour of the product $\coth(az) \cot(bz)$ with $a, b \in \mathbb{R}$ as $z \rightarrow 0$, we use [52, Eq. (30:6.2)]

$$\coth(z) = \frac{1}{z} \sum_{j=0}^{\infty} \frac{4^j B_{2j}}{(2j)!} z^{2j}, \quad |z| < \pi, \quad (20)$$

and [52, Eq. (34:6.2)]

$$\cot(z) = \frac{1}{z} - \sum_{j=1}^{\infty} \frac{4^j |B_{2j}|}{(2j)!} z^{2j-1}, \quad |z| < \pi. \quad (21)$$

Here, B_m with $m \in \mathbb{N}_0$ is a Bernoulli number [43, Section 23].

If we now insert the leading terms of (20) and (21) into the product $\coth(az) \cot(bz)$ and use $B_0 = 1$ and $B_2 = 1/6$ [43, Eq. (23.1.3)], we obtain:

$$\coth(az) \cot(bz) = \frac{1}{abz^2} + \frac{a}{3b} - \frac{b}{3a} + O(z^2), \quad z \rightarrow 0. \quad (22)$$

This suggests – consistent with the renormalisation of the effective Lagrangian – the definition of the function

$$f(z) = ab \coth(az) \cot(bz) - \frac{1}{z^2} - \frac{a^2 - b^2}{3}, \quad a, b \in \mathbb{R}, \quad (23)$$

which corresponds to the left-hand side of Eq. (18) and which satisfies

$$f(z) = O(z^2), \quad z \rightarrow 0. \quad (24)$$

For the determination of the poles of $f(z)$, we use the above equations (16) and (17) which can be reformulated as [52, Eq. (30:6.6)]

$$\coth(z) = \sum_{m=-\infty}^{\infty} \frac{z}{z^2 + m^2 \pi^2}, \quad z \neq ik\pi, \quad k \in \mathbb{Z}, \quad (25)$$

and [52, Eq. (34:6.5)]

$$\cot(z) = \sum_{m=-\infty}^{\infty} \frac{z}{z^2 - m^2\pi^2}, \quad z \neq k\pi, \quad k \in \mathbb{Z}. \quad (26)$$

From these expansions and from the definition of $f(z)$ we conclude that $f(z)$ has the simple poles

$$a_k = i \frac{k\pi}{a}, \quad k \in \mathbb{Z} \setminus \{0\}, \quad (27)$$

$$a'_k = \frac{k\pi}{b}, \quad k \in \mathbb{Z} \setminus \{0\}. \quad (28)$$

Next, we want to determine the corresponding residues

$$b_k = \operatorname{Res}_{z=a_k} f(z) = \lim_{z \rightarrow a_k} [(z - a_k) f(z)] \quad (29)$$

and

$$b'_k = \operatorname{Res}_{z=a'_k} f(z) = \lim_{z \rightarrow a'_k} [(z - a'_k) f(z)]. \quad (30)$$

For that purpose, we rewrite (25) and (26) as follows by isolating the terms that contribute to the residues at a_k and a'_k , respectively:

$$\begin{aligned} \coth(az) &= \sum_{n=-\infty}^{\infty} \frac{z/a}{(z + in\pi/a)(z - in\pi/a)} \\ &= \sum_{\substack{n=-\infty \\ n \neq \pm k}}^{\infty} \frac{z/a}{(z + in\pi/a)(z - in\pi/a)} + \frac{2z/a}{(z + ik\pi/a)(z - ik\pi/a)}, \end{aligned} \quad (31)$$

$$\begin{aligned} \cot(bz) &= \sum_{m=-\infty}^{\infty} \frac{z/b}{(z + m\pi/b)(z - m\pi/b)} \\ &= \sum_{\substack{m=-\infty \\ m \neq \pm k}}^{\infty} \frac{z/b}{(z + m\pi/b)(z - m\pi/b)} + \frac{2z/b}{(z + k\pi/b)(z - k\pi/b)}. \end{aligned} \quad (32)$$

With the help of (31), we then obtain:

$$\begin{aligned} b_k &= \lim_{z \rightarrow a_k} [(z - a_k) f(z)] \\ &= \lim_{z \rightarrow ik\pi/a} [(z - ik\pi/a) ab \coth(az) \cot(bz)] \\ &= \lim_{z \rightarrow ik\pi/a} \frac{2bz \cot(bz)}{(z + ik\pi/a)} = b \cot(ik\pi b/a). \end{aligned} \quad (33)$$

If we now use $\cot(iz) = -i \coth(z)$, we finally obtain

$$b_k = -ib \coth(k\pi b/a), \quad k \in \mathbb{Z} \setminus \{0\}. \quad (34)$$

Similarly, we obtain with the help of (32):

$$b'_k = \lim_{z \rightarrow a'_k} [(z - a'_k) f(z)] = a \cot(k\pi a/b), \quad k \in \mathbb{Z} \setminus \{0\}. \quad (35)$$

If (19) is to be used for the derivation of a partial-fraction decomposition for $f(z)$ defined by (23), it is natural to identify the closed contour \mathcal{C}_n with the rectangle having the 4 sides $\mathcal{X}_n^{(\pm)}$ and $\mathcal{Y}_n^{(\pm)}$, where

$$\mathcal{X}_n^{(\pm)} = \{z_n = x_n + iy_n \mid x_n = (2s - 1)X_n, 0 \leq s \leq 1, y_n = \pm Y_n\} \quad (36)$$

and

$$\mathcal{Y}_n^{(\pm)} = \{z_n = x_n + iy_n \mid x_n = \pm X_n, y_n = (2t - 1)Y_n, 0 \leq t \leq 1\}. \quad (37)$$

Clearly, X_n and Y_n have to be chosen in such a way that the poles $a_k = k\pi/b$ and $a'_k = ik\pi/a$ of $f(z)$ do not lie on this rectangle.

We are on the safe side if we choose X_n and Y_n in such a way that they are located in the middle between two neighbouring poles of $f(z)$:

$$X_n = \frac{\pi}{b}(n + 1/2), \quad Y_n = \frac{\pi}{a}(n + 1/2), \quad n \in \mathbb{N}. \quad (38)$$

We now have to show that $f(z)$ is bounded on the rectangle \mathcal{C}_n according to (iii) of Theorem 1. For that purpose, we now use [52, Eq. (30:11.2)]

$$\coth(x + iy) = \frac{\sinh(2x) - i \sin(2y)}{\cosh(2x) - \cos(2y)} \quad (39)$$

and [52, Eq. (34:11.2)]

$$\cot(x + iy) = \frac{\sin(2x) - i \sinh(2y)}{\cos(2x) - \cosh(2y)}. \quad (40)$$

If s in (36) satisfies $s = 1/2$, which implies $x_n = 0$ and $y_n = \pm Y_n$, or t in (37) satisfies $t = 1/2$, which implies $x_n = \pm X_n$ and $y_n = 0$, we obtain

$$\coth(\pm iaY_n) \cot(\pm ibY_n) = \frac{-i \sin(\pm 2aY_n)}{[1 - \cos(\pm 2aY_n)]} \frac{i \sinh(\pm 2bY_n)}{[1 - \cosh(\pm 2bY_n)]}, \quad (41)$$

which remains bounded on \mathcal{C}_n as $n \rightarrow \infty$, or

$$\coth(\pm aX_n) \cot(\pm bX_n) = \frac{\sinh(\pm 2aX_n)}{[\cosh(\pm 2aX_n) - 1]} \frac{\sin(\pm 2bX_n)}{[1 - \cos(\pm 2bX_n)]}, \quad (42)$$

which also remains bounded on \mathcal{C}_n as $n \rightarrow \infty$.

If s in (36) and t in (37) satisfy $s, t \neq 1/2$, $f(z)$ remains bounded on the rectangles \mathcal{C}_n because of the periodicity and boundedness of \cos and \sin for real arguments, and we find

$$\lim_{n \rightarrow \infty} \coth(az_n) = \lim_{n \rightarrow \infty} \frac{\sinh(2ax_n)}{\cosh(2ax_n)} \quad (43)$$

and

$$\lim_{n \rightarrow \infty} \cot(bz_n) = i \lim_{n \rightarrow \infty} \frac{\sinh(2by_n)}{\cosh(2by_n)}. \quad (44)$$

Thus, the constant M in (iii) of Theorem 1 can in the case of $f(z)$ be chosen according to

$$M = \max_{n \in \mathbb{N}} (|f(z_n)|), \quad z_n \in \mathcal{C}_n. \quad (45)$$

Then, the R_n satisfying (i) of Theorem 1 should be chosen according to

$$R_n = \min(X_n, Y_n), \quad (46)$$

and the constant S in (ii) of Theorem 1 should be chosen according to

$$S = 4 \max(b/a, a/b). \quad (47)$$

However, the function $f(z)$ defined in (23) does not exactly meet the requirements of Theorem 1. Instead of a sequence of simple poles a_1, a_2, a_3, \dots with corresponding residues b_1, b_2, b_3, \dots , there are actually two sequences $a_{\pm 1}, a_{\pm 2}, a_{\pm 3}, \dots$ of simple poles with corresponding residues $b_{\pm 1}, b_{\pm 2}, b_{\pm 3}, \dots$, which implies that we have a partial-fraction decomposition of the following kind:

$$F(z) = F(0) + \sum_{m=1}^{\infty} \left\{ \frac{b_m}{z - a_m} + \frac{b_m}{a_m} \right\} + \sum_{m=1}^{\infty} \left\{ \frac{b_{-m}}{z - a_{-m}} + \frac{b_{-m}}{a_{-m}} \right\}. \quad (48)$$

If we take into account that the hyperbolic cotangent is *odd* according to $\coth(-z) = -\coth(z)$, we see from (27) and (34) or from (28) and (35) that the poles and residues of $F(z)$ have to satisfy for all $k \in \mathbb{N}$ the (anti-)symmetry relations

$$a_k = -a_{-k}, \quad b_k = -b_{-k}. \quad (49)$$

If we also take into account that (24) implies

$$f(0) = 0, \quad (50)$$

we see that (48) is to be replaced by the following partial-fraction decomposition:

$$F(z) = \sum_{m=1}^{\infty} \left\{ \frac{b_m}{z - a_m} - \frac{b_m}{z + a_m} + \frac{2b_m}{a_m} \right\}. \quad (51)$$

If we now insert the poles (27) and (28) and the corresponding residues (34) and (35), respectively, into (51), we obtain the following partial-fraction decomposition:

$$\begin{aligned} & ab \coth(az) \cot(bz) - \frac{1}{z^2} - \frac{a^2 - b^2}{3} \\ &= \sum_{k=1}^{\infty} a \coth(k\pi a/b) \left\{ \frac{1}{z - k\pi/b} - \frac{1}{z + k\pi/b} + \frac{2}{k\pi/b} \right\} \\ & \quad - \sum_{k=1}^{\infty} ib \coth(k\pi b/a) \left\{ \frac{1}{z - ik\pi/a} - \frac{1}{z + ik\pi/a} + \frac{2}{ik\pi/a} \right\}. \end{aligned} \quad (52)$$

By putting the expressions in curly brackets on a common denominator, we obtain (18).

4 Electric–Magnetic Duality

Electric-magnetic duality, understood as a mutual transformation of electric and magnetic quantities, has attracted a great deal of attention since Dirac’s ideas on magnetic monopoles [54, 55] were introduced in classical electrodynamics. Whereas electric-magnetic duality can be formulated as a continuous symmetry of the classical Maxwell equations [56] (either including both types of charges or without any charges), we devote this section to a brief study of discrete

duality transformations of quantum effective actions. In particular, we consider two types of duality:

$$\begin{aligned} \text{Type I:} \quad & \mathbf{E} \rightarrow \mathbf{B}, \quad \mathbf{B} \rightarrow -\mathbf{E}, \\ \text{Type II:} \quad & a \rightarrow -ib, \quad b \rightarrow ia. \end{aligned} \tag{53}$$

(In a Lorentz frame where $\mathbf{E} \parallel \mathbf{B}$, the Type II duality implies $|\mathbf{B}| \rightarrow -i|\mathbf{E}|$ and $|\mathbf{E}| \rightarrow i|\mathbf{B}|$.) As a consequence, the invariants transform as

$$\begin{aligned} \text{Type I:} \quad & \mathcal{F} \rightarrow -\mathcal{F}, \quad \mathcal{G} \rightarrow -\mathcal{G}, \quad \implies \quad b \rightarrow a, \quad a \rightarrow -b \\ \text{Type II:} \quad & \mathcal{F} \rightarrow \mathcal{F}, \quad \mathcal{G} \rightarrow \mathcal{G}. \end{aligned} \tag{54}$$

Note that the duality of Type II preserves the invariants, so that Maxwell's equations derivable from $\mathcal{L}_{\text{cl}} = -\mathcal{F}$ are trivially invariant; by contrast, the classical Lagrangian is not invariant under Type I, but Maxwell's equations are.¹ It therefore becomes obvious that only Type I affects physical quantities (field strengths), whereas Type II signifies a certain redundancy in the parameterisation of physical quantities.

In fact, this redundancy exists even for a larger class of duality transformations of Type II,

$$a \rightarrow \pm i b \quad b \rightarrow \mp i a. \tag{55}$$

which preserves the invariants \mathcal{F} and \mathcal{G} . Since the effective Lagrangian of QED is gauge and Lorentz invariant, the effective Lagrangian for constant fields is necessarily a function of \mathcal{F} and \mathcal{G} only, $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(\mathcal{F}, \mathcal{G})$, implying that the duality transformations of Eq. (55) leave $\mathcal{L}_{\text{eff}}(\mathcal{F}, \mathcal{G})$ invariant. This statement is not at all tied to perturbation theory, and the duality (55) holds therefore to all orders in the external fields and to all loop orders. From a diagrammatic perspective, each external leg of a diagram contributing to any given loop order corresponds to a field strength tensor; and any possible contraction can be expressed in terms of \mathcal{F} and \mathcal{G} , ensuring this duality invariance. In some sense, these invariants are therefore more fundamental than a and b .

Let us now study the invariance properties of the effective Lagrangian under consideration. From Eq. (1), it is obvious that $\Delta\mathcal{L}$ is not invariant under Type I, similarly to the Maxwell Lagrangian, (even a rotation of the contour in the first quadrant does not help): in an asymptotic expansion, terms of odd order in \mathcal{F} , \mathcal{G} flip sign.

At a first glance, $\Delta\mathcal{L}$ indeed seems to be invariant under the transformation (55) as expected, since the integrand is invariant.² Nevertheless, it has recently been argued [9, 10] that $\Delta\mathcal{L}$ is uniquely invariant only under Type II, excluding explicitly the remaining transformations of Eq. (55). The argument is based on the special function representation (12) in terms of incomplete Gamma functions.

In our language, the argument could be rephrased in the following way: Taking the details of the integral contour specified by η and ϵ into account corresponds to a modification $a \rightarrow a \exp(i\delta)$, where δ is a small positive quantity. We discuss the apparent uniqueness of the replacement $a \rightarrow -ib$ by way of example (see also Appendix A).

When applying the replacement $a \rightarrow a \exp(i\delta)$ to b_n in Eq. (12), the relevant factor in the argument of the last Gamma function in b_n assumes the form $i/a \rightarrow (i/a) \exp(-i\delta)$, which has complex argument $\pi/2 - \delta$. With attention to the fact that the incomplete Gamma function has singularities along the negative real axis, it therefore becomes apparent that this argument could

¹The transformations of a and b under Type I follow from Eqs. (2,3). Note that Eq. (4) is not meant to participate in the duality transformations, owing to the nonlinear relation between the two sets of invariants.

²The integrand is even invariant under a larger class of duality transformations, including $a \rightarrow \pm ib$, $b \rightarrow \pm ia$, owing to parity invariance of QED.

only be increased by an amount of $+\pi/2$ (not $-\pi/2$), if we want to stay on the same branch of the incomplete Gamma function. This seems to fix uniquely the replacement $a \rightarrow -ib$ because in this case,

$$\frac{i}{a} \exp(-i\delta) \rightarrow \frac{i^2}{b} \exp(-i\delta) \rightarrow -\frac{1}{b} \exp(-i\delta) = -\frac{1}{b} + i\gamma, \quad (56)$$

in agreement with the last term in the definition of c_n in Eq. (12). Here, γ denotes a further infinitesimal positive quantity.

The contradiction between the required general form of the duality (55) and the seemingly unique form of Type II can be resolved by looking at the integral representation: consider the representation (1) of $\Delta\mathcal{L}$ as a function of two complex variables z_1 and z_2 :

$$\Delta\mathcal{L}(z_1, z_2) = -\frac{e^2}{8\pi^2} \lim_{\epsilon, \eta \rightarrow 0^+} \int_{i\eta}^{\infty+i\eta} \frac{ds}{s} e^{-(m^2-i\epsilon)s} \left[z_1 z_2 \coth(ez_1 s) \cot(ez_2 s) - \frac{z_1^2 - z_2^2}{3} - \frac{1}{(es)^2} \right]. \quad (57)$$

The duality of Type II (53) can be formulated as the identity $\Delta\mathcal{L}(a, b) = \Delta\mathcal{L}(-ib, ia)$. Because a and b are real and positive, this identity is valid in a strict sense only if the function $\Delta\mathcal{L}(z_1, z_2)$ is one-valued in the relevant range of the complex arguments of z_1 and z_2 .

In order to avoid the poles given by the cotangent and hyperbolic cotangent functions, the complex arguments of z_1 and z_2 in (57) cannot be varied without restriction if we keep the class of allowed contours fixed (recall that the limits of integration in (1) can also be chosen as $i\eta$ and $\infty+i\eta$, respectively). Taking into account the infinitesimal parts ϵ and η , we are led to conclude that the integral representation remains valid for the fixed class of contours in the argument range

$$\begin{aligned} -\pi/2 &\leq \arg(z_1) < \pi/2, \\ 0 &\leq \arg(z_2) < \pi \end{aligned} \quad (58)$$

for z_1 and z_2 (observe the fine difference between the $<$ and \leq signs!). The restriction given in (58) identifies (as a function of z_1 and z_2) a “physical” or “causal” branch of the effective action for a given contour. Among the four different replacements $\Delta\mathcal{L}(a, b) \rightarrow \Delta\mathcal{L}(\pm ib, \pm ia)$ or $\Delta\mathcal{L}(a, b) \rightarrow \Delta\mathcal{L}(\pm ib, \mp ia)$, it is only the replacement $\Delta\mathcal{L}(a, b) \rightarrow \Delta\mathcal{L}(-ib, ia)$ which respects the restriction set by (58) for a fixed contour. Therefore, the duality of Type II is only “unique” in connection with a precisely specified integral representation which does not allow for an unrestricted analytic continuation of its arguments. In other words, the seeming “uniqueness” is simply a shortcoming of the integral representation (and also of the special-function representation being identical to the former).

In order to achieve full invariance under the dualities (55), we have to allow for the fact that the contour also has to be readjusted (or certain poles are allowed to be crossed without picking up their contribution). From a different perspective, it is only natural to perform the duality transformation first, and then specify the details of the contour. This is perfectly justified, since the particular choice of the contour is not a *result* of the calculation (of the fermion determinant in this case), but rather an additional piece of information that has to be inserted afterwards in order to define the integral. This information arises, of course, from physical requirements: the $\pm i\epsilon$ prescription is dictated by causality, and the shift by η can, e.g., be fixed by requiring that the pair-production probability related to the imaginary part of $\Delta\mathcal{L}$ is a number between 0 and 1. The resulting integral representation will always lead to the same special-function representation (12).

From another point of view, the special-function representation and the integral representation with fixed contour remove a part of the above-mentioned redundancy in the parameterisation of the field strength invariants, which is generally present in the effective Lagrangian.

At this point, let us remark that dualities of Type II or (55) can be very useful from a technical viewpoint, although they have no physical meaning: for instance, from the effective Lagrangian for a purely magnetic field, one can extract information about the electric case with the aid of the substitution, $|\mathbf{B}| \rightarrow -i|\mathbf{E}|$ [53]. Moreover, standard-model calculations in constant electromagnetic fields can always be checked by testing their dual invariance; e.g. in [57, 58], this dual invariance is visible in neutrino amplitudes in electromagnetic fields. Of course, to be on the safe side, the duality of Type II suffices for such a check in order to avoid problems of the kind mentioned before.

The question of electric-magnetic duality becomes even more interesting for systems which are characterised by additional Lorentz covariant quantities. For example, let us consider QED with constant fields in a heat bath; the latter involves an additional Lorentz vector u_μ , the heat-bath four-velocity, allowing for one further gauge and Lorentz invariant quantity [cf. Eq. (3.143) of [1]]:

$$\mathcal{E} = (u_\mu F^{\mu\nu})^2 = u_\mu F^{\mu\nu} u^\rho F_{\rho\nu}. \quad (59)$$

Under duality of Type II, \mathcal{E} is trivially invariant, since it is linearly independent of \mathcal{F} and \mathcal{G} , and thereby independent of a and b . In fact, the known QED effective actions, depending on a , b and \mathcal{E} at finite temperature, show this invariance under Type II [1].

Under duality of Type I, \mathcal{E} transforms according to $\mathcal{E} \rightarrow \mathcal{E} + 2\mathcal{F}$; and similarly to the zero-temperature case, the finite-temperature effective action is generally not invariant under Type I. Nevertheless, the dominant low-temperature contribution arising at two-loop order astonishingly exhibits an invariance under Type I. This might be related to the fact that only transversal thermal fluctuations of the photon give rise to this contribution.

5 Acceleration of Convergence

From the asymptotic expansion of the incomplete Gamma function as $z \rightarrow \infty$ [43, Eq. (6.5.32)],

$$e^z \Gamma(a, z) = z^{a-1} \left[1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \mathcal{O}(z^{-3}) \right], \quad |\arg z| < 3\pi/2, \quad (60)$$

we obtain

$$\frac{e^{nz} \Gamma(0, nz)}{n} = \frac{1}{n^2 z} - \frac{1}{n^3 z^2} + \mathcal{O}(n^{-4}), \quad n \rightarrow \infty. \quad (61)$$

Since the cotangents in (12) rapidly approach one as $n \rightarrow \infty$, we can conclude that terms in (12) are essentially of order $\mathcal{O}(n^{-2})$ as $n \rightarrow \infty$. A more detailed analysis of the large-order behaviour of the terms in (12) can be found in Appendices A and B of [7].

An $\mathcal{O}(n^{-2})$ behaviour of the terms also occurs in the Dirichlet series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for the Riemann zeta function with $s = 2$. This is a very disconcerting observation since the Dirichlet series for $\zeta(2)$ converges quite slowly – it can be shown that increasing the number of terms of its partial sum by a factor of 10 improves the accuracy by a single decimal digit only – and in the literature on convergence acceleration it is one of the standard test systems for checking the capability of a transformation in the case of a slowly convergent monotonic series.

Thus, the series expansions (9) and (12) represent typical cases of monotonic series which converge so slowly that a straightforward evaluation by adding up one term after another is computationally very unattractive. Of course, the acceleration of the convergence of series of

that kind has been studied quite extensively in the literature and many techniques are known to improve the efficiency of numerical computations (see for example [59–61] and references therein).

Nevertheless, one should not forget that the acceleration of the convergence of a slowly convergent monotonic series can be a very challenging problem. Moreover, numerical instabilities due to rounding errors are likely to occur. A sequence transformation accelerates convergence by extracting and utilising information from a finite set of input data on the index-dependence of the truncation errors. This is normally accomplished by forming higher weighted differences. If the input data are the partial sums of a *strictly alternating* series, the formation of higher weighted differences is a remarkably stable process, but if the input data all have the *same sign*, numerical instabilities are quite likely. Thus, if the sequence to be transformed are the partial sums of a slowly convergent monotonic series, numerical instabilities are to be expected, and most convergence acceleration methods are not able to obtain transformation results that are close to machine accuracy.

In some cases, these instability problems can be overcome with the help of a condensation transformation attributable to Van Wijngaarden, which converts input data having the same sign to the partial sums of an alternating series, whose convergence can be accelerated more effectively (compare, for instance, Appendix A of [42]). The condensation transformation was first mentioned in [62, pp. 126 - 127] and only later published by Van Wijngaarden [63]. It was used by Daniel [64] in combination with the Euler transformation, and recently, it was rederived by Pelzl and King [65]. Since the transformation of a strictly alternating series by means of nonlinear sequence transformations is a stable process, it was in this way possible to evaluate special functions that are defined by extremely slowly convergent monotonic series, not only relatively efficiently but also close to machine accuracy [42], or to perform extensive quantum electrodynamical calculations [40, 41]. Unfortunately, the use of this “combined nonlinear-condensation transformation” (CNCT) [42] is not always possible: The conversion of a monotonic to an alternating series requires that terms of the input series with large indices can be computed.

However, this CNCT is well suited for the acceleration of the convergence of the series expansions (9) and (12) considered in this article.

The method consists in first rewriting the slowly convergent monotonic input series $\sum_{k=0}^{\infty} \tau(k)$ into an alternating series. In the second step, the convergence of the alternating series is accelerated via a suitable nonlinear sequence transformation. In our case, we have $\tau(k) = [a_{k+1} + d_{k+1}]$, with a_{k+1} and d_{k+1} given by Eq. (9). The partial sums

$$\sigma_n = \sum_{k=0}^n \tau(k) \quad (62)$$

increase monotonically, i.e. $\sigma_{n+1} > \sigma_n$ for all $n = 0, \dots, \infty$ if all terms satisfy $\tau(n) \geq 0$. Let us further assume that the sequence of the partial sums $\{\sigma_n\}_{n=0}^{\infty}$ converges to some limit $\sigma = \sigma_{\infty}$ as $n \rightarrow \infty$. Following Van Wijngaarden [63], we transform the original series into an alternating series $\sum_{j=0}^{\infty} (-1)^j \mathbf{A}_j$ according to

$$\sum_{k=0}^{\infty} \tau(k) = \sum_{j=0}^{\infty} (-1)^j \mathbf{A}_j, \quad (63)$$

$$\mathbf{A}_j = \sum_{k=0}^{\infty} \mathbf{b}_k^{(j)}, \quad (64)$$

$$\mathbf{b}_k^{(j)} = 2^k \tau(2^k (j+1) - 1). \quad (65)$$

The terms \mathbf{A}_j defined in Eq. (64) are all positive if the terms $\tau(k)$ of the original series are

Table 1: Evaluation of the real (dispersive) part of the QED effective Lagrangian for $|\mathbf{E}| = 30 E_{cr}$ and $|\mathbf{B}| = 30 B_{cr}$ by evaluating the special function representation (9) using a combination [42] of the Van Wijngaarden condensation transformation defined in Eqs. (63) – (65) and the nonlinear delta transformation [59]. The \mathbf{S}_n are defined in (66), and the nonlinear delta transform $\delta_n^{(0)}(1, \mathbf{S}_0)$ is defined in Eq. (8.4-4) of [59]. The result is given in units of m^4 . The apparent convergence of the delta transforms in the second column is indicated by underlining. After 21 iterations, the transforms $\delta_n^{(0)}(1, \mathbf{S}_0)$ have stabilised to the 15-figure result 8.393 398 582 100 617.

n	\mathbf{S}_n	$\delta_n^{(0)}(1, \mathbf{S}_0)$
0	11.834 710 587 368	11.834 710 587 368
1	6.388 353 476 842	<u>8.463 873 830 587</u>
2	9.741 830 922 440	<u>8.384 963 963 657</u>
3	7.413 293 009 436	<u>8.393 553 703 382</u>
4	9.141 317 648 944	<u>8.393 398 289 155</u>
5	7.803 272 984 326	<u>8.393 399 592 701</u>
6	8.870 519 831 631	<u>8.393 398 337 299</u>
7	8.000 385 721 330	<u>8.393 398 666 561</u>
8	8.721 936 912 365	<u>8.393 398 594 227</u>
9	8.115 447 030 287	<u>8.393 398 583 148</u>
10	8.630 906 154 429	<u>8.393 398 580 473</u>
11	8.188 731 860 505	<u>8.393 398 582 040</u>
12	8.571 048 192 682	<u>8.393 398 582 143</u>
13	8.238 223 421 331	<u>8.393 398 582 104</u>
14	8.529 698 152 551	<u>8.393 398 582 097</u>
15	8.273 084 942 867	<u>8.393 398 582 099</u>
16	8.500 073 809 529	<u>8.393 398 582 099</u>

all positive. The quantities \mathbf{A}_j are commonly referred to as the condensed sums, and the series $\sum_{j=0}^{\infty} (-1)^j \mathbf{A}_j$ is referred to as the Van Wijngaarden transformed series.

The transformation from a monotonic series to a strictly alternating series according to Eqs. (63), (64) and (65) is essentially a reordering of the terms $\tau(k)$ of the original series. We define the partial sums

$$\mathbf{S}_n = \sum_{j=0}^n (-1)^j \mathbf{A}_j \quad (66)$$

of the Van Wijngaarden transformed series. As illustrated in Table 1 of [42], the \mathbf{S}_n with $n \geq 0$ reproduces the partial sum σ_n , Eq. (62), which contains the first $n + 1$ terms of the original series. Formal proofs of the correctness of this rearrangement can be found in Ref. [64] or in the Appendix of Ref. [65].

The series (64) for the terms of the Van Wijngaarden transformed series can be rewritten as follows:

$$\mathbf{A}_j = \tau(j) + 2\tau(2j + 1) + 4\tau(4j + 3) + \dots \quad (67)$$

Since the terms $\tau(k)$ of the original series have by assumption the same sign, we immediately observe

$$|\mathbf{A}_j| \geq |\tau(j)|. \quad (68)$$

Consequently, the Van Wijngaarden transformation, given by Eqs. (63), (64) and (65), does not lead to an alternating series whose terms decay more rapidly in magnitude than the terms of the original monotonic series. However, an acceleration of convergence may be achieved if the partial sums (66) of the Van Wijngaarden transformed series are used as input data in a convergence acceleration process, and – as, for example, discussed in Appendix A of [42]) – the convergence of alternating series can be accelerated much more effectively than the convergence of monotonic series. For the acceleration of convergence we use the delta transformation given in Eq. (8.4-4) of [59], which was found to be a very effective accelerator for Van Wijngaarden transformed series [42].

This will now be demonstrated explicitly. From a consideration of the expressions (9) and (12) it is obvious that the computationally most demanding special cases will be encountered for large fields; in these cases, many terms of the representation (9) are needed in order to achieve convergence [accordingly, for strong fields we encounter problematic oscillations in the integrand of Eq. (1)]. We consider only one example here – Table 1 –, which describes the evaluation of the dispersive (real) part of the effective Lagrangian at field strength $|\mathbf{E}| = b = 30 E_{cr}$ and $|\mathbf{B}| = a = 30 B_{cr}$. This does not preclude the possibility that other efficient calculational methods exist for the evaluation of (9). However, we stress here that a suitable acceleration of the convergence of the special function representations (9) and (12) removes the principal numerical difficulty associated with the slow convergence of the series at large field strength. In our example – see Table 1 –, the highest index encountered in the calculation is $\hat{n} = 37748736$, the total number of evaluations of terms $[a_n + d_n]$ is 405. The ratio of these two numbers is roughly 93000, which corresponds to an acceleration of the calculation by roughly five orders of magnitude.

The CNCT transforms the slowly convergent series expansions (9) and (12) into the rapidly converging sequence of the delta transforms $\delta_n^{(0)}(1, \mathbf{S}_0)$ (see Table 1). In general, no direct interpretation is available for the delta transforms [59], just as much as Padé approximants [37] lack a direct physical interpretation. At best, the delta transforms can be viewed as the analytic continuations (“extrapolations”) of the partial sums (66) of the Van Wijngaarden transformed alternating series \mathbf{S}_n to $n \rightarrow \infty$.

6 Conclusion

We have investigated questions related to the representation of the quantum electrodynamic (QED) effective Lagrangian and its numerical evaluation. In Sec. 2, we recall our previous results given in [7, 8] for special function representations of the effective Lagrangian, and we clarify the mathematical notation used in the special function representations (9), (10) and (12). The representation (12) unifies the real and imaginary parts.

In Sec. 3, we present a detailed description of the proof of a certain mathematical identity (15) used in our investigations [7, 8]. In Sec. 4, we discuss the question as to whether the QED effective Lagrangian is invariant under certain types of electric-magnetic duality. We conclude that gauge and Lorentz symmetry guarantees invariance under a general class of dualities (55) to all loop orders; but discrete representations of the effective Lagrangian may not realize this general dual invariance in a strict sense, so that only a smaller duality subgroup (Type II) remains.

In Sec. 5, we show that the convergence over the summation index n of the special function representation (9) can be accelerated efficiently by the application of the CNCT transformation [42]. In this way, the computing time is reduced by several orders of magnitude. Based on the results of the current paper, we expect to carry out detailed studies related to various projected and ongoing experiments and astrophysical phenomena [1, 12–14, 16] involving strong

static-field conditions (or fields with frequencies that are small as compared to the electron Compton wavelength).

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A Model Example

Consider as a model example the integral

$$\mathcal{M}(a, b) = \int_{i\eta}^{\infty+i\eta} dt \exp(-t) \left(\frac{1}{1-iat} + \frac{1}{1+iat} + \frac{1}{1-bt} + \frac{1}{1+bt} \right), \quad (69)$$

where a and b are *a priori* real variables, but can be also be generalised to the complex case $a \rightarrow z_1$, $b \rightarrow z_2$. The integrand is invariant under the four different “duality transformations” $a \rightarrow \pm ib$, $b \rightarrow \pm ia$ ($\mp ia$), but the integral is not. Owing to the poles of the integrand, the imaginary parts change sign when the complex arguments a and b are varied such that one of the poles of the integrand is crossed. When considered as a function of complex arguments z_1 and z_2 , the function $\mathcal{M}(z_1, z_2)$ has branch cuts along the positive and negative imaginary z_1 axis and along the positive and negative real z_2 axis. Furthermore, by inspection of the integrand we conclude that the argument ranges for z_1 and z_2 are

$$\begin{aligned} -\pi/2 &\leq \arg(z_1) < \pi/2, \\ 0 &\leq \arg(z_2) < \pi. \end{aligned} \quad (70)$$

From these relations it becomes clear that if we want to stay on the principal branch in the complex z_1, z_2 plane, then we have to modify the arguments of a and b in accord with the restrictions given by (70). This singles out the duality of Type II

$$a \rightarrow -ib, \quad b \rightarrow ia. \quad (71)$$

We conclude the discussion of the model example by pointing out that it can be expressed as

$$\begin{aligned} \mathcal{M}(a, b) = \lim_{\epsilon \rightarrow 0^+} &\left\{ \frac{i}{a} \left[\exp\left(\frac{i}{a}\right) \Gamma\left(0, \frac{i}{a}\right) - \exp\left(-\frac{i}{a}\right) \Gamma\left(0, -\frac{i}{a}\right) \right] \right. \\ &\left. + \frac{1}{b} \left[\exp\left(\frac{1}{b}\right) \Gamma\left(0, \frac{1}{b}\right) - \exp\left(-\frac{1}{b}\right) \Gamma\left(0, -\frac{1}{b} + i\epsilon\right) \right] \right\}. \end{aligned} \quad (72)$$

There is a certain analogy to the special function representation (12).

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