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QUADRATIC FUNCTIONALS AND SMALL BALL PROBABILITIES FOR THE *m*-FOLD INTEGRATED BROWNIAN MOTION

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Let the Gaussian process $X_m(t)$ be the m-fold integrated Brownian motion for positive integer m. The Laplace transform of the quadratic functional of $X_m(t)$ is found by using an appropriate self-adjoint integral operator. The result is then used to show the power of a general connection between small ball probabilities for the Gaussian process. The connection is discovered by introducing an independent random shift. The interplay between our results and the principal eigenvalues for nonuniform elliptic generators on an unbounded domain is also discussed.

1. Introduction. Let $W(t)$, $t \geq 0$, be the standard Brownian motion starting at 0. Denote by $X_0(t) = W(t)$ and

$$X_m(t) = \int_0^t X_{m-1}(s)\,ds, \qquad t \geq 0,\ m \geq 1,$$

the m-fold integrated Brownian motion for positive integer m. Upon integration by parts, we also have the representation

$$X_m(t) = \frac{1}{m!}\int_0^t (t-s)^m\,dW(s), \quad m \geq 0.$$

The Gaussian process $X_m(t)$ has been studied from various points of view in Shepp (1966), Wahba (1978), Cheleyat-Maurel and Elie (1981), Lachal (1997a, 1997b, 2001) and Lin (2001). It is clear that the $\mathbb{R}^{m+1}$-valued process $(X_0(t), X_1(t), \dots, X_m(t))$ is Markov and the transition density can be found easily by computing the expectation and covariances of the Gaussian process involved; see McKean (1963) for $m = 1$ and Lachal (1997) for $m \geq 1$. Its generator is given in (5.1). In particular, when $m = 1$, the process $(X_0(t), X_1(t)) = (W(t), \int_0^t W(s)\,ds)$ is often called the Kolmogorov diffusion since its study was apparently initiated by Kolmogorov (1934); see Lachal (1998) and Groeneboom, Jongbloed and Wellner (1999) for other related works and references in this case.

Other significant use of X_m, as $m \to \infty$, can be found in Li and Shao (2002) where a limiting representation is given for the decay exponent of the probability that a random polynomial has no real root.

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This work is concerned with the quadratic functionals of the real process $X_m(t)$ as well as the asymptotic behavior of the small ball probability. In particular, we developed a method for estimating the small ball probability such as $\mathbb{P}(\sup_{0\le t\le 1}|X_m(t)|\le\varepsilon)$ as $\varepsilon\to 0$ via the appropriate quadratic functionals. This is our primary motivation for studying the quadratic functionals for $X_m(t)$, besides their own importance and various connections discussed in Section 5.

Another motivation for studying the quadratic functionals and small ball probabilities for $X_m(t)$ originated in a study of Khoshnevisan and Shi (1998) for the lim inf behavior of the integrated Brownian motion $X_1(t)$. In particular, they showed that

$$\mathbb{E}\exp\biggl(-\frac{\theta^2}{2}\int_0^1 X_1^2(s)\,ds\biggr)=\biggl(\frac{2}{\cosh^2\sqrt{\theta/2}+\cos^2\sqrt{\theta/2}}\biggr)^{1/2} \tag{1.1}$$

and

$$\lim_{\varepsilon\to 0}\varepsilon^{2/3}\log\mathbb{P}\Bigl(\sup_{0\le t\le 1}|X_1(t)|\le\varepsilon\Bigr)=-\kappa_1, \tag{1.2}$$

with

$$3/8\le\kappa_1\le 2c^2, \tag{1.3}$$

where the constant $c>0$ satisfies

$$\lim_{\varepsilon\to 0}\varepsilon^{1/3}\log\mathbb{P}\Bigl(\sup_{0\le t\le 1}|S(t)|\le\varepsilon\Bigr)=-c \tag{1.4}$$

and $S(t)$ is a symmetric $(1/3)$-stable process.

In this paper, we first find the Laplace transform of the quadratic functional of $X_m(t)$, that is,

$$\mathbb{E}\exp\biggl(-\lambda\int_0^1 X_m^2(t)\,dt\biggr),\qquad m\ge 0, \tag{1.5}$$

for any $\lambda>0$. This may look easy and it seems possible to find it by using the standard Karhunen–Loève expansion for Gaussian quadratic functionals. Indeed, we can find (1.1) and the well-known case $m=0$,

$$\mathbb{E}\exp\biggl\{-\lambda\int_0^1 W(s)^2\,ds\biggr\}=\bigl(\cosh\sqrt{2\lambda}\bigr)^{-1/2}$$

by the Karhunen–Loève expansion. However, we could not carry out the detailed and complicated calculations along that line since the eigenvalues cannot be found explicitly in a reasonable form. In fact, we see from our formula for (1.5) given in Theorem 2.1 that the two cases $m=0$ and $m=1$ are indeed very special and hence partially explain why the standard Karhunen–Loève expansion approach does not work well in the general case. On the other hand, the proof of (1.1) given in Khoshnevisan and Shi (1998) is based on a stochastic Fubini argument and then some special facts on the rotation invariance of Brownian motion together with the explicit formula

$$\mathbb{E}\exp\left\{i\lambda\int_0^1 W(t)W(1-t)\,dt\right\} = \left(\frac{1+i}{\cosh\sqrt{\lambda}+i\cos\sqrt{\lambda}}\right)^{1/2} \tag{1.6}$$

given in Klyachko and Solodyannikov (1987). In our evaluation of (1.5) given in Theorem 2.1, we begin with a stochastic Fubini argument that we have learned from Donati-Martin and Yor (1991); see also Yor (1992) and Khoshnevisan and Shi (1998). Then we analyze an associated self-adjoint operator in a way different from that used in Klyachko and Solodyannikov (1987) for (1.6). All the difficulties we encountered are due to the inherent difficulties of the problem.

Second, based on the exact formula for (1.5) given in Theorem 2.1, we obtain the following asymptotic result. The equivalence follows from de Bruijn's exponential Tauberian theorem given in [Bingham, Goldie and Teugel (1987), Theorem 4.12.9]; see also Section 3.3 in the survey of Li and Shao (2001).

THEOREM 1.1. *For each integer $m \ge 0$,*

$$\begin{aligned}\lim_{\lambda\to\infty}\lambda^{-1/(2m+2)}\log\mathbb{E}\exp\left\{-\lambda\int_0^1 X_m^2(t)\,dt\right\}\\ = -2^{-(2m+1)/(2m+2)}\left(\sin\frac{\pi}{2m+2}\right)^{-1}\end{aligned} \tag{1.7}$$

and thus, equivalently, as $\varepsilon\to 0$,

$$\begin{aligned}&\log\mathbb{P}\left(\int_0^1 X_m^2(t)\,dt \le \varepsilon^2\right)\\ &\quad\sim -2^{-1}(2m+1)\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-(2m+2)/(2m+1)}\varepsilon^{-2/(2m+1)}.\end{aligned} \tag{1.8}$$

Note that by the scaling relationship $X_m(ct) = c^{(2m+1)/2}X_m(t)$ in law for any $c>0$, we have, for any $\lambda>0$,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{-\lambda\int_0^t X_m^2(s)\,ds\right\}\\ = -2^{-(2m+1)/(2m+2)}\left(\sin\frac{\pi}{2m+2}\right)^{-1}\lambda^{1/(2m+2)}.\end{aligned}$$

The estimates given in (1.8) are small ball probabilities for the Gaussian process $X_m(t)$ under the L_2-norm. More generally, the small ball probability (or small deviation) studies the behavior of

$$\log\mathbb{P}(\|X\|\le\varepsilon) \tag{1.9}$$

for a random process under various norms $\|\cdot\|$ as $\varepsilon \to 0$. In the last few years, there has been considerable progress on the small ball estimate for Gaussian processes. As it was established in Kuelbs and Li (1993) [see also Li and Linde (1999) for further developments], the behavior of (1.9) for a Gaussian random element X is determined up to a constant by the metric entropy of the unit ball of the reproducing kernel Hilbert space associated with X and vice versa. For other connections and applications of small ball probabilities, we refer readers to a recent survey paper by Li and Shao (2001). In addition, various connections and interplays between our results and the principal eigenvalues are discussed in Section 5 for degenerated (nonuniform elliptic) generators on an unbounded domain. This provides a concrete and relatively simple example that is not covered by general theory as far as we know. We hope this special example, an $\mathbb{R}^{m+1}$-valued Gaussian and Markov process, will attract some interest in the direction of this paper.

The third and probably the most important contribution in this paper is the following general connection between small ball probabilities. It can be used to estimate small ball probabilities under any norm via a relatively easier L_2-norm estimate.

Let X and Y be any two centered Gaussian random vectors in a separable Banach space E with norm $\|\cdot\|$. We recall the definition of the reproducing kernel Hilbert space of $\mu = \mathcal{L}(X)$. Since $\int f^2(x)\mu(dx) < +\infty$ for each $f \in E^*$, where E^* is the topological dual of E, one can define the map $\phi : E^* \to E$ by

$$\phi(f) = \int_E x f(x)\mu(dx), \qquad f \in E^*,$$

where the integral is defined as a Bochner integral. The image space $\phi(E^*)$ becomes an inner product space under the inner product

$$\langle x, y\rangle_\mu = \int f(x)g(x)\mu(dx),$$

where $f, g \in E^*$ satisfy $x = \phi(f)$ and $y = \phi(g)$. We use $|\cdot|_\mu$ to denote the inner product norm and call the completion H_μ of $\phi(E^*)$ (under the norm $|\cdot|_\mu$) the reproducing kernel Hilbert space of $\mu = \mathcal{L}(X)$.

THEOREM 1.2. *For any* $\lambda > 0$ *and* $\varepsilon > 0$,

$$\mathbb{P}(\|Y\| \le \varepsilon) \ge \mathbb{P}(\|X\| \le \lambda\varepsilon)\mathbb{E}\exp\{-2^{-1}\lambda^2|Y|_\mu^2\}. \tag{1.10}$$

Note that we need $Y \in H_\mu \subset E$ almost surely. Otherwise, for $f \notin H_\mu$, $|f|_\mu = \infty$ and the result is trivial. Thus, the result can also be stated as follows. Let $(H, |\cdot|_H)$ be a Hilbert space and let Y be a Gaussian vector in H. Then, for any linear operator $L : H \to E$ and the Gaussian vector X in E with covariance operator LL^*,

$$\mathbb{P}(\|LY\| \le \varepsilon) \ge \mathbb{P}(\|X\| \le \lambda\varepsilon)\mathbb{E}\exp\{-2^{-1}\lambda^2|Y|_H^2\}$$

for any $\lambda > 0$ and $\varepsilon > 0$.

The proof of Theorem 1.2 is based on both directions of the well-known shift inequalities (4.1). The key new idea is the use of an independent random shift. To see the power of Theorem 1.2, we have the following consequence of Theorems 1.1 and 1.2 for the sup-norm. Other norms such as the Hölder norm and the L_p-norm can also be considered similarly, but we omit the exact statements; see the discussion on (4.5) for more details.

THEOREM 1.3. *We have, for* $m \geq 1$,

$$\lim_{\varepsilon \to 0} \varepsilon^{2/(2m+1)} \log \mathbb{P}\Big(\sup_{0 \le t \le 1} |X_m(t)| \le \varepsilon \Big) = -\kappa_m, \tag{1.11}$$

with

$$\begin{aligned} &\frac{2m+1}{2}\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-(2m+2)/(2m+1)} \\ &\quad \le \kappa_m \\ &\quad \le \frac{2m+1}{2}\left(\frac{\pi}{2}\right)^{2/(2m+1)}\left(2m \sin\frac{\pi}{2m}\right)^{-2m/(2m+1)} \end{aligned} \tag{1.12}$$

and $\lim_{m\to\infty} m^{-1}\kappa_m = \pi^{-1}$. *In addition, the following upper bound holds for* κ_m *based on* κ_{m-1}:

$$\big(4\kappa_m/(2m+1)\big)^{2m+1} \le \big(4\kappa_{m-1}/(2m-1)\big)^{2m-1}. \tag{1.13}$$

The particular case of $m = 1$, or the so-called integrated Brownian motion given in (1.2) and (1.3), was studied in Khoshnevisan and Shi (1998) using local time techniques. Here our upper bound $\kappa_1 \le (2\pi)^{2/3} \cdot (3/8)$ is explicit. In general, the exact value of κ_m in (1.11) is unknown for $m \ge 1$. The value $\kappa_0 = \pi^2/8$ is well known.

Finally, we mention the following easy application of Theorems 1.1 and 1.3 for the lim inf behavior of $X_m(t)$. The proof is standard and similar to that for $X_1(t)$ outlined in Khoshnevisan and Shi (1998) and we omit the details in this paper.

THEOREM 1.4. *We have, with probability* 1,

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t}\left(\frac{\log\log t}{t}\right)^{2m+1} \int_0^t |X_m(s)|^2\,ds \\ &\quad = \left(\frac{2m+1}{2}\right)^{2m+1}\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-(2m+2)} \end{aligned}$$

and

$$\liminf_{t\to\infty}\left(\frac{\log\log t}{t}\right)^{(2m+1)/2} \sup_{s\le t}|X_m(s)| = \kappa_m^{(2m+1)/2},$$

where κ_m *is given in Theorem* 1.3.

The rest of the paper is organized as follows. In Section 2, we find the exact formula for (1.5) given as Theorem 2.1. In Section 3, we present the proof of Theorem 1.1, which requires detailed asymptotic analysis. The proofs of Theorems 1.2 and 1.3 are presented in Section 4. Finally, in Section 5, we discuss the interplay between our results and the principal eigenvalue problems. Clearly, there is a need for further research.

2. Exact Laplace transform via a stochastic Fubini argument. To begin, we set, for a real number $\theta > 0$,

$$\phi(\theta) = \mathbb{E}\exp\Big(-\frac{\theta^2}{2}\int_0^1 X_m^2(t)\,dt\Big). \tag{2.1}$$

Let us introduce the Cameron–Martin space

$$H = \Big\{f \in L^2(0,1) : f' \in L^2(0,1) \text{ and } \forall\, t \in (0,1),\ f(t) = \int_0^t f'(s)\,ds\Big\}$$

together with the inner product defined, for any $f, g \in H$, by $\langle f, g\rangle_H = \int_0^1 f'(t)g'(t)\,dt$. Set

$$f_m(t) = \int_0^t \frac{(t-s)^m}{m!}\,df(s) \quad \text{and} \quad g_m(t) = \int_0^t \frac{(t-s)^m}{m!}\,dg(s).$$

We have

$$\begin{aligned}\int_0^1 f_m(t)g_m(t)\,dt &= \int_0^1 \Big(f_m(t)\int_0^t \frac{(t-s)^m}{m!}\,dg(s)\Big)\,dt\\ &= \int_0^1 \Big(\int_s^1 \frac{(t-s)^m}{m!} f_m(t)\,dt\Big)\,dg(s)\\ &= \langle \mathcal{A}(f), g\rangle_H,\end{aligned}$$

where

$$\mathcal{A}(f)(t) = \int_t^1 \frac{(u-t)^m}{m!} f_m(u)\,du = \int_0^1 k(s,t)\,df(s)$$

and

$$k(s,t) = \int_{s\vee t}^1 \frac{(u-s)^m}{m!}\frac{(u-t)^m}{m!}\,du.$$

Clearly, $\mathcal{A}$ is a self-adjoint operator on H and

$$\phi(\theta) = \mathbb{E}\exp\Big(-\frac{\theta^2}{2}\langle \mathcal{A}(W), W\rangle_H\Big) = \big(\det(I + \theta^2\mathcal{A})\big)^{-1/2} = \prod_{k\ge 1}(1 + \nu_k\theta^2)^{-1/2},$$

where the ν_k's are the eigenvalues of the operator $\mathcal{A}$, defined by the integral equation $\mathcal{A}(\phi) = \nu\phi$. It is easily seen that this equation is equivalent to the boundary value problem, see Lachal (2001),

$$\begin{aligned}
&\phi^{(2m+2)}(t) = \frac{(-1)^{m+1}}{\nu}\phi(t), && t \in (0,1),\\
&\phi^{(j)}(1) = 0, && 0 \le j \le m,\\
&\phi^{(j)}(0) = 0, && m+1 \le j \le 2m+1.
\end{aligned}$$

It is difficult to evaluate explicitly the eigenvalues in this way for all m. Similarly, we can follow the approach studied in Chiang, Chow and Lee (1986, 1994) to the point of explicit evaluation of all the eigenvalues above. To overcome the difficulties of finding all the eigenvalues, we derive the Laplace transform by a different method.

To motivate and show how we find a different operator to work with, we introduce an auxiliary independent Brownian motion $\widetilde{W}(t)$. By the stochastic Fubini theorem mentioned in the Introduction, we have

$$\begin{aligned}
\phi(\theta) &= \mathbb{E}\exp\biggl(-\frac{\theta^2}{2}\int_0^1 X_m^2(t)\,dt\biggr)\\
&= \mathbb{E}\exp\biggl(i\theta\int_0^1 X_m(t)\,d\widetilde{W}(t)\biggr)\\
&= \mathbb{E}\exp\biggl(i\theta\int_0^1\int_0^t X_{m-1}(s)\,ds\,d\widetilde{W}(t)\biggr)\\
&= \mathbb{E}\exp\biggl(i\theta\int_0^1 X_{m-1}(s)\bigl(\widetilde{W}(1)-\widetilde{W}(s)\bigr)\,ds\biggr)\\
&= \mathbb{E}\exp\biggl(i\theta\int_0^1 X_{m-1}(s)\bigl(\widetilde{W}(1-s)\bigr)\,ds\biggr),
\end{aligned}$$

since $\{\widetilde{W}(1) - \widetilde{W}(1-t);\ 0 \le t \le 1\}$ is a Brownian motion independent of $\{X_{m-1}(t); 0 \le t \le 1\}$.

Next, we can write the convolution in the form

$$\begin{aligned}
&\int_0^1 X_{m-1}(u)\bigl(\widetilde{W}(1-u)\bigr)\,du\\
&\quad= \int_0^1\int_0^u \frac{1}{(m-1)!}(u-s)^{m-1}\,dW(s)\int_0^{1-u} d\widetilde{W}(t)\,du\\
&\quad= \int_0^1\int_0^1 \frac{1}{(m-1)!}\biggl(\int_0^1 \mathbb{1}_{(0,u)}(s)(u-s)^{m-1}\mathbb{1}_{(0,1-u)}(t)\,du\biggr)\,dW(s)\,d\widetilde{W}(t)\\
&\quad= \int_0^1\int_0^1 K(s,t)\,dW(s)\,d\widetilde{W}(t),
\end{aligned}$$

where

$$K(s,t) = (m!)^{-1}\max(0, 1-s-t)^m = (m!)^{-1}(1-s-t)_+^m. \tag{2.2}$$

Now consider the compact continuous self-adjoint operator

$$A(f)(t) = \int_0^1 K(s,t) f(s)\, ds \tag{2.3}$$

and denote by λ_k and ϕ_k, $k \geq 1$, its eigenvalue and orthonormal eigenfunctions. Then

$$\int_0^1 \int_0^1 K(s,t)\, dW(s)\, d\widetilde{W}(t) = \sum_{k\geq 1} \lambda_k \xi_k \widetilde{\xi}_k$$

and

$$K(s,t) = \sum_{k\geq 1} \lambda_k \phi_k(s)\phi_k(t),$$

where $\xi_k = \int_0^1 \phi_k(t)\, dW(t)$ and $\widetilde{\xi}_k = \int_0^1 \phi_k(t)\, d\widetilde{W}(t)$ are independent standard normal random variables. Thus, we have, by combining things together,

$$\begin{aligned}
\phi(\theta) &= \mathbb{E}\exp\left(i\theta \int_0^1 X_{m-1}(s)\big(\widetilde{W}(1-s)\big)\, ds\right) \\
&= \mathbb{E}\exp\left(i\theta \sum_{k\geq 1} \lambda_k \xi_k \widetilde{\xi}_k\right) \\
&= \prod_{k\geq 1} \mathbb{E}\exp(-2^{-1}\theta^2 \lambda_k^2 \xi_k^2) \\
&= \prod_{k\geq 1} (1 - i\theta\lambda_k)^{-1/2}(1 + i\theta\lambda_k)^{-1/2}.
\end{aligned}$$

In fact, as pointed out by one of the referees, the operator A is exactly the square root of the operator $\mathcal{A}$ and the above relationship follows easily from $\nu_k = \lambda_k^2$, $\forall\, k \geq 1$. To be more precise, the kernel K_2 of the operator A^2 coincides with the kernel k of the operator $\mathcal{A}$, following from

$$K_2(s,t) = \int_0^1 K(s,x)K(x,t)\, dx = k(s,t).$$

In the rest of this section, we carry out the spectral study of K and the computations seem to be less ugly and more appropriate than those of k.

Let

$$\psi(\tau) = \prod_{k\geq 1} (1 - \tau\lambda_k).$$

Then

$$\frac{\phi'(\theta)}{\phi(\theta)} = -\frac{i}{2}\left(\frac{\psi'(i\theta)}{\psi(i\theta)} - \frac{\psi'(-i\theta)}{\psi(-i\theta)}\right) = \Im\left(\frac{\psi'(i\theta)}{\psi(i\theta)}\right), \tag{2.4}$$

where $\Im(\cdot)$ represents the imaginary part. On the other hand, $\psi(\tau)$ can be represented by the resolvent kernel

$$\Gamma(s,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(s,t) = \sum_{k\geq 1} \frac{\lambda_k \phi_k(s)\phi_k(t)}{1-\lambda\lambda_k} \tag{2.5}$$

of the operator A given in (2.3), where

$$K_n(s,t) = \int_0^1 \cdots \int_0^1 K(s,x_1)K(x_1,x_2)\cdots K(x_{n-1},t)\,dx_1\cdots dx_{n-1}$$

are the iterated kernels of A. That is, we have

$$\frac{\psi'(\tau)}{\psi(\tau)} = -\sum_{k\geq 1} \frac{\lambda_k}{1-\tau\lambda_k} = \int_0^1 \Gamma(t,t;\tau)\,dt. \tag{2.6}$$

It remains for us to compute the resolvent kernel $\Gamma(s,t;\lambda)$ given in (2.5) and then use (2.6) to find $\psi(\tau)$. This, indeed, was the approach used to find the Laplace transform of $\int_0^1 W(t)W(1-t)\,dt$ in the paper by Klyachko and Solodyannikov (1987). In our problem, however, we run into serious difficulties in finding a reasonable expression for $\Gamma(s,t;\lambda)$ that is suitable for evaluating (2.5).

To overcome these difficulties, we observe from (2.6) that

$$\frac{\psi'(\tau)}{\psi(\tau)} = -\sum_{k\geq 1} \frac{\lambda_k}{1-\tau\lambda_k} = -\sum_{n=0}^{\infty} \tau^n \int_0^1 K_{n+1}(t,t)\,dt. \tag{2.7}$$

We now compute the integrals on the right-hand side. Recall that $K(s,t) = (m!)^{-1}(1-s-t)_+^m$ for $0 \leq s,t \leq 1$. It is convenient to make the substitution $y_i = 1/2 - u_i$ to obtain

$$\begin{aligned}
&\int_0^1 K_{n+1}(u_0,u_0)\,du_0 \\
&\qquad = \int_0^1 \cdots \int_0^1 K(u_0,u_1)K(u_1,u_2)\cdots K(u_{n-1},u_n)K(u_n,u_0)\,du_0\cdots du_n \\
&\qquad = (m!)^{-(n+1)} \int_D (y_0+y_1)^m (y_1+y_2)^m \cdots (y_n+y_0)^m\,dy_0\,dy_1\cdots dy_n,
\end{aligned}$$

where the region D is determined by the inequalities

$$|y_i| \leq 1/2, \qquad y_i + y_{i+1} \geq 0, \qquad 0 \leq i \leq n,$$

with $y_{n+1} = y_0$. We can partition the region D into disjoint subsets D_k, $0 \le k \le n$, so that $\max_{0\le i\le n} y_i = y_k$ in D_k. That is, for $0 \le k \le n$,

$$D_k = D \cap \left\{(y_1, \dots, y_n) : \max_{0\le i\le n} y_i = y_k\right\}.$$

Note that the integrals over all these regions are the same, and therefore it is sufficient to evaluate the integral over one of them, say D_0. Using the substitutions $y_i = y_0 t_i$, $1 \le i \le n$, we have

$$\begin{aligned}
&\int_0^1 K_{n+1}(t,t)\,dt \\
&\quad = (n+1)(m!)^{-(n+1)} \\
&\qquad \times \int_{D_0} (y_0+y_1)^m (y_1+y_2)^m \cdots (y_n+y_0)^m \, dy_0\, dy_1 \cdots dy_n \\
&\quad = (n+1)(m!)^{-(n+1)} \int_0^{1/2} y_0^{(m+1)(n+1)-1}\, dy_0 \\
&\qquad \times \int_{|t_i|\le 1;\, t_i+t_{i+1}\ge 0} (1+t_1)^m (t_1+t_2)^m \cdots (t_n+1)^m \, dt_1 \cdots dt_n \\
&\quad = (m+1)^{-1} 2^{-(m+1)(n+1)} (m!)^{-(n+1)} P_{n+1}(1),
\end{aligned} \tag{2.8}$$

where $P_k(x)$, $k \ge 1$, is the polynomial defined by the recursion relationships

$$P_1(x) = (1+x)^m, \qquad P_{k+1}(x) = \int_{-x}^1 P_k(t)(t+x)^m \, dt. \tag{2.9}$$

Thus, we need to find the generation function for the polynomials $P_k(x)$. It is given in the following lemma along with other basic properties of the polynomials $P_k(x)$, $k \ge 1$.

LEMMA 2.1. (i) $\deg P_k(x) = (m+1)k - 1$, $k \ge 1$.

(ii) $P_{k+1}^{(l)}(x) = (m!/(m-l)!) \int_{-x}^1 P_k(t)(t+x)^{m-l}\, dt$, $l = 0, 1, 2, \dots, m$.

(iii) $P_{k+1}^{(m+1)}(x) = m! P_k(-x)$, $k \ge 1$.

(iv) $P_k^{(l)}(-1) = 0$, $l = 0, 1, \dots, m$. *More generally*, $P_k^{(q)}(-1) = 0$ *for* $q = 2j(m+1) + l$ *and* $P_k^{(q)}(1) = 0$ *for* $q = (2j+1)(m+1) + l$, *where* $0 \le l \le m$, $j \ge 0$.

(v) *The generation function for the polynomials* $P_k(x)$ *is*

$$\begin{aligned}
G(x,y) &\equiv \sum_{k=1}^{\infty} P_k(x) y^{(m+1)k-1} \\
&= m! \sum_{l=0}^{m} (-1)^{m+l} F_l(xy) \frac{W(F_0, \dots, F_{l-1}, F_{l+1}, \dots, F_m)(-y)}{W(F_0, F_1, \dots, F_m)(-y)},
\end{aligned} \tag{2.10}$$

where the analytic function $F_l(z)$ *is given by*

$$F_l(z) = \sum_{j=0}^{\infty} \frac{(-1)^{jl+(m+1)(j-1)j/2}(m!)^j}{((m+1)j+l)!} z^{(m+1)j+l}, \qquad 0 \le l \le m, \tag{2.11}$$

and the notation

$$W(f_0, f_1, \dots, f_m)(z) = \det\bigl(f_j^{(i)}(z)\bigr)_{0\le i,j\le m}$$

is the Wronskian determinant for the functions f_i, $0 \le i \le m$.

(vi) *In particular, for any complex number* z *with* $D(z) \ne 0$,

$$G(1, z) = \sum_{k=1}^{\infty} P_k(1) z^{(m+1)k-1} = m! \frac{\widehat{D}(z)}{D(z)},$$

where $D(z) = W(F_0, F_1, \dots, F_m)(-z)$ *and* $\widehat{D}(z)$ *is the determinant of the matrix obtained by replacing the last row of the matrix* $(F_l^{(q)}(-z))_{0\le q,l\le m}$ *with the row vector* $((F_0(z), \dots, F_m(z))$.

PROOF. The first three properties follow from the recursion definition given in (2.9) by differentiation. Property (iv) follows from properties (ii) and (iii).

To show (v), we first need a recursive relationship on the coefficients of $P_k(x)$ represented by

$$P_k(x) = \sum_{l=0}^{(m+1)k-1} a_{k,l} x^l = \sum_{j=0}^{k-1} \sum_{l=0}^{m} a_{k,(m+1)j+l} x^{(m+1)j+l}. \tag{2.12}$$

Substituting (2.12) into property (iii), we have, after simplifying the expression and equating the coefficients,

$$a_{k+1,(m+1)(j+1)+l} = \frac{(-1)^{(m+1)j+l}((m+1)j+l)!m!}{((m+1)(j+1)+l)!} a_{k,(m+1)j+l} \tag{2.13}$$

for any $k \ge 1$, $j \ge 0$ and $0 \le l \le m$.

Next, note that

$$\begin{aligned} G(x, y) &= \sum_{k=1}^{\infty} P_k(x) y^{(m+1)k-1} \\ &= \sum_{l=0}^{m} \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} a_{k,(m+1)j+l} x^{(m+1)j+l} y^{(m+1)k-1} \qquad (2.14) \\ &= \sum_{l=0}^{m} \sum_{j=0}^{\infty} Q_{j,l}(y) x^{(m+1)j+l}, \end{aligned}$$

where

$$Q_{j,l}(y) = \sum_{k=j+1}^{\infty} a_{k,(m+1)j+l}\, y^{(m+1)k-1}.$$

Using the recursive relationship in (2.13), we see that

$$\begin{aligned} Q_{j+1,l}(y) &= \sum_{k=j+1}^{\infty} a_{k+1,(m+1)(j+1)+l}\, y^{(m+1)(k+1)-1} \\ &= \sum_{k=j+1}^{\infty} \frac{(-1)^{(m+1)j+l}((m+1)j+l)!m!}{((m+1)(j+1)+l)!} a_{k,(m+1)j+l}\, y^{(m+1)(k+1)-1} \\ &= \frac{(-1)^{(m+1)j+l}((m+1)j+l)!m!}{((m+1)(j+1)+l)!} y^{m+1} Q_{j,l}(y). \end{aligned}$$

Thus, by iterating the above relationship, we obtain

$$Q_{j,l}(y) = \frac{(-1)^{jl+(m+1)(j-1)j/2}(m!)^j l!}{((m+1)j+l)!} y^{(m+1)j} Q_{0,l}(y). \tag{2.15}$$

Now returning to (2.14), we have, for $G(x, y)$,

$$\begin{aligned} G(x, y) &= \sum_{l=0}^{m} \sum_{j=0}^{\infty} \frac{(-1)^{jl+(m+1)(j-1)j/2}(m!)^j l!}{((m+1)j+l)!} y^{(m+1)j} Q_{0,l}(y) x^{(m+1)j+l} \\ &= \sum_{l=0}^{m} F_l(xy) l! Q_{0,l}(y) y^{-l} \qquad (2.16) \\ &= \sum_{l=0}^{m} F_l(xy) G_l(y), \end{aligned}$$

where F_l is as defined in (2.11) and $G_l(y) = l! Q_{0,l}(y) y^{-l}$. Next, we need to find $G_l(y)$, $0 \le l \le m$. They are not easy to find them via the generating function idea given above. But observe that we have, from the definition of $G(x, y)$ in (2.10) and property (iv),

$$\begin{aligned} \frac{\partial^q}{\partial x^q} G(x, y)\Big|_{x=-1} &= 0, \qquad q = 0, 1, \ldots, m-1, \\ \frac{\partial^m}{\partial x^m} G(x, y)\Big|_{x=-1} &= m! y^m, \end{aligned}$$

which are, from (2.16),

$$\begin{aligned}\sum_{l=0}^{m} F_l^{(q)}(-y)G_l(y) &= 0, \qquad 0 \le q \le m-1,\\ \sum_{l=0}^{m} F_l^{(m)}(-y)G_l(y) &= m!.\end{aligned} \tag{2.17}$$

Since F_l are linear independent, we see that the Wronskian determinant $W(F_0, F_1, \dots, F_m)(-y) \neq 0$ and thus, from (2.17),

$$G_l(y) = (-1)^{m+l} m! \frac{W(F_0, \dots, F_{l-1}, F_{l+1}, \dots, F_m)(-y)}{W(F_0, F_1, \dots, F_m)(-y)}, \tag{2.18}$$

and this finishes the proof of the lemma. □

Summarizing what we have so far, (2.4), (2.7), (2.8) and Lemma 2.1, we obtain the following result for the Laplace transform of the quadratic functional of the m-fold integrated Brownian motion by observing

$$\frac{\phi'(\theta)}{\phi(\theta)} = \frac{1}{(m+1)\theta} \Im\left(i \sum_{k\ge1} P_k(1)\left(\frac{i\theta}{m!2^{m+1}}\right)^k\right)$$

and

$$\sum_{k\ge1} P_k(x) y^k = y^{1/(m+1)} G(x, y^{1/(m+1)}).$$

THEOREM 2.1. *Let $\phi(\theta)$ be given in* (2.1) *for $\theta > 0$ real. Then we have*

$$\begin{aligned}\phi'(\theta)/\phi(\theta) &= (2m+2)^{-1}(m!/\theta)^{m/(m+1)}\\ &\quad \times \Re\{i^{1/(m+1)} \widehat{D}(2^{-1}(i\theta/m!)^{1/(m+1)})/D(2^{-1}(i\theta/m!)^{1/(m+1)})\},\end{aligned}$$

where the notation $\Re(\cdot)$ is for the real part of a complex number and the functions $D(z)$ and $\widehat{D}(z)$ are defined in Lemma 2.1(vi).

Note that for the special case $m = 1$, we have $F_0(y) = \cosh y$, $F_1(y) = \sin y$, and by setting $a = 2^{-1}\sqrt{\theta/2}$,

$$\begin{aligned}D(a(1+i)) &= (\cosh^2 a - \sin^2 a) - i(\cosh^2 a - \cos^2 a),\\ \widehat{D}(a(1+i)) &= \sin(2a) + i \sinh(2a).\end{aligned}$$

Therefore, Theorem 2.1 implies

$$\frac{\phi'(\theta)}{\phi(\theta)} = -\frac{1}{4\sqrt{2\theta}} \frac{\sinh\sqrt{2\theta} - \sin\sqrt{2\theta}}{\cosh^2\sqrt{\theta/2} + \cos^2\sqrt{\theta/2}}$$

from which (1.1) follows.

3. Proof of Theorem 1.1. It is clear that we need to obtain the asymptotic of $\phi'(\theta)/\phi(\theta)$ and hence $\log\phi(\theta)$ for $\theta \to \infty$. To this end, the following lemma for analytic functions over the complex with power series representation is critical in order to find the asymptotics of F_l, $W(F_0, F_1, \ldots, F_m)(-y)$ and G_l. General results of this type can be found in Wright (1940).

LEMMA 3.1. *For any integer $n \geq 0$ and q and complex number z,*

$$\sum_{jn+q\geq 0} \frac{1}{(jn+q)!} z^j = \frac{1}{n} \sum_{k=1}^{n} Z_k^{-q} e^{Z_k},$$

where

$$Z_k = |z|^{1/n} \exp\{i(2k\pi + \arg z)/n\}, \qquad k = 1, 2, \ldots, n.$$

Consequently,

$$\sum_{jn+q\geq 0} \frac{1}{(jn+q)!} z^j = \frac{1}{n} \sum_{|(2k\pi+\arg z)/n|\leq \pi/2} Z_k^{-q} e^{Z_k} + o\bigl(\exp\{-\delta|z|^{1/n}\}\bigr) \tag{3.1}$$

as $|z| \to \infty$, where δ can be any positive number less than

$$\min\{|\cos((2k\pi + \arg z)/n)|;\ \pi/2 < |(2k\pi + \arg z)/n| \leq \pi\}.$$

PROOF. First, note that, for each integer j,

$$\sum_{k=1}^{n} Z_k^j = \begin{cases} 0, & j \neq 0 \bmod(n), \\ nz^p, & j = 0 \bmod(n) \text{ and } j = pn. \end{cases}$$

Hence,

$$\sum_{k=1}^{n} Z_k^{-q} e^{Z_k} = \sum_{k=1}^{n} Z_k^{-q} \sum_{j=0}^{\infty} \frac{1}{j!} Z_k^j = \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{k=1}^{n} Z_k^{j-q} = n \sum_{pn+q\geq 0} \frac{1}{(pn+q)!} z^p.$$

□

PROOF OF THEOREM 1.1. By variable substitution and l'Hospital's principle, we only need to prove

$$\lim_{\theta\to\infty} \theta^{m/(m+1)} \frac{\phi'(\theta)}{\phi(\theta)} = -\left(2(m+1)\sin\frac{\pi}{2(m+1)}\right)^{-1}. \tag{3.2}$$

We apply Theorem 2.1. Note that, for all $0 \leq q, l \leq m$,

$$\begin{aligned} F_l^{(q)}(z) = {} & z^{l-q} \sum_{2(m+1)j+l-q\geq 0} \frac{(-1)^{j(m+1)}(m!)^{2j}}{[2(m+1)j+l-q]!} z^{2(m+1)j} \\ & + (-1)^l z^{m+1+l-q} \\ & \times \sum_{2(m+1)j+m+1+l-q\geq 0} \frac{(-1)^{j(m+1)}(m!)^{2j+1}}{[2(m+1)j+m+1+l-q]!} z^{2(m+1)j}. \end{aligned}$$

Let

$$I = \{k; |k + (1 - (-1)^m)/4| \le (m+1)/2\}. \tag{3.3}$$

Then I contains exactly $m+1$ integers. Let

$$\begin{aligned} Z_k &= \frac{1}{2}\theta^{1/(m+1)} \exp\left\{ i \frac{2k\pi + 2^{-1}(1 - (-1)^m)\pi}{2(m+1)} \right\} \\ &= \frac{1}{2}\theta^{1/(m+1)} \varepsilon_m^k e^{i\pi(1-(-1)^m)/4(m+1)}, \qquad k \in I, \end{aligned}$$

where

$$\varepsilon_m = \exp\{i\pi/(m+1)\}. \tag{3.4}$$

Applying (3.1) of Lemma 3.1 with $n = 2(m+1)$ and $z = (-1)^m(2^{-(m+1)}\theta)^2$ [hence $\arg z = (1 - (-1)^m)\pi/2$], we have after simplification that

$$\begin{aligned} &F_l^{(q)}\big(-2^{-1}(i\theta/m!)^{1/(m+1)}\big) \\ &\quad = \frac{1}{2(m+1)} H_m^{q-l} \sum_{k\in I} \varepsilon_m^{k(q-l)} \big[1 + (-1)^{m+k+l+1} i^{(1+(-1)^m)/2}\big] \\ &\qquad \times \exp\{Z_k\} + o\big(\exp\{-\delta\theta^{1/(m+1)}\}\big) \\ &\quad = f_{ql} + o\big(\exp\{-\delta\theta^{1/(m+1)}\}\big) \qquad \text{(say)} \end{aligned}$$

as $\theta \to \infty$, where $\delta > 0$ is a constant and $H_m = -(m!)^{1/(m+1)} e^{-(1+(-1)^m)\pi/4(m+1)}$.

To estimate $D(2^{-1}(i\theta/m!)^{1/(m+1)})$, we break it into two terms, according to the above decomposition. Since Z_k has a positive real part for each $k \in I$,

$$D\big(2^{-1}(i\theta/m!)^{1/(m+1)}\big) = \det(f_{ql})_{0\le q,l\le m} + o\left(\exp\left\{\sum_{k\in I} Z_k\right\}\right).$$

We can also break each row in the determinant of the the first term on the right-hand side, according to the decomposition

$$f_{ql} = \frac{1}{2(m+1)} H_m^{q-l} \sum_{k\in I} \varepsilon_m^{k(q-l)} \big[1 + (-1)^{m+k+l+1} i^{(1+(-1)^m)/2}\big] \exp\{Z_k\}.$$

We thus obtain a sum of $(m+1)^{m+1}$ determinants of order $(m+1)\times(m+1)$. Each of the determinants that has two or more rows indexed by the same k vanishes, as can also be seen from a computation similar to the one given below. The others can be obtained by using the fact

$$\begin{aligned} &\det\Big(\big(1 + (-1)^{m+k_q+l+1} i^{(1+(-1)^m)/2}\big) H_m^{q-l} \varepsilon_m^{k_q(q-l)}\Big)_{0\le q,l\le m} \\ &\quad = \det\Big(\big(1 + (-1)^{m+k_q+l+1} i^{(1+(-1)^m)/2}\big) \varepsilon_m^{k_q(q-l)}\Big)_{0\le q,l\le m}, \end{aligned}$$

where the equality follows from taking factor H_m^q from the qth row ($q = 0, 1, \dots, m$) and H_m^{-l} from the lth column ($l = 0, 1, \dots, m$), and $\{k_0, \dots, k_m\}$ is a permutation of the integers in I. Therefore,

$$D\big(2^{-1}(i\theta/m!)^{1/(m+1)}\big) = \exp\left\{\sum_{k\in I} Z_k\right\}\big[\big(2(m+1)\big)^{-(m+1)} \cdot \Delta + o(1)\big],$$

where

$$\Delta = \sum_{k_0,\dots,k_m} \det\Big(\big(1 + (-1)^{m+k_q+l+1} i^{(1+(-1)^m)/2}\big)\varepsilon_m^{k_q(q-l)}\Big)_{0\le q,l\le m}.$$

Similarly,

$$\begin{aligned} &D\left(2^{-1}\left(\frac{i\theta}{m!}\right)^{1/(m+1)}\right)\\ &\qquad = -\exp\left\{\sum_{k\in I} Z_k\right\}\bigg[\big(2(m+1)\big)^{-(m+1)}(m!)^{-m/(m+1)}\\ &\qquad\qquad\qquad \times \exp\left(-i\frac{1+(-1)^m}{4(m+1)}\pi\right)\widehat{\Delta} + o(1)\bigg], \end{aligned}$$

where

$$\widehat{\Delta} = \sum_{k_0,\dots,k_m} \varepsilon_m^{k_m} \det\Big(\big(1 + (-1)^{m+k_q+l+1} i^{(1+(-1)^m)/2}\big)\varepsilon_m^{k_q(q-l)}\Big)_{0\le q,l\le m}.$$

Let $p_0, \dots, p_m$ be the integers in I with increasing order. Given a permutation $\{k_0, \dots, k_m\}$ of $\{p_0, \dots, p_m\}$, we see that

$$\begin{aligned} &\det\Big(\big(1 + (-1)^{m+k_q+l+1} i^{(1+(-1)^m)/2}\big)\varepsilon_m^{k_q(q-l)}\Big)_{0\le q,l\le m}\\ &\quad = \left(\prod_{n=0}^m \varepsilon_m^{nk_n}\right)\det\Big(\big(1 + (-1)^{m+k_q+l+1} i^{(1+(-1)^m)/2}\big)\varepsilon_m^{-k_q l}\Big)_{0\le q,l\le m}\\ &\quad = (-1)^{a(k_0,\dots,k_m)}\left(\prod_{n=0}^m \varepsilon_m^{nk_n}\right)\\ &\qquad\quad \times \det\Big(\big(1 + (-1)^{m+p_q+l+1} i^{(1+(-1)^m)/2}\big)\varepsilon_m^{-p_q l}\Big)_{0\le q,l\le m}\\ &\quad = (-1)^{a(k_0,\dots,k_m)} \prod_{n=0}^m \varepsilon_m^{nk_q} C \qquad \text{(say)}, \end{aligned}$$

where $a(k_0, \dots, k_m)$ is the permutation number of $\{k_0, \dots, k_m\}$ with respect to $\{p_0, \dots, p_m\}$. Here the determinant $C \ne 0$ is given in Lemma 3.2. Thus, by the definition of determinant,

$$\begin{aligned}\Delta &= C\sum_{k_0,\dots,k_m}(-1)^{a(k_0,\dots,k_m)}\prod_{n=0}^{m}\varepsilon_m^{nk_q}\\ &= C\det(\varepsilon_m^{qp_l})_{0\le q,l\le m}\\ &= C\prod_{k=1}^{m}\varepsilon_m^{kp_0}\det(\varepsilon_m^{ql})_{0\le q,l\le m}\\ &= C\prod_{k=1}^{m}\varepsilon_m^{kp_0}\prod_{0\le j<k\le m}(\varepsilon_m^k-\varepsilon_m^j),\end{aligned}$$

where the last step follows from the well-known formula for Vandermonde's determinant.

Similarly,

$$\widehat{\Delta} = C\sum_{k_0,\dots,k_m}\varepsilon_m^{k_m}(-1)^{a(k_0,\dots,k_m)}\prod_{n=0}^{m}\varepsilon_m^{nk_q} = C\det(r_{ql})_{0\le q\le m;\ 0\le l\le m},$$

where

$$r_{ql} = \varepsilon_m^{qp_l}, \qquad 0\le q\le m-1,\ l\le m,$$

and

$$r_{ml} = \varepsilon_m^{(m+1)p_l}, \qquad 0\le l\le m.$$

By the formula for Vandermonde's determinant,

$$\widehat{\Delta} = C\prod_{k=1}^{m-1}\varepsilon_m^{kp_0}\varepsilon_m^{(m+1)p_0}\prod_{0\le j<k\le m-1}(\varepsilon_m^k-\varepsilon_m^j)\prod_{j=0}^{m-1}\big(\varepsilon_m^{m+1}-\varepsilon_m^j\big).$$

Using the fact that $C\neq 0$ from Lemma 3.2, we see $\Delta\neq 0$ and $\widehat{\Delta}\neq 0$. Hence, we have

$$\lim_{\theta\to\infty}\frac{\widehat{D}(2^{-1}(i\theta/m!)^{1/(m+1)})}{D(2^{-1}(i\theta/m!)^{1/(m+1)})} = -(m!)^{-m/(m+1)}\exp\biggl(-i\frac{1+(-1)^m}{4(m+1)}\pi\biggr)\frac{\widehat{\Delta}}{\Delta}.$$

Further, from the above computations of Δ and $\widehat{\Delta}$,

$$\frac{\widehat{\Delta}}{\Delta} = \varepsilon_m^{p_0}\frac{\prod_{j=0}^{m-1}(\varepsilon_m^{m+1}-\varepsilon_m^j)}{\prod_{j=0}^{m-1}(\varepsilon_m^{m}-\varepsilon_m^j)} = \varepsilon_m^{p_0}\frac{(\varepsilon_m^{m+1}-1)\varepsilon_m^{m-1}}{\varepsilon_m^m-\varepsilon_m^{m-1}} = \frac{2\varepsilon_m^{p_0}}{1-\varepsilon_m}.$$

Using the fact that $p_0 = -m/2-(1-(-1)^m)/4$, one can easily obtain

$$\lim_{\theta\to\infty}\frac{\widehat{D}(2^{-1}(i\theta/m!)^{1/(m+1)})}{D(2^{-1}(i\theta/m!)^{1/(m+1)})} = -(m!)^{-m/(m+1)}i^{-1/(m+1)}\biggl(\sin\frac{\pi}{2(m+1)}\biggr)^{-1}.$$

Finally, (3.2) follows from Theorem 2.1 and the proof is complete after establishing the following fact.

LEMMA 3.2. *Given $m \geq 1$, let I and ε_m be defined by* (3.3) *and* (3.4), *respectively, and let $p_0, \dots, p_m$ all be integers in I with increasing order. Then*

$$C \equiv \det\Big(\big(1 + (-1)^{m+p_q+l+1} i^{(1+(-1)^m)/2}\big)\varepsilon_m^{-p_q l}\Big)_{0 \le q,l \le m} \neq 0.$$

PROOF. Note that $(-1)^{p_0} i^{(1+(-1)^m)/2} = (-1)^m i^{-m+1}$. Hence,

$$C = \det\Big(\big(1 + (-1)^{q+l+1} i^{-m+1}\big)\varepsilon_m^{-ql}\Big)_{0 \le q,l \le m} \prod_{l=0}^{m} \varepsilon_m^{-lp_0}.$$

We first consider the case when $m = 2k$ is even. To simplify the notation, we use "$a \simeq b$" for the relationship between two numbers a and b such that $a = cb$ for some $c \neq 0$. We have

$$C \simeq \det\big(\big(1 + (-1)^{q+l} i\big)\varepsilon_m^{-ql}\big)_{0 \le q,l \le 2k}.$$

To manipulate the determinant on the right-hand side, we introduce the notation $[l]_2$ for any nonnegative integer l given by $[l]_2 = 0$ if l is even, and $[l]_2 = 1$ if l is odd. Dividing each row by $1 + i$ and using $1 - i = -i(1 + i)$, we have

$$\begin{aligned} C &\simeq \det\big((-i)^{[l+q]_2} \varepsilon_m^{-ql}\big)_{0 \le q,l \le 2k} = \det\big((-1)^{l+q} (-i)^{[l+q]_2} \varepsilon_m^{-ql}\big)_{0 \le q,l \le 2k} \\ &\simeq \det\big(i^{[l+q]_2} \varepsilon_m^{-ql}\big)_{0 \le q,l \le 2k}. \end{aligned}$$

Then multiplying $i^{[q]_2}$ by the qth row for $q = 0, 1, \dots, m$, that is, multiplying i by the odd rows, we obtain

$$C \simeq \det\big(i^{[l+q]_2+[q]_2} \varepsilon_m^{-ql}\big)_{0 \le q,l \le 2k} \simeq \det\big((-1)^{ql} \varepsilon_m^{-ql}\big)_{0 \le q,l \le 2k} \neq 0,$$

where the second step follows from the fact that

$$i^{[l+q]_2+[q]_2} = (-1)^{(l+1)(q+1)} i^{[l]_2+1},$$

taking out common factors from each row and column, and the last step is due to Vandermonde's formula.

Next, consider the case when $m = 2k - 1$ is odd. We need only to prove

$$\det\big(\big(1 + (-1)^{q+l}\big)\varepsilon_m^{-ql}\big)_{0 \le q,l \le 2k-1} \neq 0 \tag{3.5}$$

and

$$\det\big(\big(1 + (-1)^{q+l+1}\big)\varepsilon_m^{-ql}\big)_{0 \le q,l \le 2k-1} \neq 0. \tag{3.6}$$

We only prove (3.5) since the proof of (3.6) is similar. First, we move the original rows with $q = 0, 2, \dots, 2k-2$ of the determinant in (3.5) to the first k rows without changing their relative order (so the rest of the rows will occupy the next k rows with their relative order being kept); second, we do the same column interchanges. So we obtain

$$\det\big(\big(1 + (-1)^{q+l}\big)\varepsilon_m^{-ql}\big)_{0 \le q,l \le 2k-1} \simeq \det\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}_{2k \times 2k} = \det(A) \cdot \det(B) \neq 0,$$

where A and B are two $k \times k$ matrices given by

$$A = \det(\varepsilon_m^{-4rs})_{0\le r,s\le k-1} \quad \text{and} \quad B = \det\left(\varepsilon_m^{-(2r+1)(2s+1)}\right)_{0\le r,s\le k-1},$$

and the fact that $\det(A) \cdot \det(B) \neq 0$ can easily be seen from the formula for Vandermonde's determinant. $\square$

4. Proofs of Theorems 1.2 and 1.3. The proof of Theorem 1.2 is simple and based on both directions of the following well-known shift inequalities for symmetric convex sets [see, e.g., Dudley, Hoffmann-Jørgensen and Shepp (1979) and de Acosta (1983)].

LEMMA 4.1. *Let μ be a centered Gaussian measure in a separable Banach space E with norm $\|\cdot\|$. Then, for any $f \in H_\mu$, the reproducing kernel Hilbert space of μ, and $r > 0$,*

$$\exp\{-|f|_\mu^2/2\}\mu(x : \|x\| \le r) \le \mu(x : \|x - f\| \le r) \le \mu(x : \|x\| \le r). \tag{4.1}$$

Furthermore,

$$\mu(x : \|x - f\| \le \varepsilon) \sim \exp\{-|f|_\mu^2/2\}\mu(x : \|x\| \le \varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{4.2}$$

The upper bound follows from Anderson's inequality $\mu(A + x) \le \mu(A)$ for every convex symmetric subset A of E and every $x \in E$; see Anderson (1955). The lower bound follows from the Cameron–Martin formula

$$\mu(A - f) = \int_A \exp\{-\tfrac{1}{2}|f|_\mu^2 - \langle x, f\rangle_\mu\}\, d\mu(x) \tag{4.3}$$

for Borel subsets A of E, $f \in H_\mu$, together with Hölder's inequality and the symmetry of $\langle x, f\rangle_\mu$ on $A = \{x : \|x\| \le r\}$. Note that $\langle x, f\rangle_\mu$ can be defined as the stochastic inner product for μ almost all x in E. A particularly nice proof of (4.3) is contained in Proposition 2.1 of de Acosta (1983). Refinements of (4.1) are given in Kuelbs, Li and Linde (1994) and Kuelbs, Li and Talagrand (1994), which play important roles in the studies of the functional form of Chung's law of the iterated logarithm.

PROOF OF THEOREM 1.2. Without loss of generality, assume X and Y are independent. Then, by applying (4.1), we obtain

$$\mathbb{P}(\|Y\| \le \varepsilon) \ge \mathbb{P}(\|X - \lambda Y\| \le \lambda\varepsilon) \ge \mathbb{P}(\|X\| \le \lambda\varepsilon)\mathbb{E}\exp\{-2^{-1}\lambda^2|Y|_\mu^2\},$$

which completes the proof. $\square$

PROOF OF THEOREM 1.3. The existence of limits in (1.11) can be found in Li and Linde (1998) based on subadditivity. The lower bound for κ_m in (1.12) follows easily from

$$\mathbb{P}\left(\sup_{0\le t\le 1} |X_m(t)| \le \varepsilon\right) \le \mathbb{P}\left(\int_0^1 |X_m(t)|^2\, dt \le \varepsilon^2\right)$$

and the L_2-estimate given in Theorem 1.1.

The upper bound for κ_m in (1.12) follows from Theorem 1.2, the L_2-estimate given in (1.7) and the well-known estimate

$$\log \mathbb{P}\Big(\sup_{0\le t\le 1} |W(t)| \le \varepsilon\Big) \sim -(\pi^2/8)\varepsilon^{-2}.$$

To be more precise, take $Y = X_m(t)$ and $X = W(t)$ in Theorem 1.2. Then, for any norm $\|\cdot\|$ on $C[0,1]$ and any $\lambda = \lambda_\varepsilon > 0$,

$$\mathbb{P}(\|X_m\| \le \varepsilon) \ge \mathbb{P}(\|W(t)\| \le \lambda\varepsilon)\mathbb{E}\exp\Big\{-\frac{\lambda^2}{2}\int_0^1 X_{m-1}^2(s)\,ds\Big\}, \tag{4.4}$$

since $|f|_\mu^2 = \int_0^1 (f'(s))^2\,ds$ for Wiener measure $\mu = \mathcal{L}(W)$. Taking $\|\cdot\|$ to be the sup-norm on $C[0,1]$ and $\lambda = \lambda_\varepsilon = \alpha\varepsilon^{-2m/(2m+1)}$ in (4.4) with $\alpha > 0$, it follows from the existence of the constants that

$$\begin{aligned}
-\kappa_m &= \lim_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |X_m(t)| \le \varepsilon\Big)\\
&\ge \lim_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |W(t)| \le \alpha\varepsilon^{1/(2m+1)}\Big)\\
&\quad + \lim_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log \mathbb{E}\exp\Big\{-\frac{\alpha^2}{2}\varepsilon^{-4m/(2m+1)}\int_0^1 X_{m-1}^2(s)\,ds\Big\}\\
&= -\frac{\pi^2}{8}\alpha^{-2} - \frac{1}{2}\alpha^{1/m}\Big(\sin\frac{\pi}{2m}\Big)^{-1}.
\end{aligned}$$

Now picking the best $\alpha > 0$, we obtain

$$\begin{aligned}
\kappa_m &\le \min_{\alpha>0}\Big(\frac{\pi^2}{8}\alpha^{-2} + \frac{1}{2}\alpha^{1/m}\Big(\sin\frac{\pi}{2m}\Big)^{-1}\Big)\\
&= \frac{2m+1}{2}\Big(\frac{\pi}{2}\Big)^{2/(2m+1)}\Big(2m\sin\frac{\pi}{2m}\Big)^{-2m/(2m+1)},
\end{aligned}$$

which is the upper bound for κ_m in (1.12).

It is of interest to note that both bounds rely on easier L_2-estimates and the constant bounds for κ_m are the sharpest known. Furthermore, we can also consider the L_p-norm on $C[0,1]$ with $1 \le p < \infty$ for $X_m(t)$. It is shown in Li (2001) that

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log \mathbb{P}\Big(\Big(\int_0^1 |X_m(t)|^p\,dt\Big)^{1/p} \le \varepsilon\Big) = -C_{m,p}. \tag{4.5}$$

Thus, bounds on $C_{m,p}$ can be given, similar to what we did for the above sup-norm case. We omit the details here.

Finally, we prove (1.13) based on Theorem 1.2, similar to the upper bound argument above. This time, we take $Y = X_m(t)$ and $X = X_{m-1}(t)$ in Theorem 1.2. Then, for any $\lambda = \lambda_\varepsilon > 0$,

$$\mathbb{P}\Big(\sup_{0\le t\le 1} |X_m(t)| \le \varepsilon\Big) \ge \mathbb{P}\Big(\sup_{0\le t\le 1} |X_{m-1}(t)| \le \lambda\varepsilon\Big)\mathbb{E}\exp\Big\{-\frac{\lambda^2}{2}\int_0^1 W^2(s)\,ds\Big\},$$

since $|f|_\mu^2 = \int_0^1 (f^{(m)}(s))^2\,ds$ for the Gaussian measure $\mu = \mathcal{L}(X_{m-1})$. Taking $\lambda = \lambda_\varepsilon = \beta\varepsilon^{-2/(2m+1)}$ with $\beta > 0$, it follows from the existence of the constants and the L_2-estimate given in (1.7) that

$$\begin{aligned}
-\kappa_m &= \lim_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |X_m(t)| \le \varepsilon\Big) \\
&\ge \lim_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log \mathbb{P}\Big(\sup_{0\le t\le 1} |X_{m-1}(t)| \le \beta\varepsilon^{(2m-1)/(2m+1)}\Big) \\
&\quad + \lim_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log \mathbb{E}\exp\Big\{-\frac{\beta^2}{2}\varepsilon^{-4/(2m+1)}\int_0^1 W^2(s)\,ds\Big\} \\
&= -\beta^{-2/(2m-1)}\kappa_{m-1} - \frac{\beta}{2}.
\end{aligned}$$

Now picking the best $\beta > 0$, we obtain, for $m \ge 1$,

$$\begin{aligned}
\kappa_m &\le \min_{\beta>0} \big(\beta^{-2/(2m-1)}\kappa_{m-1} + \beta/2\big) \\
&= \big((2m-1)/4\big)^{2/(2m+1)} \cdot (2m+1)(2m-1)^{-1} \cdot \kappa_{m-1}^{(2m-1)/(2m+1)},
\end{aligned}$$

which can be rewritten as (1.13). In the case $m = 2$, we have

$$\kappa_2 \le (5/3)\cdot(3/4)^{2/5}\cdot\kappa_1^{3/5}.$$

Note that, based on the estimate $\kappa_1 \le (3/8)\cdot(2\pi)^{2/3}$, the estimate given in (1.12), namely $\kappa_2 \le (5/8)\cdot(2\pi)^{2/5}$, is better than the foregoing one. $\square$

5. Remarks on related problems and approaches. Consider the $(m+1)$-dimensional diffusion process $X(t) = (X_0(t), \dots, X_m(t))$ with generator

$$\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x_0^2} + \sum_{k=1}^m x_{k-1}\frac{\partial}{\partial x_k}. \tag{5.1}$$

The hypoellipticity of this operator can be checked with the help of the Hörmander condition [see Chaleyat-Maurel and Elie (1981) for details] and could possibly yield some information about the unicity of the solution for certain boundary value problems related to $\mathcal{L}$.

Consider the first exit time of the unbounded domain $D = \{(x_0, \dots, x_m)\colon |x_m| < 1\}$:

$$\tau = \inf\{t \ge 0 : |X_m(t)| \ge 1\} = \inf\{t \ge 0 : X(t) \notin D\}.$$

Our main results are then equivalent to

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{-\int_0^t X_m^2(s)\,ds\right\} = -2^{-(2m+1)/(2m+2)}\left(\sin\frac{\pi}{2m+2}\right)^{-1} \tag{5.2}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\{\tau \ge t\} = -\kappa_m, \tag{5.3}$$

with bounds on κ_m given in (1.12) and (1.13).

Define the linear operator

$$\widetilde{\mathcal{L}}u = \frac{1}{2}\frac{\partial^2 u}{\partial x_0^2} + \sum_{k=1}^{m} x_{k-1}\frac{\partial u}{\partial x_k} - x_m^2 u, \qquad u \in C_{\mathrm{b}}^2(\mathbb{R}^{m+1}),$$

and let $\sigma(\widetilde{\mathcal{L}})$ be the spectrum set of $\widetilde{\mathcal{L}}$, where $C_{\mathrm{b}}^2(\mathbb{R}^{m+1})$ is the space of bounded, twice differentiable functions on $\mathbb{R}^{m+1}$. Further, let $C_{\mathrm{b}}(\mathbb{R}^{m+1})$ be the Banach space of bounded functions on $\mathbb{R}^{m+1}$ with the norm $\|f\| = \sup_x |f(x)|$ and define a semigroup of linear bounded operators T_t $(t \ge 0)$ on $C_{\mathrm{b}}(\mathbb{R}^{m+1})$ by

$$T_t f(x) = \mathbb{E}_x\left(\exp\left\{-\int_0^t X_m^2(s)\,ds\right\} f\big(X_0(t), \dots, X_m(t)\big)\right),$$
$$x = (x_0, \dots, x_m) \in \mathbb{R}^{m+1}.$$

Then $\widetilde{\mathcal{L}}$ is the infinitesimal generator of T_t. By the Gelfand theorem [see, e.g., Proposition 3.8, page 197, in Conway (1990)],

$$\lim_{t\to\infty} \frac{1}{t} \log \|T_t\| = \sup\{\log|\sigma(T_1)|\}. \tag{5.4}$$

On the other hand, it is not hard to see

$$\|T_t\| = \sup_x \mathbb{E}_x \exp\left\{-\int_0^t X_m^2(s)\,ds\right\} = \mathbb{E}\exp\left\{-\int_0^t X_m^2(s)\,ds\right\}, \tag{5.5}$$

where the second equality follows from Anderson's inequality for centered Gaussian measures and the simple fact that $\mathbb{E}Y = \int_0^\infty \mathbb{P}(Y > y)\,dy$ for $Y \ge 0$. Thus we have, from (5.4) and (5.5),

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\left\{-\int_0^t X_m^2(s)\,ds\right\} = \sup\{\log|\sigma(T_1)|\}$$

and, as a consequence of Theorem 1.1 or (5.2),

$$\sup\{\log|\sigma(T_1)|\} = -2^{-(2m+1)/(2m+2)}\left(\sin\frac{\pi}{2m+2}\right)^{-1}.$$

It is tempting to conclude directly via the spectrum map theorem that

$$\sup\{\log|\sigma(T_1)|\} = \sup \Re\{\sigma(\widetilde{\mathcal{L}})\}.$$

But there is no general results for unbounded $\widetilde{\mathcal{L}}$; see Davies [(1980), Theorem 2.16; (1995)], Goldstein (1985) and Ethier and Kurtz (1986).

Similarly, if we view $\mathcal{L}$ as the linear operator defined on the space

$$C^2_{0,\mathrm{b}}(\bar{D}) = \{f \in C^2_{\mathrm{b}}(\bar{D}) : f \equiv 0 \text{ on } \partial D\},$$

then we can also relate the constant κ_m to the corresponding spectral radius.

It is interesting to note that the technique of spectrum theory can be applied in broader situations, especially on the existence of the constant. For example, we can apply the above arguments to show easily the existence of the small ball constant under the L_p-norm for X_m by considering

$$T_{t,p} f(x) = \mathbb{E}_x\biggl(\exp\biggl\{-\int_0^t |X_m(s)|^p\, ds\biggr\} f\bigl(X_0(t), \ldots, X_m(t)\bigr)\biggr), \qquad x = (x_0, \ldots, x_m).$$

In particular, this justifies Theorem 4.1 in Khoshnevisan and Shi (1998) for the case $m = 1$, where the suggested arguments for the proof do not work. A different and direct method for proving the existence of small ball constants under various norms, based mainly on scaling rather than subadditivity, is given in Li (2001) for Gaussian processes, including X_m.

Our result can also be related to, respectively, the following eigenvalue problems:

$$\widetilde{\mathcal{L}}u = -\lambda u \tag{5.6}$$

and

$$\begin{aligned} \mathcal{L}u &= -\lambda u \qquad \text{in } D,\\ u &= 0 \qquad \text{on } \partial D. \end{aligned} \tag{5.7}$$

In the uniform elliptic operator case, Donsker and Varadhan (1976), see also Pinsky (1996), show by the Feymann–Kac formula that solutions for the above eigenvalue problems exist and their principal eigenvalues are the constants we are looking for. However, this approach does not seem to work in our case, mainly because we do not know whether or not the solutions (principal eigenvalues) exist. If they exist, they should be the constants we want, in the light of (5.2) and (5.3). Recall that $-\lambda_0$ is the principal eigenvalue of (5.6) if it satisfies (5.6) for some $u_0 > 0$ in the domain of $\widetilde{\mathcal{L}}$. Then

$$T_t u_0(x) = e^{-t\lambda_0} u_0(x)$$

by solving the differential equation

$$\frac{d}{dt} T_t u_0(x) = -\lambda T_t u_0(x) \qquad \text{with } T_0 u_0(x) = u_0(x),$$

and it is hoped that, as $t \to \infty$, $\mathbb{E}\exp\{-\int_0^t X_m^2(s)\,ds\}$ behaves like, up to some controllable factors,

$$\mathbb{E}\biggl(\exp\biggl\{-\int_0^t X_m^2(s)\,ds\biggr\}u_0\bigl(X_0(t),\dots,X_m(t)\bigr)\biggr) = e^{-t\lambda_0}u_0(0) = T_t u_0(0).$$

Note that we encounter some difficulties here since we have an unbounded domain from above, but not from below. In the literature on eigenvalue problems, it seems that little is known in the degenerate case. We hope the special case considered here will attract some interest. Maybe the recent work of Feng and Kurtz (2001) can be applied here.

The third connection we know is the following. Write

$$u(x,t) = \mathbb{E}_x \exp\biggl\{-\int_0^t X_m^2(s)\,ds\biggr\}, \qquad x = (x_0,\dots,x_m) \in \mathbb{R}^{m+1},$$

and

$$v(x,t) = P_x\{\tau \ge t\}, \qquad x = (x_0,\dots,x_m) \in \mathbb{R}^{m+1}.$$

Then u and v solve, respectively,

$$\frac{\partial u}{\partial t} = \widetilde{\mathcal{L}}u, \qquad u(x,0) = 1, \tag{5.8}$$

and

$$\begin{aligned} &\frac{\partial v}{\partial t} = \mathcal{L}v && \text{in } D,\\ &v(x,0) = 1, && x \in D,\\ &v(x,t) = 0, && x \in \partial D. \end{aligned} \tag{5.9}$$

So (5.2) and (5.3) can be written as

$$\lim_{t\to\infty} \frac{1}{t}\log u(0,t) = -2^{-(2m+1)/(2m+2)}\biggl(\sin\frac{\pi}{2m+2}\biggr)^{-1} \tag{5.10}$$

and

$$\lim_{t\to\infty} \frac{1}{t}\log v(0,t) = -\kappa_m. \tag{5.11}$$

The trouble is, however, we do not know whether or not the bounded solutions of (5.8) and (5.9) are unique. Is it true that all solutions satisfy (5.10) and (5.11) if they are not unique?

A closely related and useful technique in studying certain asymptotic problems is the logarithmic transformation $V = -\log v(x,t)$, which changes (5.11) into a nonlinear evolution equation for V. This can then be viewed as a stochastic control problem; see, for example, Fleming and Soner (1993).

Finally, we point out that we do not have any variational representation for the constants κ_m other than the existence with bounds given in (1.12) and (1.13). It would be very nice to have an analytical expression for κ_m. Unfortunately, we do not quite have the technical machinery to carry out any of the approaches mentioned in this section.

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