

QR-SUBMANIFOLDS OF MAXIMAL QR-DIMENSION IN QUATERNIONIC PROJECTIVE SPACE

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ABSTRACT. The purpose of this paper is to study n -dimensional QR -submanifolds of maximal QR -dimension isometrically immersed in a quaternionic projective space and to give sufficient conditions in order for such a submanifold to be a tube over a quaternionic invariant submanifold.

1. Introduction

Let M be a connected real n -dimensional submanifold of real codimension p of a quaternionic Kähler manifold \overline{M} with quaternionic Kähler structure $\{F, G, H\}$. If there exists an r -dimensional normal distribution ν of the normal bundle TM^\perp such that

$$\begin{aligned} F\nu_x &\subset \nu_x, \quad G\nu_x \subset \nu_x, \quad H\nu_x \subset \nu_x, \\ F\nu_x^\perp &\subset T_x M, \quad G\nu_x^\perp \subset T_x M, \quad H\nu_x^\perp \subset T_x M \end{aligned}$$

at each point x in M , then M is called a *QR-submanifold of r QR-dimension*, where ν^\perp denotes the complementary orthogonal distribution to ν in TM^\perp ([1, 10]). Real hypersurfaces, which are typical examples of QR -submanifold with $r = 0$, have been investigated by many authors ([2, 9, 10, 11, 12, 14]) in connection with the shape operator and the induced almost contact 3-structure (for definition, see [7]).

In this paper we shall study QR -submanifolds of maximal QR -dimension isometrically immersed in a quaternionic projective space $QP^{(n+p)/4}$ and prove the following theorem which is an extension of theorem proved in [12, Theorem 10] to the case of QR -submanifolds with maximal QR -dimension :

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THEOREM 1.1. *Let M be an n -dimensional QR -submanifold of maximal QR -dimension in a quaternionic projective space $QP^{(n+p)/4}$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and the equalities appeared in (3.4) hold on M , then M is locally isometric to*

$$\pi(S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2)) \quad (r_1^2 + r_2^2 = 1)$$

for some integers n_1, n_2 with $4n_1 + 4n_2 = n - 3$, where π is the Hopf fibration $S^{n+4}(1) \rightarrow QP^{(n+1)/4}$.

Next, under the same assumptions as in Theorem 1.1, we bring into use an integral formula ([15]) which leads to an inequality among the Ricci curvature, the scalar curvature and the mean curvature of M . Using this inequality, we provide the following theorem as quaternionic analogue to theorem given in [4, Theorem 4.2]. Theorem 1.2 is also a generalization of Lawson's result ([11, Theorem 4]) for higher codimension, but avoiding the condition of minimality:

THEOREM 1.2. *Let M be an n -dimensional compact QR -submanifold of $(p - 1)$ QR -dimension in a quaternionic projective space $QP^{(n+p)/4}$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and the inequality*

$$\frac{1}{3}\{\text{Ric}(U, U) + \text{Ric}(V, V) + \text{Ric}(W, W)\} + \rho - n^2\|\mu\|^2 \geq (n^2 + 8n - 1),$$

then M is isometric to

$$\pi(S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2}))$$

for some integers n_1, n_2 with $4n_1 + 4n_2 = n - 3$, where π is the Hopf fibration $S^{n+4} \rightarrow QP^{(n+1)/4}$.

2. Preliminaries

Let \overline{M} be a real $(n + p)$ -dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting with tensor fields of type $(1,1)$ over \overline{M} satisfying the following conditions (a), (b), and (c) :

- (a) In any coordinate neighborhood \overline{U} , there is a local basis $\{F, G, H\}$ of V such that

$$(2.1) \quad \begin{cases} F^2 = -I, & G^2 = -I, & H^2 = -I, \\ FG = -GF = H, & GH = -HG = F, & HF = -FH = G. \end{cases}$$

- (b) There is a Riemannian metric g which is hermite with respect to all of F , G and H .
- (c) For the Riemannian connection $\bar{\nabla}$ with respect to g

$$(2.2) \quad \begin{pmatrix} \bar{\nabla} F \\ \bar{\nabla} G \\ \bar{\nabla} H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix}$$

where p , q and r are local 1-forms defined in \bar{U} . Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in \bar{U} ([5, 6]).

For canonical local bases $\{F, G, H\}$ and $\{F', G', H'\}$ of V in coordinate neighborhoods \bar{U} and \bar{U}' , it follows that in $\bar{U} \cap \bar{U}'$

$$\begin{pmatrix} F' \\ G' \\ H' \end{pmatrix} = (s_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \quad (x, y = 1, 2, 3)$$

where s_{xy} are local differentiable functions with $(s_{xy}) \in SO(3)$ as a consequence of (2.1). As is well known ([5, 6]), every quaternionic Kähler manifold is orientable.

Now let M be an n -dimensional QR-submanifold of maximal QR-dimension, that is, of $(p-1)$ QR-dimension isometrically immersed in $\vee M$. Then by definition there is a unit normal vector field ξ such that $\nu_x^\perp = \text{Span}\{\xi\}$ at each point x in M . We set

$$(2.3) \quad U = -F\xi, \quad V = -G\xi, \quad W = -H\xi.$$

Denoting by \mathcal{D}_x the maximal quaternionic invariant subspace

$$T_x M \cap FT_x M \cap GT_x M \cap HT_x M$$

of $T_x M$, we have $\mathcal{D}_x^\perp \supset \text{Span}\{U, V, W\}$, where \mathcal{D}_x^\perp means the complementary orthogonal subspace to \mathcal{D}_x in $T_x M$. But, using (2.1), we can prove that $\mathcal{D}_x^\perp = \text{Span}\{U, V, W\}$ ([1, 10]). Thus we have

$$T_x M = \mathcal{D}_x \oplus \text{Span}\{U, V, W\}, \quad \forall x \in M,$$

which together with (2.1) and (2.3) implies

$$FT_x M, GT_x M, HT_x M \subset T_x M \oplus \text{Span}\{\xi\}.$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha\}_{\alpha=1,\dots,p}$ ($\xi_1 := \xi$) of normal vectors to M , we have

$$(2.4) \quad \begin{aligned} FX &= \phi X + u(X)\xi, & GX &= \psi X + v(X)\xi, \\ HX &= \theta X + w(X)\xi, \end{aligned}$$

$$(2.5) \quad \begin{aligned} F\xi_\alpha &= -U_\alpha + P_1\xi_\alpha, & G\xi_\alpha &= -V_\alpha + P_2\xi_\alpha, \\ H\xi_\alpha &= -W_\alpha + P_3\xi_\alpha \end{aligned}$$

($\alpha = 1, \dots, p$). Then it is easily seen that $\{\phi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on $T_x M$ and $T_x M^\perp$, respectively. Moreover, the hermitian property of $\{F, G, H\}$ implies

$$(2.6) \quad \begin{aligned} g(X, \phi U_\alpha) &= -u(X)g(N_1, P_1\xi_\alpha), \\ g(X, \psi V_\alpha) &= -v(X)g(N_1, P_2\xi_\alpha), \\ g(X, \theta W_\alpha) &= -w(X)g(N_1, P_3\xi_\alpha), \quad \alpha = 1, \dots, p, \\ g(U_\alpha, U_\beta) &= \delta_{\alpha\beta} - g(P_1\xi_\alpha, P_1\xi_\beta), \\ g(V_\alpha, V_\beta) &= \delta_{\alpha\beta} - g(P_2\xi_\alpha, P_2\xi_\beta), \\ g(W_\alpha, W_\beta) &= \delta_{\alpha\beta} - g(P_3\xi_\alpha, P_3\xi_\beta), \quad \alpha, \beta = 1, \dots, p. \end{aligned}$$

Also, from the hermitian properties

$$\begin{aligned} g(FX, \xi_\alpha) &= -g(X, F\xi_\alpha), & g(GX, \xi_\alpha) &= -g(X, G\xi_\alpha), \\ g(HX, \xi_\alpha) &= -g(X, H\xi_\alpha), \end{aligned}$$

it follows that

$$g(X, U_\alpha) = u(X)\delta_{1\alpha}, \quad g(X, V_\alpha) = v(X)\delta_{1\alpha}, \quad g(X, W_\alpha) = w(X)\delta_{1\alpha}$$

and hence

$$(2.8) \quad \begin{aligned} g(U_1, X) &= u(X), & g(V_1, X) &= v(X), & g(W_1, X) &= w(X), \\ U_\alpha &= 0, & V_\alpha &= 0, & W_\alpha &= 0, \quad \alpha = 2, \dots, p. \end{aligned}$$

On the other hand, comparing (2.3) and (2.5) with $\alpha = 1$, we have $U_1 = U$, $V_1 = V$, $W_1 = W$, which together with (2.3) and (2.8) implies

$$(2.9) \quad \begin{aligned} g(U, X) &= u(X), & g(V, X) &= v(X), & g(W, X) &= w(X), \\ u(U) &= 1, & v(V) &= 1, & w(W) &= 1. \end{aligned}$$

Here and in the sequel we use the notations U, V, W instead of U_1, V_1, W_1 .

Next, applying F to the first equation of (2.4) and using (2.5), (2.8), and (2.9), we have

$$\phi^2 X = -X + u(X)U, \quad u(X)P_1\xi = -u(\phi X)\xi.$$

Similarly we have

$$(2.10) \quad \begin{aligned} \phi^2 X &= -X + u(X)U, & \psi^2 X &= -X + v(X)V, \\ \theta^2 X &= -X + w(X)W, \end{aligned}$$

$$(2.11) \quad \begin{aligned} u(X)P_1\xi &= -u(\phi X)\xi, & v(X)P_2\xi &= -v(\psi X)\xi, \\ w(X)P_3\xi &= -w(\theta X)\xi, \end{aligned}$$

from which, taking account of the skew-symmetry of P_1, P_2 , and P_3 and using (2.6) with $\alpha = 1$, we also have

$$(2.12) \quad \begin{aligned} u(\phi X) &= 0, & v(\psi X) &= 0, & w(\theta X) &= 0, \\ \phi U &= 0, & \psi V &= 0, & \theta W &= 0, \\ P_1\xi &= 0, & P_2\xi &= 0, & P_3\xi &= 0. \end{aligned}$$

So (2.5) can be rewritten of the form

$$(2.13) \quad \begin{aligned} F\xi &= -U, & G\xi &= -V, & H\xi &= -W, \\ F\xi_\alpha &= P_1\xi_\alpha = \sum_{\beta=2}^p P_{1\alpha\beta}\xi_\beta, & G\xi_\alpha &= P_2\xi_\alpha = \sum_{\beta=2}^p P_{2\alpha\beta}\xi_\beta, \\ HN_\alpha &= P_3N_\alpha = \sum_{\beta=2}^p P_{3\alpha\beta}\xi_\beta \quad (\alpha = 2, \dots, p). \end{aligned}$$

Applying G and H to the first equation of (2.4) and using (2.1), (2.4), and (2.13), we have

$$\begin{aligned} \theta X + w(X)\xi &= -\psi(\phi X) - v(\phi X)\xi + u(X)V, \\ \psi X + v(X)\xi &= \theta(\phi X) + w(\phi X)\xi - u(X)W, \end{aligned}$$

and consequently

$$(2.14) \quad \begin{aligned} \psi(\phi X) &= -\theta X + u(X)V, & v(\phi X) &= -w(X), \\ \theta(\phi X) &= \psi X + u(X)W, & w(\phi X) &= v(X). \end{aligned}$$

Similarly the other equations of (2.4) yield

$$(2.15) \quad \begin{aligned} \phi(\psi X) &= \theta X + v(X)U, & u(\psi X) &= w(X), \\ \theta(\psi X) &= -\phi X + v(X)W, & w(\psi X) &= -u(X), \end{aligned}$$

$$(2.16) \quad \begin{aligned} \phi(\theta X) &= -\psi X + w(X)U, & u(\theta X) &= -v(X), \\ \psi(\theta X) &= \phi X + w(X)V, & v(\theta X) &= u(X). \end{aligned}$$

From the first three equations of (2.13), we can also easily obtain

$$(2.17) \quad \begin{aligned} \psi U &= -W, & v(U) &= 0, & \theta U &= V, & w(U) &= 0, \\ \phi V &= W, & u(V) &= 0, & \theta V &= -U, & w(V) &= 0, \\ \phi W &= -V, & u(W) &= 0, & \psi W &= U, & v(W) &= 0. \end{aligned}$$

The equations (2.8)–(2.10), (2.12), and (2.14)–(2.17) tell us that M admits the so-called almost contact 3-structure (for definition, see [7]) and consequently $n = 4m + 3$ for some integer m .

Now let ∇ be the Levi-Civita connection on M and let ∇^\perp the normal connection induced from $\overline{\nabla}$ in the normal bundle TM^\perp of M . Then Gauss and Weingarten formulae are given by

$$(2.18) \quad \overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.19) \quad \overline{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla_X^\perp \xi_\alpha, \quad \alpha = 1, \dots, p$$

for vector fields X, Y tangent to M . Here h denotes the second fundamental form and A_α the shape operator corresponding to ξ_α . They are related by

$$h(X, Y) = \sum_{\alpha=1}^p g(A_\alpha X, Y) \xi_\alpha$$

Furthermore, we put

$$(2.20) \quad \nabla_X^\perp \xi_\alpha = \sum_{\beta=1}^p s_{\alpha\beta}(X) \xi_\beta,$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of ∇^\perp .

Differentiating the first equation of (2.4) covariantly and using (2.2), (2.4), (2.18), and (2.19), we have

$$(2.21) \quad \begin{aligned} (\nabla_Y \phi)X &= r(Y)\psi X - q(Y)\theta X + u(X)A_1 Y - g(A_1 Y, X)U, \\ (\nabla_Y u)X &= r(Y)v(X) - q(Y)w(X) + g(\phi A_1 Y, X). \end{aligned}$$

From the other equations of (2.4) we also have

$$(2.22) \quad \begin{aligned} (\nabla_Y \psi)X &= -r(Y)\phi X + p(Y)\theta X + v(X)A_1Y - g(A_1Y, X)V, \\ (\nabla_Y v)X &= -r(Y)u(X) + p(Y)w(X) + g(\psi A_1Y, X), \end{aligned}$$

$$(2.23) \quad \begin{aligned} (\nabla_Y \theta)X &= q(Y)\phi X - p(Y)\psi X + w(X)A_1Y - g(A_1Y, X)W, \\ (\nabla_Y w)X &= q(Y)u(X) - p(Y)v(X) + g(\theta A_1Y, X). \end{aligned}$$

Next, differentiating the first equation of (2.13) covariantly and using (2.2), (2.13), (2.18), and (2.19), we have

$$(2.24) \quad \begin{aligned} \nabla_Y U &= r(Y)V - q(Y)W + \phi A_1Y, \\ g(A_\alpha U, Y) &= -\sum_{\beta=2}^p s_{1\beta}(Y)P_{1\beta\alpha}, \quad \alpha = 2, \dots, p. \end{aligned}$$

From the other equations of (2.13), we have similarly

$$(2.25) \quad \begin{aligned} \nabla_Y V &= -r(Y)U + p(Y)W + \psi A_1Y, \\ g(A_\alpha V, Y) &= -\sum_{\beta=2}^p s_{1\beta}(Y)P_{2\beta\alpha}, \quad \alpha = 2, \dots, p, \end{aligned}$$

$$(2.26) \quad \begin{aligned} \nabla_Y W &= q(Y)U - p(Y)V + \theta A_1Y, \\ g(A_\alpha W, Y) &= -\sum_{\beta=2}^p s_{1\beta}(Y)P_{3\beta\alpha}, \quad \alpha = 2, \dots, p. \end{aligned}$$

Finally if the ambient manifold $\vee M$ is of constant Q -sectional curvature c , the equations of Gauss and Codazzi are given by

$$(2.27) \quad \begin{aligned} R(X, Y)Z &= \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y \\ &\quad + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &\quad + g(\psi Y, Z)\psi X - g(\psi X, Z)\psi Y - 2g(\psi X, Y)\psi Z \\ &\quad + g(\theta Y, Z)\theta X - g(\theta X, Z)\theta Y - 2g(\theta X, Y)\theta Z\} \\ &\quad + \sum_{\alpha} g(A_\alpha Y, Z)A_\alpha X - \sum_{\alpha} g(A_\alpha X, Z)A_\alpha Y, \end{aligned}$$

$$\begin{aligned}
(2.28) \quad & g((\nabla_X A_1)Y - (\nabla_Y A_1)X, Z) \\
&= \frac{c}{4} \{g(\phi Y, Z)u(X) - g(\phi X, Z)u(Y) - 2g(\phi X, Y)u(Z) \\
&\quad + g(\psi Y, Z)v(X) - g(\psi X, Z)v(Y) - 2g(\psi X, Y)v(Z) \\
&\quad + g(\theta Y, Z)w(X) - g(\theta X, Z)w(Y) - 2g(\theta X, Y)w(Z)\} \\
&\quad + \sum_{\beta} \{g(A_{\beta}X, Z)s_{\beta 1}(Y) - g(A_{\beta}Y, Z)s_{\beta 1}(X)\},
\end{aligned}$$

respectively. Moreover, (2.9), (2.10), and (2.27) yield

$$\begin{aligned}
(2.29) \quad & \text{Ric}(X, Y) \\
&= \frac{c}{4} \{(n+8)g(X, Y) - 3(u(X)u(Y) + v(X)v(Y) + w(X)w(Y))\} \\
&\quad + \sum_{\alpha} \{(\text{tr} A_{\alpha})g(A_{\alpha}X, Y) - g(A_{\alpha}^2 X, Y)\},
\end{aligned}$$

$$(2.30) \quad \rho = \frac{c}{4}(n+9)(n-1) + n^2 \|\mu\|^2 - \sum_{\alpha} \text{tr} A_{\alpha}^2,$$

where Ric and ρ denote the Ricci tensor and the scalar curvature, respectively, and

$$(2.31) \quad \mu = \frac{1}{n} \sum_{\alpha} (\text{tr} A_{\alpha}) \xi_{\alpha}$$

is the mean curvature vector ([3]).

3. Some properties of the shape operator A_1

In this section, let M be an n -dimensional QR -submanifold of maximal QR dimension in a quaternionic space form $\overline{M}^{(n+p)/4}(c)$ with constant Q -sectional curvature c . In what follows, we assume that the distinguished normal vector field $\xi_1 := \xi$ is parallel with respect to the normal connection ∇^{\perp} , that is,

$$\nabla_X^{\perp} \xi = 0$$

for any vector field X tangent to M . Then it follows from (2.20) that $s_{1\beta} = 0$ and consequently (2.24)–(2.26) imply

$$(3.1) \quad A_\alpha U = 0, \quad A_\alpha V = 0, \quad A_\alpha W = 0, \quad \alpha = 2, \dots, p.$$

Moreover, since $s_{1\beta} = 0$, (2.20) yields

$$(3.2) \quad \nabla_X^\perp \xi_\alpha = \sum_{\beta=2}^p s_{\alpha\beta}(X) \xi_\beta, \quad \alpha = 2, \dots, p.$$

From now on we denote by A the shape operator A_1 corresponding to the normal vector $\xi = \xi_1$ and prepare a lemma for later use.

LEMMA 3.1. *Let M be an n -dimensional QR-submanifolds of maximal QR-dimension in a quaternionic space form $\overline{M}^{(n+p)/4}(c)$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and if the equalities*

$$(3.4) \quad \begin{aligned} h(X, \phi Y) &= h(\phi X, Y), \quad h(X, \psi Y) = h(\psi X, Y), \\ h(X, \theta Y) &= h(\theta X, Y), \end{aligned}$$

hold for any vector fields X, Y tangent to M , then

$$(3.5) \quad A\phi = \phi A, \quad A\psi = \psi A, \quad A\theta = \theta A,$$

and $A_\alpha = 0$ for $\alpha = 2, \dots, p$.

Proof. Since $n = 4m + 3$ and $p = 4t + 1$ for some integers m and t , and since the subspace ν is quaternionic invariant (see also (2.13)), we can take a local orthonormal basis $\{\xi, \xi_a, \xi_{a^*}, \xi_{a^{**}}, \xi_{a^{***}}\}_{a=1, \dots, t}$ of normal vectors M such that

$$\begin{aligned} \xi_{1^*} &:= F\xi_1, \dots, \xi_{t^*} := F\xi_t, & \xi_{1^{**}} &:= G\xi_1, \dots, \xi_{t^{**}} := G\xi_t, \\ \xi_{1^{***}} &:= H\xi_1, \dots, \xi_{t^{***}} &:= H\xi_t. \end{aligned}$$

Then we can express the second fundamental form h as

$$\begin{aligned} h(X, Y) &= g(AX, Y)\xi + \sum_{a=1}^t \{g(A_a X, Y)\xi_a + g(A_{a^*} X, Y)\xi_{a^*} \\ &\quad + g(A_{a^{**}} X, Y)\xi_{a^{**}} + g(A_{a^{***}} X, Y)\xi_{a^{***}}\}. \end{aligned}$$

Hence the assumption (3.4) implies

$$\begin{aligned}
 (3.6) \quad & A\phi = \phi A, \quad A\psi = \psi A, \quad A\theta = \theta A, \\
 & A_{a^*}\phi = \phi A_{a^*}, \quad A_{a^*}\psi = \psi A_{a^*}, \quad A_{a^*}\theta = \theta A_{a^*}, \\
 & A_{a^{**}}\phi = \phi A_{a^{**}}, \quad A_{a^{**}}\psi = \psi A_{a^{**}}, \quad A_{a^{**}}\theta = \theta A_{a^{**}}, \\
 & A_{a^{***}}\phi = \phi A_{a^{***}}, \quad A_{a^{***}}\psi = \psi A_{a^{***}}, \quad A_{a^{***}}\theta = \theta A_{a^{***}}.
 \end{aligned}$$

On the other hand,

$$\xi_{a^*} = F\xi_a, \quad \xi_{a^{**}} = G\xi_a, \quad \xi_{a^{***}} = H\xi_a \quad (a = 1, \dots, t)$$

give, respectively,

$$\begin{aligned}
 (3.7) \quad & \bar{\nabla}_X F\xi_a = -A_{a^*}X + \nabla_X^\perp \xi_{a^*}, \\
 & \bar{\nabla}_X G\xi_a = -A_{a^{**}}X + \nabla_X^\perp \xi_{a^{**}}, \\
 & \bar{\nabla}_X H\xi_a = -A_{a^{***}}X + \nabla_X^\perp \xi_{a^{***}}, \quad a = 1, \dots, t.
 \end{aligned}$$

Hence, using (2.2), (2.13), (2.19), and (3.2), it follows from the first equation of (3.7) that

$$\begin{aligned}
 & -A_{a^*}X + \nabla_X^\perp \xi_{a^*} \\
 & = r(X)G(\xi_a) - q(X)H(\xi_a) \\
 & \quad + F(-A_aX + \sum_{b=1}^t \{s_{ab^*}(X)\xi_{b^*} + s_{ab^{**}}(X)\xi_{b^{**}} + s_{ab^{***}}(X)\xi_{b^{***}}\}),
 \end{aligned}$$

from which, taking the tangential part, we can easily obtain

$$(3.8) \quad \phi A_a = A_{a^*}, \quad a = 1, \dots, t.$$

Similarly, from the other equations of (3.7), we have

$$(3.9) \quad \psi A_a = A_{a^{**}}, \quad \theta A_a = A_{a^{***}}, \quad a = 1, \dots, t.$$

Therefore, for any vectors X, Y tangent to M , it is clear from (3.8) that

$$g(A_{a^*}\phi X, Y) = -g(A_a\phi X, \phi Y)$$

and consequently

$$g(A_{a^*}\phi X, Y) = g(A_{a^*}\phi Y, X) = -g(\phi A_{a^*}X, Y),$$

that is,

$$A_{a^*}\phi = -\phi A_{a^*},$$

which and (3.6) imply $\phi A_{a^*} = 0$. Thus we have from (3.8) that $\phi^2 A_a = 0$, which together with (2.10) and (3.1) yields $A_a = 0$. Hence (3.9) yields

$$A_a = 0, \quad A_{a^*} = 0, \quad A_{a^{**}} = 0, \quad A_{a^{***}} = 0. \quad \square$$

4. Codimension reduction and proof of Theorem 1.1

In this section we assume that the ambient manifold is a quaternionic projective space Q of constant Q -sectional curvature 4. Let

$$N_0(x) = \{\eta \in T_x M^\perp : A_\eta = 0\}$$

and let $H_0(x)$ be the maximal quaternionic invariant subspace of $N_0(x)$, that is,

$$H_0(x) = N_0(x) \cap FN_0(x) \cap GN_0(x) \cap HN_0(x).$$

Then Kwon and the first author of this paper [8] have proved the following theorem :

LEMMA 4.1. *Let M be an n -dimensional real submanifold of an $(n+p)$ -dimensional quaternionic projective space $QP^{(n+p)/4}$. If the orthogonal complement $H_1(x)$ of $H_0(x)$ in TM^\perp is invariant under the parallel translation with respect to the normal connection and q is the constant dimension of $H_1(x)$, then there exists a real $(n+q)$ -dimensional totally geodesic quaternionic projective space $QP^{(n+p)/4}$ such that $M \subset QP^{(n+p)/4}$.*

In our cases, $N_0(x) = \text{Span}\{\xi_2(x), \dots, \xi_p(x)\}$. In fact, as a consequence of Lemma 3.1, $A_\alpha = 0$ for $\alpha = 2, \dots, p$. Hence

$$\text{Span}\{\xi_2(x), \dots, \xi_p(x)\} \subset N_0(x).$$

On the other hand, for any η in $N_0(x)$, we can put $\eta = \sum_{\alpha=1}^p \lambda^\alpha \xi_\alpha$. But

$$A_\eta = \sum_{\alpha=1}^p \lambda^\alpha A_\alpha = \lambda^1 A_1 = 0$$

since $A_\alpha = 0$ for $\alpha = 2, \dots, p$. Hence $\lambda^1 = 0$ and consequently

$$\eta \in \text{Span}\{\xi_2(x), \dots, \xi_p(x)\}.$$

Hence we have

$$N_0(x) = H_0(x) = \text{Span}\{\xi_2(x), \dots, \xi_p(x)\}.$$

Thus $H_1(x) = \text{Span}\{\xi(x)\}$ and so our assumption yields that $H_1(x)$ is invariant under parallel translation with respect to the normal connection. Therefore we can apply Lemma 4.1 and obtain the following theorem.

THEOREM 4.2. *Let M be an n -dimensional QR -submanifold of maximal QR -dimension in a quaternionic projective space $QP^{(n+p)/4}$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and the equalities appeared in (3.4) hold on M , then there exists a real $(n+1)$ -dimensional totally geodesic quaternionic projective space $QP^{(n+1)/4}$ such that $M \subset QP^{(n+1)/4}$.*

Proof of Theorem 1.1. From now on we shall give the proof of the theorem stated in Section 1. By means of Theorem 4.2 the submanifold M can be regarded as a real hypersurface of $QP^{(n+1)/4}$ which is totally geodesic in $QP^{(n+p)/4}$. Tentatively we denote $QP^{(n+1)/4}$ by M' and by i_1 the immersion of M into M' and by i_2 the totally geodesic immersion of M' into $QP^{(n+p)/4}$. Then it is clear from (2.18) that

$$(4.1) \quad \nabla'_{i_1 X} i_1 Y = i_1 \nabla_X Y + h'(X, Y) = i_1 \nabla_X Y + g(A'X, Y)\xi',$$

where ∇' is the induced connection on M' from that of $QP^{(n+p)/4}$, h' the second fundamental form of M in M' and A' the corresponding shape operator to a unit normal vector field ξ' to M in M' . Since $i = i_2 \circ i_1$ and M' is totally geodesic in $QP^{(n+p)/4}$, we can easily see that (2.18) and (4.1) imply

$$(4.2) \quad \xi = i_2 \xi', \quad A_1 = A'.$$

Since M' is a quaternionic invariant submanifold of $QP^{(n+p)/4}$, for any vector field X tangent to M

$$(4.3) \quad Fi_2 X = i_2 F'X, \quad Gi_2 X = i_2 G'X, \quad Hi_2 X = i_2 H'X$$

are valid, where $\{F', G', H'\}$ is the induced quaternionic Kähler structure on M' . Thus it follows from (4.2), (4.3), and the first equation of (2.4) that

$$FiX = Fi_2 \circ i_1 X = i_2 F' i_1 X = i_2 (i_1 \phi' X + u'(X)\xi') = i\phi' X + u'(X)\xi$$

for any vector field X tangent to M . Comparing this equation with the first equation of (2.4), we have $\phi = \phi'$ and $u = u'$. Similarly we have

$$\phi = \phi', \quad \psi = \psi', \quad \theta = \theta', \quad u = u', \quad v = v', \quad w = w',$$

which and Lemma 3.1 imply

$$A'\phi' = \phi'A', \quad A'\psi' = \psi'A', \quad A'\theta' = \theta'A'.$$

Now applying the theorem proved in [12, Theorem 10, pp. 57] by the first author of this paper, we may conclude that M is locally isometric to

$$\pi(S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2)) \quad (r_1^2 + r_2^2 = 1)$$

for some n_1, n_2 with $4n_1 + 4n_2 = n - 3$, where π is the Hopf fibration $S^{n+4} \rightarrow QP^{(n+1)/4}$. \square

5. An integral formula for the model space $\pi(S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2}))$

Let M be an n -dimensional QR-submanifold of maximal QR dimension in a quaternionic space form $\vee M^{(n+p)/4}(c)$ with constant Q-sectional curvature c . We put

$$T := \nabla_U U + \nabla_V V + \nabla_W W - (\operatorname{div} U)U - (\operatorname{div} V)V - (\operatorname{div} W)W$$

and take an orthonormal basis $\{U, V, W, e_a, e_{a^*}, e_{a^{**}}, e_{a^{***}}\}_{a=1, \dots, m}$ of tangent vectors to M such that

$$e_{a^*} := \phi e_a, \quad e_{a^{**}} := \psi e_a, \quad e_{a^{***}} = \theta e_a.$$

Then it follows from (2.8), (2.9), (2.12), and (2.24)–(2.26) that

$$(5.1) \quad T = \phi AU + \psi AV + \theta AW,$$

$$(5.2) \quad g(T, U) = g(T, V) = g(T, W) = 0.$$

Here and in the sequel we also denote by A the shape operator A_1 corresponding to the distinguished normal vector $\xi = \xi_1$. We note that T is a global vector field defined on M . For later use we compute

$$\operatorname{div} T = \sum_{i=1}^n g(e_i, \nabla_{e_i} T).$$

Differentiating (5.1) covariantly and using (2.21)–(2.26), we have

$$\begin{aligned} \nabla_X T &= \{u(AU) + v(AV) + w(AW)\}AX \\ &\quad - g(A^2U, X)U - g(A^2V, X)V - g(A^2W, X)W \\ &\quad + \phi A\phi AX + \psi A\psi AX + \theta A\theta AX \\ &\quad + \phi(\nabla_X A)U + \psi(\nabla_X A)V + \theta(\nabla_X A)W, \end{aligned}$$

from which, taking account of (2.8)–(2.10), (2.12), and (2.14)–(2.17),

$$\begin{aligned}
 \operatorname{div} T &= \{u(AU) + v(AV) + w(AW)\} \operatorname{tr} A - u(A^2U) - v(A^2V) - w(A^2W) \\
 &\quad + \operatorname{tr}(\phi A \phi A) + \operatorname{tr}(\psi A \psi A) + \operatorname{tr}(\theta A \theta A) - g((\nabla_V A)W - (\nabla_W A)V, U) \\
 &\quad - g((\nabla_W A)U - (\nabla_U A)W, V) - g((\nabla_U A)V - (\nabla_V A)U, W) \\
 &\quad - \sum_{a=1}^m \{g((\nabla_{e_a} A)e_{a^*} - (\nabla_{e_{a^*}} A)e_a + (\nabla_{e_{a^{**}}} A)e_{a^{***}} \\
 &\quad - (\nabla_{e_{a^{***}}} A)e_{a^{**}}, U) + g((\nabla_{e_a} A)e_{a^{**}} - (\nabla_{e_{a^{**}}} A)e_a + (\nabla_{e_{a^{***}}} A)e_{a^*} \\
 &\quad - (\nabla_{e_{a^*}} A)e_{a^{***}}, V) + g((\nabla_{e_a} A)e_{a^{***}} - (\nabla_{e_{a^{***}}} A)e_i + (\nabla_{e_{a^*}} A)e_{a^{**}} \\
 &\quad - (\nabla_{e_{a^{**}}} A)e_{a^*}, W)\},
 \end{aligned}$$

or equivalently

$$\begin{aligned}
 (5.3) \quad \operatorname{div} T &= \{u(AU) + v(AV) + w(AW)\} \operatorname{tr} A \\
 &\quad - u(A^2U) - v(A^2V) - w(A^2W) + \frac{3(n-3)}{4}c \\
 &\quad + \operatorname{tr}(\phi A \phi A) + \operatorname{tr}(\psi A \psi A) + \operatorname{tr}(\theta A \theta A)
 \end{aligned}$$

because of (2.28) with $s_{\beta 1} = 0$. On the other hand, using (2.8)–(2.10) and (2.12), we can easily obtain that

$$\begin{aligned}
 &\|\phi A - A\phi\|^2 + \|\psi A - A\psi\|^2 + \|\theta A - A\theta\|^2 \\
 &= 6 \operatorname{tr} A^2 - 2\{u(A^2U) + v(A^2V) + w(A^2W)\} \\
 &\quad + 2\{\operatorname{tr}(\phi A \phi A) + \operatorname{tr}(\psi A \psi A) + \operatorname{tr}(\theta A \theta A)\},
 \end{aligned}$$

which together with (5.3) yields

$$\begin{aligned}
 (5.4) \quad \operatorname{div} T &= \frac{1}{2} \{\|\phi A - A\phi\|^2 + \|\psi A - A\psi\|^2 + \|\theta A - A\theta\|^2\} \\
 &\quad + \frac{3(n-3)}{4}c - 3 \operatorname{tr} A^2 + \operatorname{tr} A \{u(AU) + v(AV) + w(AW)\}.
 \end{aligned}$$

On the other hand, it follows from (2.29)–(2.31) and (3.1) that

$$\begin{aligned}
 \operatorname{Ric}(U, U) &= \frac{c}{4}(n+5) + (\operatorname{tr} A)u(AU) - u(A^2U), \\
 \operatorname{Ric}(V, V) &= \frac{c}{4}(n+5) + (\operatorname{tr} A)v(AV) - v(A^2V), \\
 \operatorname{Ric}(W, W) &= \frac{c}{4}(n+5) + (\operatorname{tr} A)w(AW) - w(A^2W),
 \end{aligned}$$

$$\operatorname{tr} A^2 = -\rho + \frac{c}{4}(n+9)(n-1) + n^2 \|\mu\|^2 - \sum_{\alpha=2}^p \operatorname{tr} A_{\alpha}^2,$$

from which and (5.4), we have

$$\begin{aligned}
 \operatorname{div} T &= \frac{1}{2} \{ \|\phi A - A\phi\|^2 + \|\psi A - A\psi\|^2 + \|\theta A - A\theta\|^2 \} \\
 &\quad + \operatorname{Ric}(U, U) + \operatorname{Ric}(V, V) + \operatorname{Ric}(W, W) \\
 (5.4)' \quad &\quad + 3 \left\{ \rho - \frac{c}{4} (n^2 + 8n - 1) - n^2 \|\mu\|^2 \right\} + 3 \sum_{\alpha=2}^p \operatorname{tr} A_\alpha^2 \\
 &\quad + \|AU\|^2 + \|AV\|^2 + \|AW\|^2.
 \end{aligned}$$

Thus we have

LEMMA 5.1. *Let M be an n -dimensional compact QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and the inequality*

$$\frac{1}{3} \{ \operatorname{Ric}(U, U) + \operatorname{Ric}(V, V) + \operatorname{Ric}(W, W) \} + \rho - n^2 \|\mu\|^2 \geq n^2 + 8n - 1,$$

holds on M , then

$$A\phi = \phi A \quad A\psi = \psi A, \quad A\theta = \theta A,$$

$$A_\alpha = 0, \quad \alpha = 2, \dots, p$$

and $AU = AV = AW = 0$.

Proof. Applying Stokes's theorem to (5.4)', we can easily obtain the conclusions. \square

Combining Lemma 4.1 and Lemma 5.1, we have

THEOREM 5.2. *Let M be an n -dimensional compact QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and the inequality*

$$\frac{1}{3} \{ \operatorname{Ric}(U, U) + \operatorname{Ric}(V, V) + \operatorname{Ric}(W, W) \} + \rho - n^2 \|\mu\|^2 \geq n^2 + 8n - 1,$$

holds on M , then there exists an $(n+1)$ -dimensional totally geodesic quaternionic projective space $QP^{(n+1)/4}$ such that $M \subset QP^{(n+1)/4}$

Proof. By means of Lemma 5.1

$$A_\alpha = 0, \quad \alpha = 2, \dots, p.$$

As shown in the proof of Theorem 4.2, applying Lemma 4.1, we may conclude that there exists an $(n+1)$ -dimensional totally geodesic quaternionic projective space $QP^{(n+1)/4}$ such that $M \subset QP^{(n+1)/4}$. \square

Proof of Theorem 1.2. From now on we shall give the proof of Theorem 1.2 stated in Section 1. By means of Theorem 5.2 the submanifold M can be regarded as a real hypersurface of $QP^{(n+1)/4}$ which is totally geodesic in $QP^{(n+1)/4}$. By the same method as in the proof of Theorem 1.1, we can easily see that

$$A'\phi' = \phi'A', \quad A'\psi' = \psi'A', \quad A'\theta' = \theta'A',$$

where A' is the shape operator to a unit normal vector field ξ' to M in $QP^{(n+1)/4}$ and $\{\phi', \psi', \theta'\}$ denote the almost contact 3-structure induced on M from the quaternionic Kähler structure on $QP^{(n+1)/4}$. Now, applying the theorem proved in [12, Theorem 10, pp. 57] by the first author of this paper, we may conclude that M is isometric to

$$\pi(S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2)) \quad (r_1^2 + r_2^2 = 1)$$

for some n_1, n_2 with $4n_1 + 4n_2 = n - 3$, that is, M is a tube over a totally geodesic submanifold QP^{n_1} . Moreover, $A'U = A'V = A'W = 0$ implies that the radius of the tube is $\frac{\pi}{4}$ (for details, see [2]). Thus M is isometric to

$$\pi(S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2})) \quad (4n_1 + 4n_2 = n - 3). \quad \square$$

REMARK. We consider special generalized Clifford tori in

$$S^{n+4} := \{(x_1, \dots, x_{n+5}) \in \mathcal{R}^{(n+5)} \mid \sum_{i=1}^{n+5} x_i^2 = 1\}$$

defined by

$$\begin{aligned} & S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2}) \\ &= \{(x_1, \dots, x_{n+5}) \in S^{n+4} \mid \sum_{i=1}^{4n_1+4} x_i^2 = \frac{1}{2}, \sum_{i=4n_1+5}^{n+5} x_i^2 = \frac{1}{2}\}, \end{aligned}$$

where $4n_1 + 4n_2 = n - 3$ and $n = 4s + 3$ for some integer s . Then, since $S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2})$ is a real hypersurface of S^{n+4} , its shape operator \tilde{A} is of the form

$$\tilde{A} = \text{diag}(1, -1)$$

for suitable orthonormal basis. The multiplicities of 1 and -1 are $4n_1 + 3$ and $4n_2 + 3$, respectively ([13]). By choosing the spheres so that they lie in quaternionic subspace, we have fibrations

$$S^3 \rightarrow S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2}) \rightarrow M_{n_1, n_2}^Q$$

compatible with the Hopf fibration $\pi : S^{n+4} \rightarrow QP^{(n+1)/4}$, where we put

$$M_{n_1, n_2}^Q = \pi(S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2})).$$

In this case we can easily see that the geodesic distance from QP^{n_1} to M_{n_1, n_2}^Q is $\frac{\pi}{4}$ and that its principal curvatures are 1, -1 , and 0 with multiplicities $n - 3 - 4n_1, 4n_1, 3$, respectively (for details, see [2]). Furthermore, let ξ be a unit normal vector field of M_{n_1, n_2}^Q and let $\{F, G, H\}$ be the canonical quaternionic Kähler structure of $QP^{(n+1)/4}$. Then

$$U = -F\xi, \quad V = -G\xi, \quad W = -H\xi$$

are principal vectors corresponding to the principal curvature 0, that is

$$AU = 0, \quad AV = 0, \quad AW = 0,$$

where A denote the shape operator of M_{n_1, n_2}^Q in $QP^{(n+1)/4}$. Applying (2.29)–(2.31) to the real hypersurface M_{n_1, n_2}^Q , we obtain

$$\begin{aligned} \text{tr}A &= n - 3 - 8n_1, \quad \text{tr}A^2 = n - 3, \\ \text{Ric}(U, U) + \text{Ric}(V, V) + \text{Ric}(W, W) &= 3(n + 5), \\ \rho &= n^2 + 7n - 6 + (n - 3 - 8n_1)^2. \end{aligned}$$

Hence, for M_{n_1, n_2}^Q , we have

$$\frac{1}{3}\{\text{Ric}(U, U) + \text{Ric}(V, V) + \text{Ric}(W, W)\} + \rho - n^2\|\mu\|^2 = n^2 + 8n - 1.$$

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